Introduction to Nonlinear Dynamics

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Module -05
Lecture-16
1-Dimensional Flows, Flows on the Circle, Lecture 2

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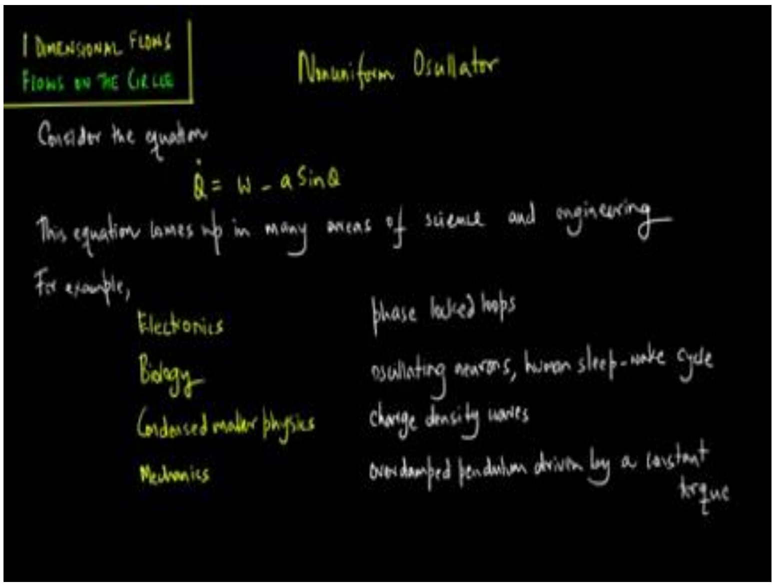

We are all still on the case of one dimensional flows, where we are dealing with flows on the circle and in this lecture our focus will be on the non-uniform oscillator. Now consider the following equation tita dot = omega - a sine tita. And this equation actually comes up in numerous areas of science and engineering. For example, in electronics we have phase locked loops, in biology you have oscillating neurons and the human sleep wake circle.

In condensed matter physics, you have charged density waves and in mechanics you have the over damped pendulum, which is driven by a constant torque. Now these are just few examples of nonlinear oscillators which arise in science and engineering.

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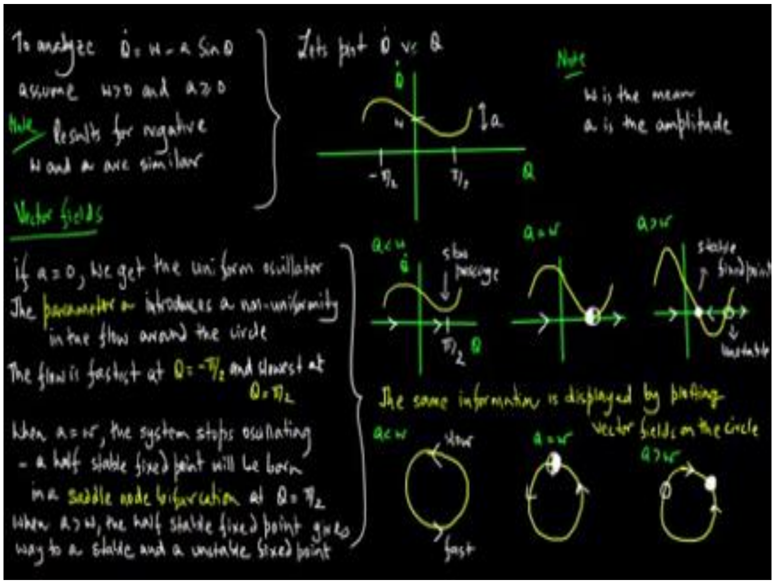

To analyse tita dot = omega - a sine tita. We assume that omega is greater than zero, a is greater than or equal to zero. Note that the results for negative omega and a are actually similar. So now let us plot tita dot versus tita, so we plot tita dot versus tita. Note that omega is the mean and a is the amplitude, now let us consider the vector fields for the system. If a = 0, we get the uniform oscillator, the parameter a introduces a non uniformity in the flow around the circle.

The flow is the fastest at tita = -pi on two and the slowest at tita = pi on 2. So, when a is less than omega and we plot tita dot versus tita, we highlight the area of the slope passage. When a = omega the system stops oscillating and a half stable fixed point will be born in a saddle node bifurcation at tita = pi on 2. So consider a = omega, in this case we get a half stable fixed point. Where a is greater than omega, the half stable fixed point gives way to a stable and a unstable fixed point.

So, let us plot a greater than omega and we note that we have an unstable fixed point and a stable fixed point. The same information is also displayed by plotting vector fields on the circle. So, when a is less than omega, we plot the circle and highlight the fast and slow passage points. When a = omega, we get a half stable fixed point and when a greater than omega we have a stable and an unstable fixed point.

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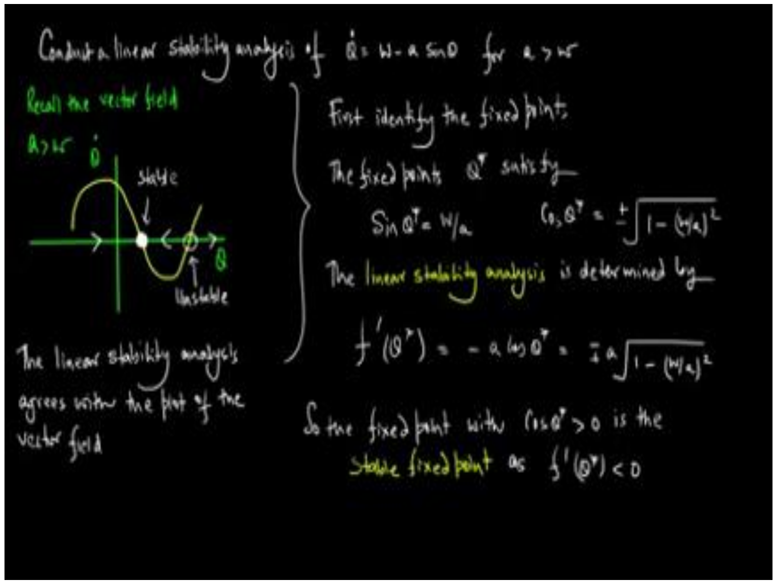

So, we conduct a linear stability analysis of tita dot = omega - a sine tita, for a greater than omega. So, let us recall the vector field for the case where a is greater than omega and plot tita dot versus tita. Remember that we had one unstable fixed point and one stable fixed point. So, we first identify the fixed points. The fixed points tita star satisfy sine tita star = omega / a and cos tita star is equal to plus minus of the square root of 1- omega by a square.

So, from the previous analysis, we know that the linear stability analysis is determined by f prime of tita star which is equal to minus a cos tita star, which is equal to minus plus a times square root 1 - omega a square. So, the fixed point with cos tita star greater than zero is the stable fixed point as f prime of tita star is less than zero. So, we find that the linear stability analysis agrees with the plot of the vector field as of course it should.

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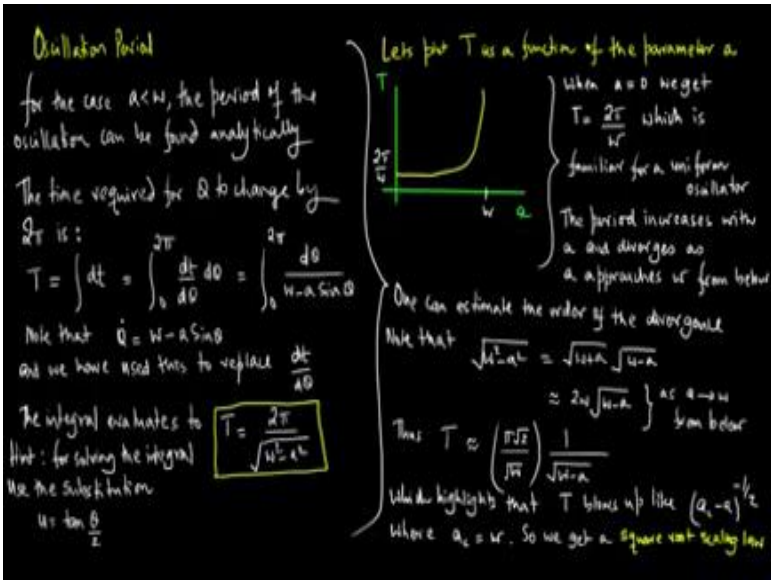

We now look at the oscillation period, for the case a less than omega. The period of the oscillation can actually be found analytically, the time required for tita to change by 2pi is capital T = integral dt = 0 to 2pi dt by d tita times d tita which is equal to zero to 2pi d tita / omega - a sine tita. Now note that tita dot = omega - a sine tita, as we have used this to actually replace dt / d tita. So the integral now evaluates to capital T =2pi / omega square - a square square root and so now we have a formula for capital T.

Here is a hint for solving the integral one would move to use the substitution u = tan tita by 2. Now let us plot capital T as a function of the parameter a when a = 0, we get capital T =2pi by omega, which is familiar for a uniform oscillator. The period actually increases with a and diverges as a approaches omega from below. One can actually estimate the order of the divergence.

Now note that omega square - a square square root = omega + a square root times omega - a square root which is approximately is equal to 2omega times omega - a square root as a tends to omega from below. And thus capital T is approximately is equal to pi times square root of 2/ square root of omega times 1 upon square root of omega - a. Now this highlights to us that the capital tou goes up like a critical -a to the power of -1/2, where a critical = omega. So we end up getting a square root like scaling low.

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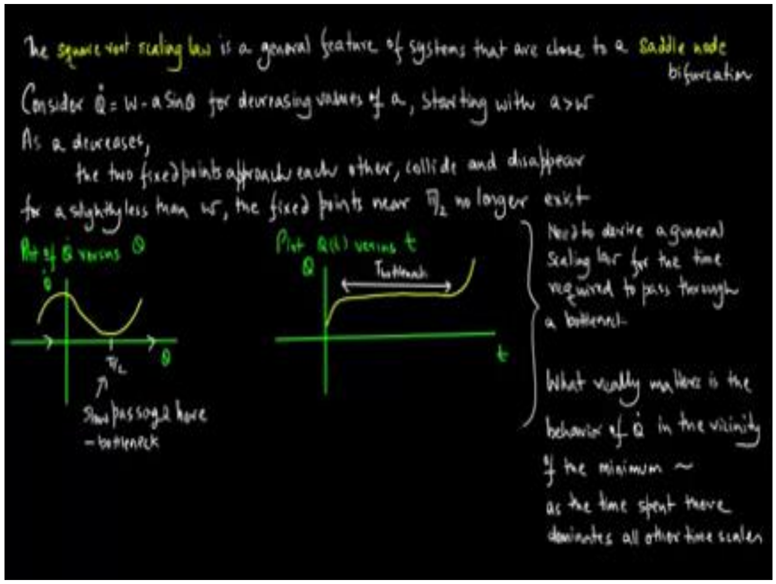

The square root scaling low is in fact a general feature of systems that are close to a saddle node bifurcation. Consider tita dot = omega - a sine tita for decreasing values of a, starting with a greater than omega. As a decreases, the two fixed points approach each other collide and disappear, for a slightly less than omega, the fixed points near pi on 2 actually no longer exists. So now we plot q's of tita versus tita and we highlight the area of slow passage, which represents the bottle neck.

Now let us plot q's of t versus t and the long stretch is where we have the bottle neck. Now we need to derive a general scaling law for the time that is required to pass through a bottle neck and what really matters is the behaviour of the q dot in the vicinity of the minimum as the time that is spent there really dominates all other times scales.

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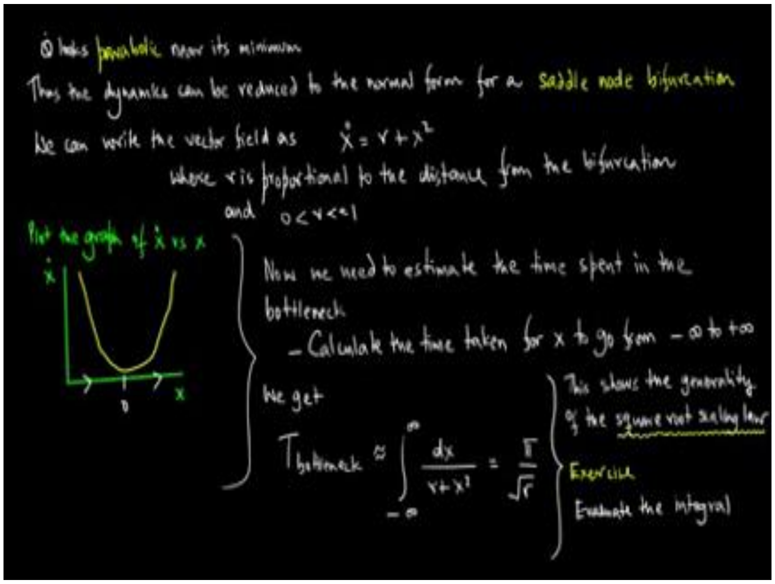

Now tita dot looks parabolic near its minimum, thus the dynamics can be reduced to the normal form for a saddle node bifurcation. We can write the vector field as x dot = r + x square, where r is proportional to distance from the bifurcation and r is much less than 1 and greater than 0. So, let us plot the graph of x dot versus x that is the simple-minded plot which we have for the x dot versus x that is the plot.

Now we need to estimate the time spent in the bottle neck, ie calculate the time taken for x to go from minus infinity to plus infinity. So, we get T bottle neck is approximately integral for minus infinity to infinity dx / r + x square, which turns out to be pi / square root of r. So, this shows the generality of the square root scaling low. So, we leave it as an exercise, for you to actually evaluate the integral.

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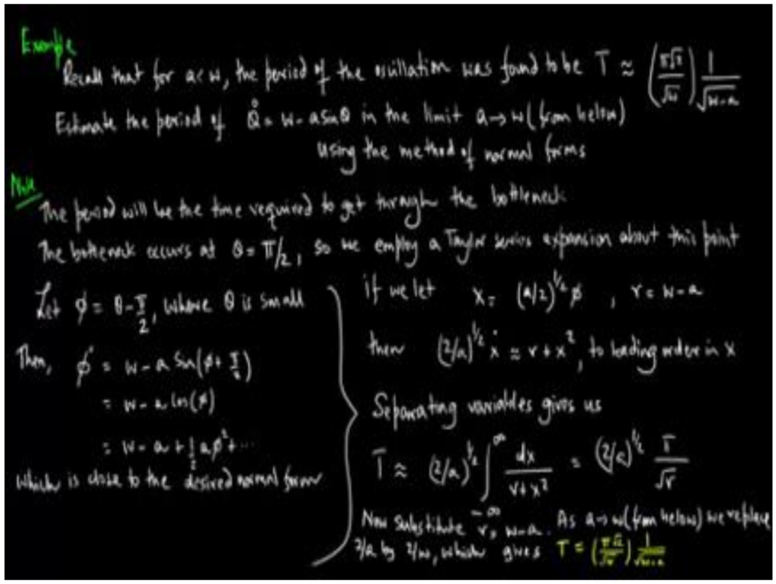

Now let us consider an example, recall that for a is less than omega, the period for the oscillation was found to be capital T = pi times the square root 2/square root omega times 1 upon square root of omega - a. So, we now estimate the period of tita dot = omega -a sine tita, in the limit that a tends to omega from below, using the method of the normal forms. Now note that the period will be the time required to get through the bottle neck.

As the bottle neck occurs at tita = pi on 2, we employ a Taylor series expansion about this point. So let fy = tita - pi on 2, where tita is small. Then fy dot = omega - a sine fy + pi on 2, which is equal to omega - a cos fy, which is equal to omega - a +1/2 a fy square + higher order terms, which is actually now close to the desired normal form. So, if we let x = a by 2 to the power of 1/2 times fy and r = omega – a, then 2 by a to the 1/2 x dot is approximately is equal to r + x square to leading order in x.

So, separating variables, gives us capital T is approximately equal to 2 by a to the 1/2 integral of minus infinity to infinity dx / r + x square, which is equal to 2 by a to the 1/2 pi / the square root of r. So now we go ahead and substitute r = omega - a. Now as a tends to omega from below, we go ahead and replace 2 by a by 2 by omega, which finally gives us capital T is approximately equal to pi times square root of 2 /the square root of omega times 1 / the square root of omega - a.

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In this short lecture, we introduce the non-uniform oscillator, so this is an equation of the form tita dot = omega - a sine tita. So, if a was is equal to zero, we are back to the uniform oscillator. So, to that end the introduction of the parameter a introduces a non uniformity around the flow in the circle. Now this equation actually shows up in numerous areas of science and technology. For example, in phase locked loops, oscillating neurons, charged density waves and so and hence forth.

So the interesting cases that show up in the equation are when a is actually less than omega in which case you have no fixed points. When a is equal to omega, where you have one half stable fixed point and when a is greater than omega, where you actually have two fixed points all of them are stable and one of them is unstable. And in this lecture, we did a sort of preliminary analysis of this equation and looked that the vector field of the situation.