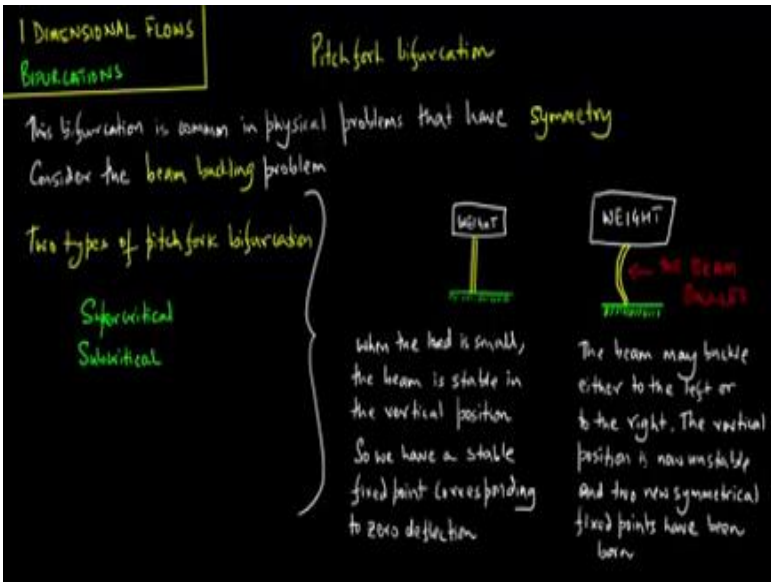
Introduction to Nonlinear Dynamics
Prof. Gaurav Raina
Department of Electrical Engineering
Indian Institute of Technology, Madras

Module -04
Lecture-13
1-Dimensional Flows, Bifurcations, Lecture 5

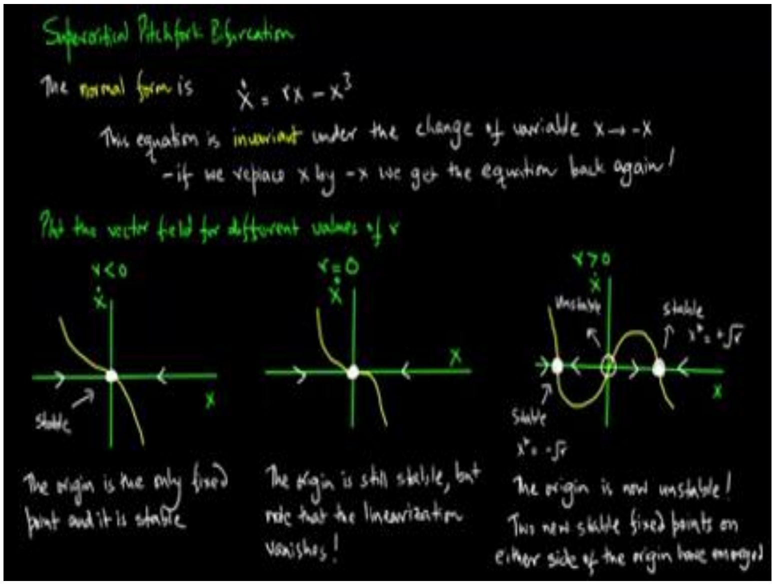
(Refer Slide Time: 00:01)



In this lecture, we talk about the pitchfork bifurcation. This bifurcation is rather common in physical problems that have symmetry, consider the beam buckling problem. We had a small weight on top of a beam as long as the weight was small the beam was stable. So, when the load is small the beam is stable in the vertical position. So, we have a stable fixed point corresponding to zero deflection. Now consider the case where we have a larger weight that is placed, which forces the beam to buckle.

So, we find that the beam buckles under the weight and the beam may in fact buckle either to the left or to the right. The vertical position is now unstable and two new symmetrical fixed points have been born. There are two types of pitchfork bifurcations one is a supercritical and the other is a subcritical.

(Refer Slide Time: 01:43)

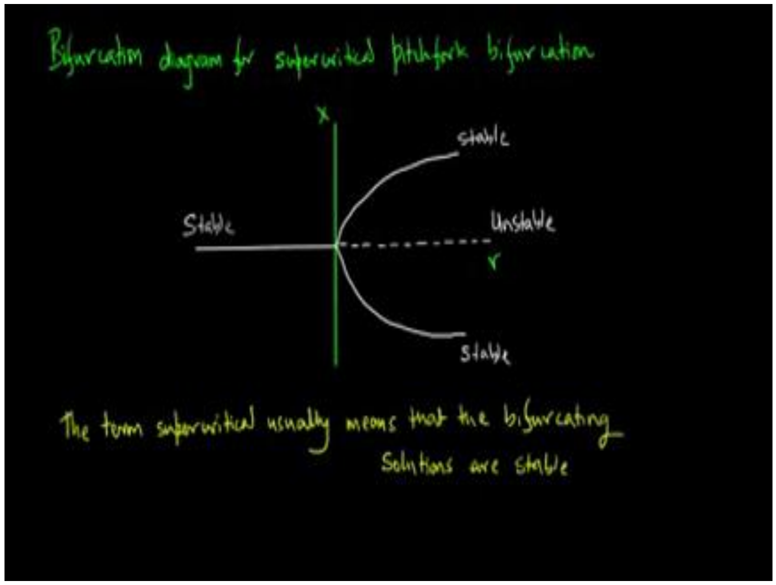


Now consider a supercritical pitchfork bifurcation. The normal form is x dot = rx - x cubed and this equation is in variant under the change of variable x to -x, what that means, is that if we replace x by -x we get the equation back again. So, let us plot the vector field for different values of r. So, if r <0, we plot x dot versus x and we find that we have a stable fixed point. The origin is the only fixed point and it is stable. Now consider r =0, plot x dot versus x.

We again have only one stable fixed point, the origin is still stable, but note that the linearization actually vanishes. Now consider r >0 and we plot x dot versus x and here is where we get some interesting dynamics. We get two stable fixed points and an unstable fixed point. So we are stable at x dot is equal to plus square root of r and stable at x* is equal to minus square root of r and origin is unstable.

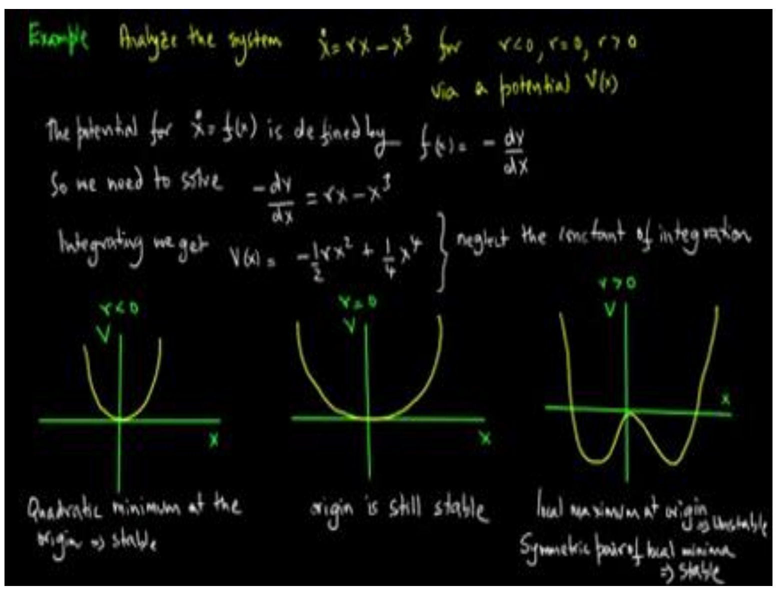
So the origin is actually now unstable, two new stable fixed points on either side of the origin have now emerged. Now we have the interesting situation where for r <0, we had a stable fixed point. For r=0, the origin was still stable, but if you look at the linearization, then the linearization would vanish at the fixed point. And that the r>0, we found that the origin turned unstable and we had these two-new stable fixed points which emerged on either side of the origin.

(Refer Slide Time: 04:33)



Now let us consider the bifurcation diagram for a supercritical pitchfork bifurcation. As is the norm, we go ahead and plot x versus r that is stable and dashed line is unstable. So, straight line is stable and that is another stable branch. The term supercritical, usually means that the bifurcating solutions themselves are stable.

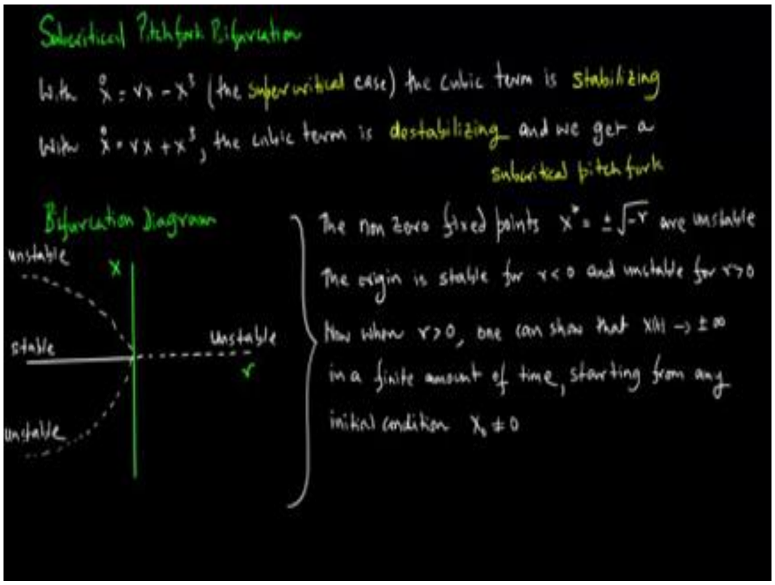
(Refer Slide Time: 05:18)



Let us consider an example, analyse the system x dot = rx - x cubed for r<0, equal to zero and greater than zero via a potential function V of x. The potential for x dot = f of x is defined by f of x = - dv dx. So, we need to solve -dv dx = rx -x cubed. So, we integrate to get V of x = - 1/2 of r x squared + 1/4 x to the four, where we have neglected the constant of integration.

First consider r less than zero, we plot v versus x. We get a quadratic and you see that the quadratic has a minimum at the origin and so the origin is stable. And r = 0 again plotting v versus x, the minimum is at the origin, so the origin is still stable. For r>0, we get something more complicated. We have a local maximum at the origin implying an unstable fixed point and symmetric pair of local minima implying stable fixed points.

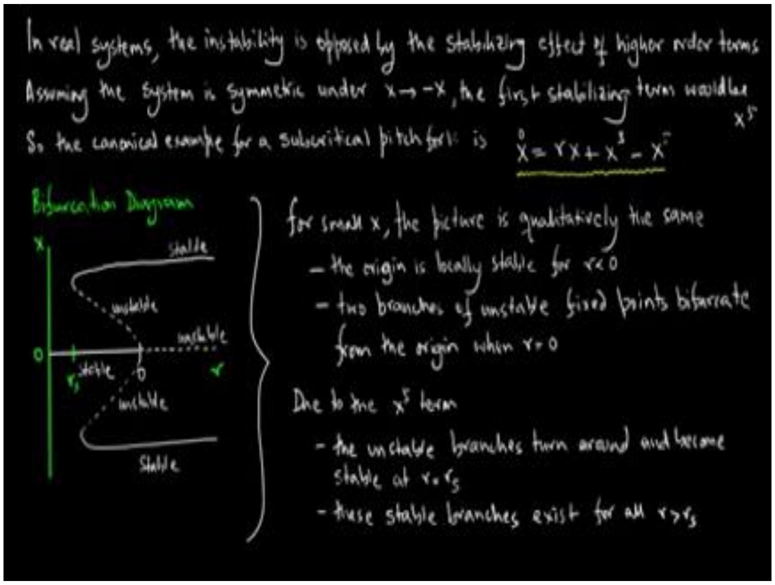
(Refer Slide Time: 07:02)



Now let us consider a subcritical pitchfork bifurcation. Where x dot = rx - x cubed. The supercritical case the cubic term is stabilising with x dot = rx + x cube. The cubic term is actually destabilising and we get a subcritical pitchfork bifurcation. We now plot the bifurcation diagram for the subcritical pitchfork. So, it is customary to plot x versus r, the dashed lines represent the unstable branches, so that is also unstable.

The non-zero fixed points x* is equal to plus minus square root of minus r are unstable and the origin is stable for r less than zero an unstable for r greater than zero. Now when r is greater than zero, one can show that x of t would tend to plus or minus infinity in a finite amount of time, starting from any initial conditions x of not, not equal to zero.

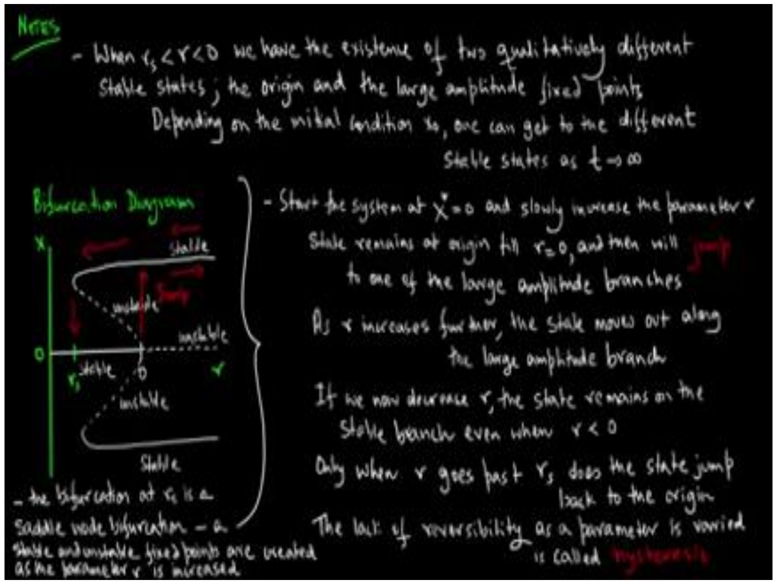
(Refer Slide Time: 08:40)



In real systems the instability is actually opposed by the stabilising effect of higher order terms. Assuming that the system is symmetric under x to – x, the first stabilising term would be x to the 5. So, the economical example for a subcritical pitchfork is x dot = rx + x cube - x to the five. Now we plot the bifurcation diagram for the subcritical pitchfork. So, we plot x versus r and it turns out be quite interesting looking bifurcation diagram, as this customary the straight lines are stable and the dotted lines are unstable branches.

For small x the picture is qualitatively the same, the origin is locally stable for r less than zero, two branches of unstable fixed points actually bifurcate from the origin when r =0. Now due to the presence of the x to the 5 term, the unstable branches actually turn around and become stable at r = rs and these stable branches when exists for all r greater than rs.

(Refer Slide Time: 11:06)

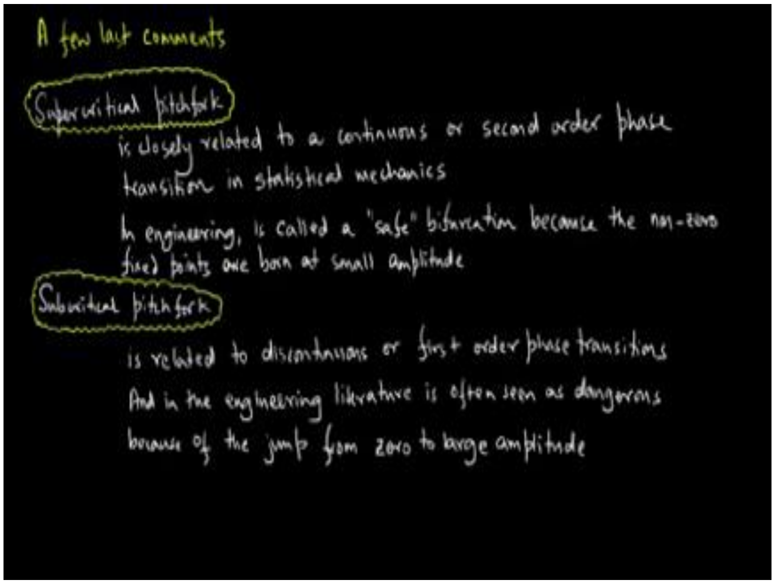


Now here is a notes about bifurcation diagram, when rs is less than r is less than zero. We have the existence of two qualitatively different stable states the origin and the large amplitudes fixed points. Depending on the initial condition x not, one can get to the different stable states as t tends to infinity. So, if you start the system at x* = 0 and slowly increase the parameter r it remains stable at the origin till r = 0 and then will actually jump to one of the large amplitudes branches.

As r increases further the state moves out along the large amplitude branch, so that is the jumps that will actually occur. If we now decrease r the state remains on the stable branch even when r is less than zero and it is only when r actually goes past rs does this state actually jump back to the origin. So, the lack of reversibility as a parameter is varied is called hysteresis. So, this lack of reversibility it can now been seen in the bifurcation diagram.

The bifurcation at rs is a saddle node bifurcation, stable and unstable fixed points are created as the parameter r is increased.

(Refer Slide Time: 13:51)



We offer a few last comments a supercritical pitchfork bifurcation is closely related to continuous or second order phase transitions in statistical mechanics. In the engineering literature, it is called a safe bifurcation because the non-zero fixed points are born at small amplitude. Now let us make some comments about the subcritical pitchfork.

Now because these are the two types you got a subcritical and supercritical. So, let just go ahead and highlight them. So a subcritical is related to discontinuous or first order phase transitions and in the engineering literature is often seen as dangerous because of the jump from zero to large amplitude.

(Refer Slide Time: 15:09)

This lecture was about a bifurcation called a pitchfork bifurcation. Now there are lots of situation in the physical world, where you have symmetry in them. Let us give an example. Now recall the beam buckling problem. So, we had the beam and on top of the beam, you went placed the weight. When the weight crossed a certain threshold, the beam actually buckled under the weight.

So, the weight actually acts as the control parameter which when it cross a certain threshold the beam actually buckle. Having said that it was not clear that it buckled to left or it will buckle to the right, so this is very simple example motivating some symmetry in real world systems. The pitchfork bifurcation comes in two forms one is a supercritical pitchfork bifurcation and the other is a subcritical pitchfork bifurcation.

In the supercritical case, the normal form for the supercritical is x dot = rx - x cube. So, what we did was, we had an example and we actually looked at the bifurcation diagram, for this particular example and what we found was that the parameter as it changed, fixed points were not destroyed. But the stability of the fixed point actually can change ok.

Then we went on to the subcritical case, in the subcritical case you can start with the equation of the form x dot = rx + x cube, so the positive x cube is destabilising. But in the real world what would happen is that you would actually have a another high order term, which would act as the stabilising force. So, the equations would be of the form x dot = rx + x cube - x to the 5 ok. Now when you look at the bifurcation diagram of this particular system and there are a few interesting things that actually show up.

Number one is that you have a jump in the bifurcation diagram. So, if the equilibrium is at a particular state and the parameter actually crosses a certain threshold and the system can take a fairly large jump to another branch. The second interesting thing that shows up is the potential lack of reversibility as parameters vary.

So was that basically means is that if a parameter was at one place and then it moved to another place you had a certain change in the solutions and then when it actually came back it did not actually go back to original state, but actually went to some other state. So, this lack of reversibility is referred to as hysteresis. So, the ability for the bifurcation diagram to exhibit a jump and to exhibit hysteresis is what we found in the subcritical case.