Introduction to Nonlinear Dynamics

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Module -04
Lecture-14
1-Dimensional Flows, Bifurcations, Lecture 6

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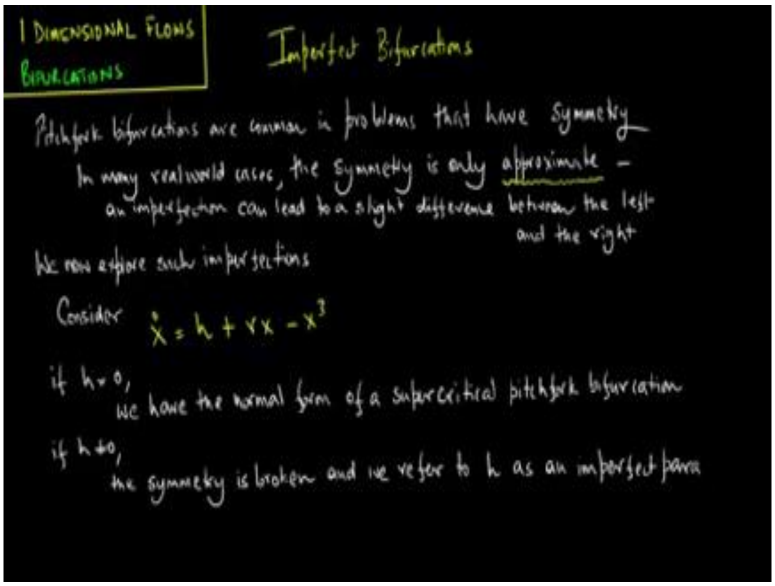

In this lecture, we deal with imperfect bifurcations, pitchfork bifurcation are fairly common in problems that have symmetry. In many real world cases the symmetry is only approximate and in imperfection can actually lead to a slight difference between the left and the right. We now go ahead and explore such imperfections. Consider x dot = h + rx - x cubed if h =0. We have the normal form of a supercritical pitchfork bifurcation and if h not equal to zero the symmetry is broken and we refer to h as an imperfect parameter.

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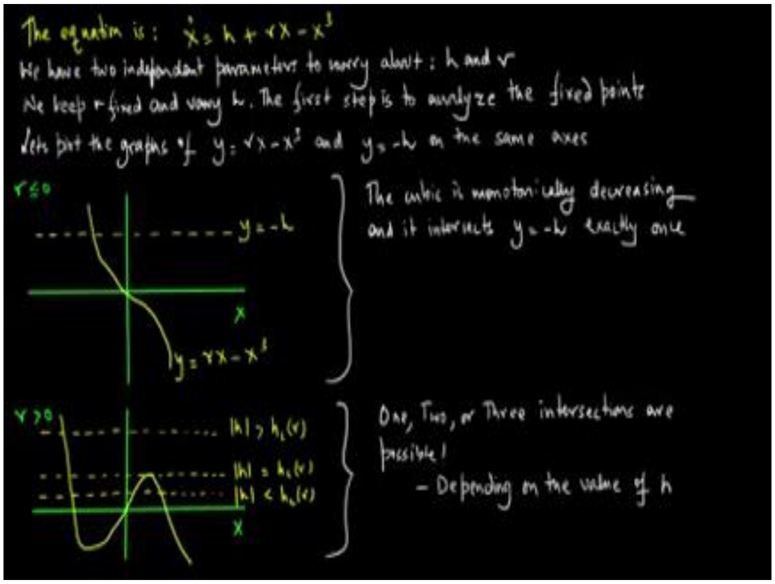

The equation is x dot = h + rx - x cubed and we have two independent parameters to worry about h and r. So, we keep r fixed and go ahead and vary h, the first step is to analyse the fixed points. So, let us plot the graphs of y = rx - x cubed and y = -h on the same axis, so for r less than equal to zero that is the line y = -h and that is the curve for y = rx - x cubed.

Note that the cubic is monotonically decreasing and it intersects with y = -h exactly once. For r greater than zero, we get much more interesting scenario that shows up. So that is the curve and that is one intersection and that is another intersection and when we get a third intersection. So, we can get one, two or three intersections and all of these are possible depending on the value of h.

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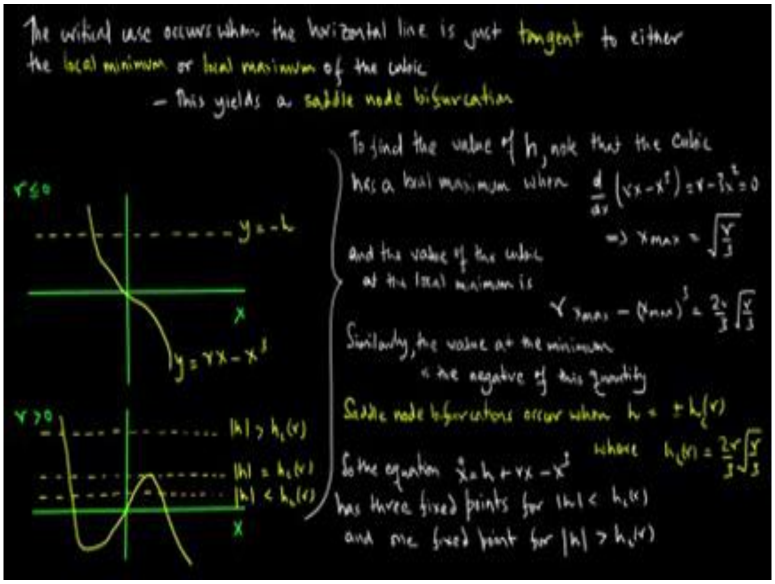

So, the critical case actually occurs when the horizontal line is just tangent to either the local minimum or the local maximum of the cubic. So, this yield a saddle node bifurcation, to find the value of h note that the cubic has a local maximum, when d dx of rx - x cubed which is equal to r - 3x squared =0. And so, x max is equal to square root of r by three and the value of the cubic at the local minimum is r times x max - x max cube, which is equal to 2r by 3 times square toot of r by 3.

So similarly, the value at the minimum is the negative of this quantity and saddle node bifurcations occur, when h is equal to plus minus h critical which is the function of r where h critical = 2r by 3 times square root of r by 3. So, the equation x dot =h+rx - x cubed has three fixed points for the absolute value of h less than h critical and has one fixed point for the absolute value of h greater than h critical.

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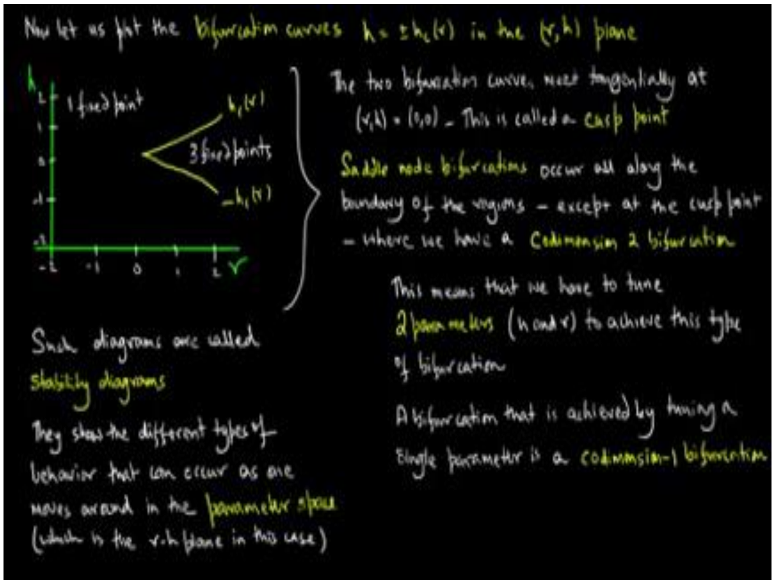

Now let us plot the bifurcation curves, h is equal to plus minus h critical in the rh plane, so that is h versus r that is h critical and that is minus h critical that is why we have three fixed points, in that we have one fixed point. So, the two bifurcation curves meet tangentially at rh =00, this is called the cusp point. So, saddle node bifurcation occur all along the boundary of the regions except at the cusp point.

So, this is where we have a co dimension two bifurcations. This means that we have to tune two parameters h and r to achieve this type of bifurcation. A bifurcation that is achieved by tuning a single parameter is a co dimension one bifurcation. Now such diagrams are referred to as stability diagrams and they show the different types of behaviour that can occur as one moves around in the parameter space, which is the rh plane in this case.

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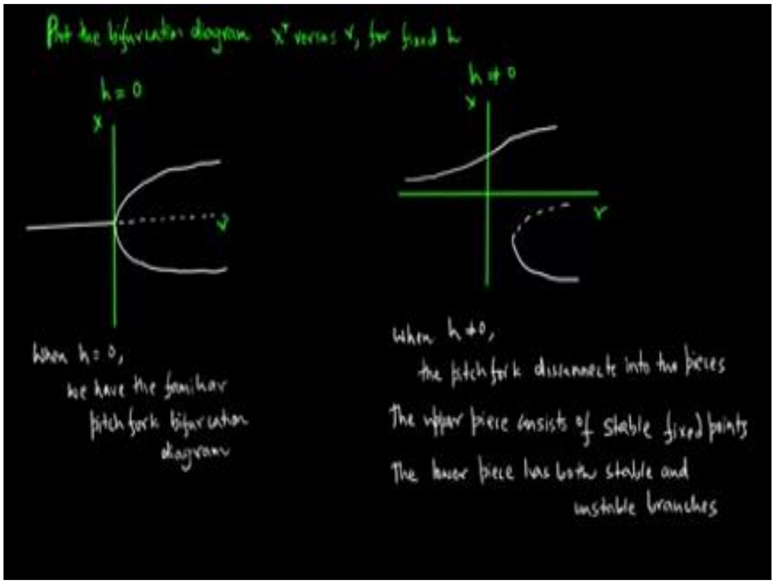

Let us plot the bifurcation diagram for x star versus r, for fixed h. So, consider the case h = 0 and plotting x versus r, so when h = 0, we have the familiar pitchfork bifurcation. When h is not equal to zero and you get something little more interesting. So, when h is not equal to zero the pitchfork disconnects into two pieces the upper piece consists of stable fixed points and the lower piece has both stable and unstable branches.

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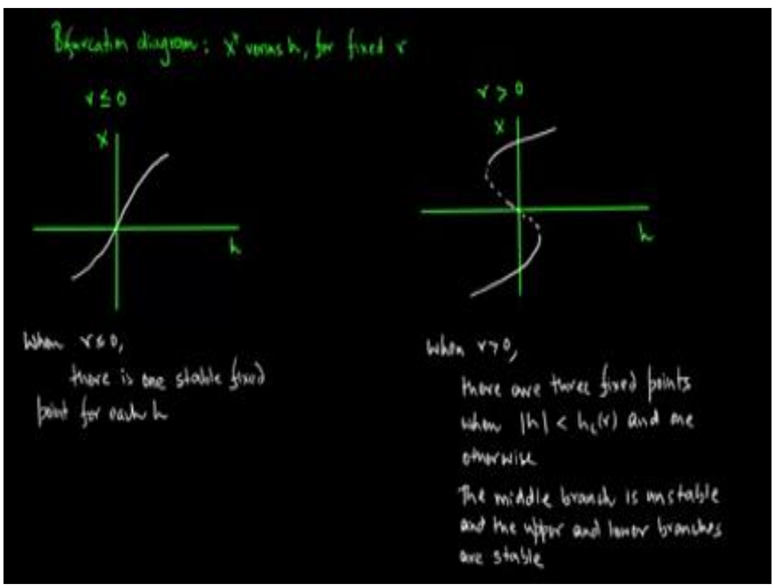

Consider the bifurcation diagram for x star versus h for fixed r. We first consider the case where r is less than or equal to zero and we plot x versus h. So, when r is less than or equal to zero, there is one stable fixed point for each h. Now we consider the case r>0 which proves to more exciting when r>0. There are three fixed points, when the absolute value of h is less than h critical and one otherwise. Note that the middle branch is unstable and the upper and the lower branches are stable.

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The topic of discussion in this lecture was imperfect bifurcation. Now we get pitchfork bifurcations in situations which have symmetry. The issue is that in the real world the symmetry may not be exact but in fact may only be approximate. Now let us consider following dynamical system, let x dot = h + rh - x cube, where h and r are two parameters. Now in h = 0, then we get the familiar pitchfork bifurcation. But h can also be non-zero in which case we get an imperfection coming into the system and so h is actually referred to as imperfect parameter.

So now we have two parameters r and h and reasonable sensible way, analyse the system is to keep r fixed and then vary h and then keep h fixed and vary r, additionally was also sensible is to plot the bifurcation curves in the rh plane itself. And that is the reasonable sensible thing to do when we have more than one parameter in the system and when you plot curves in the rh plane these are sometimes referred to as stability charts or as stability diagrams.