Introduction to Nonlinear Dynamics

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Module -04
Lecture-12
1-Dimensional Flows, Bifurcations, Lecture 4

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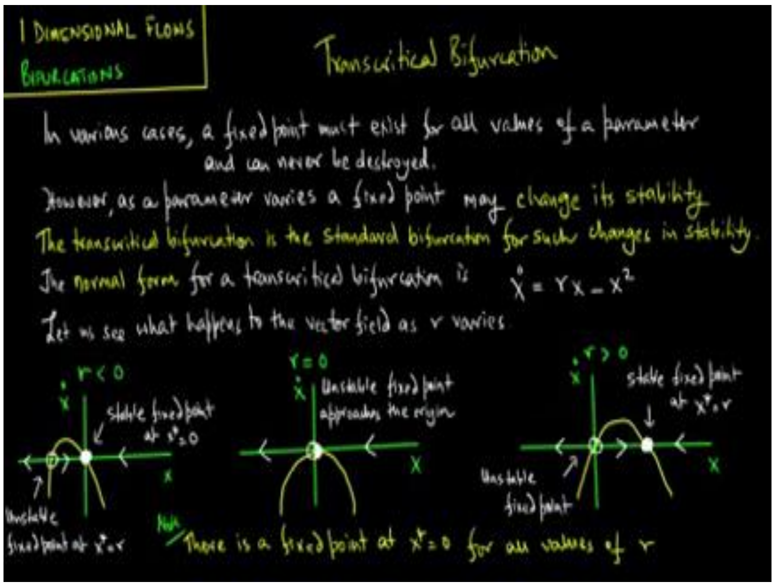

In this lecture, we deal with transcritical bifurcation, in various cases a fixed point must exist for all values of a parameter and can actually never be destroyed. However as a parameter varies a fixed point may actually change its stability. The transcritical bifurcation is the standard bifurcation, for such a change in stability. The normal form for a transcritical bifurcation is x dot = rx -x square. So let us see what happens to the vector field as r varies.

So for is r less than zero, we plot x dot versus x, as the graph of x dot versus x. And we have two fixed points, one is an unstable fixed point at x* = r and other is a stable fixed point at x*=0. These are the two fixed points of the system, one is stable and the other is unstable. For r =0, that is the plot of x dot versus x and we find that we have one fixed point in the system. Note that the unstable fixed point actually approaches the origin.

For r greater than zero, we go ahead and plot x dot verses x again, those are the two fixed points of the system. So we have an unstable fixed point at the origin and a stable fixed point at x* = r. Note that there is a fixed point at x* =0 for all values of r. So, when you look at the three diagrams together, we note that when r was less than zero there were two fixed points, one stable and one unstable and the origin was the stable fixed point.

When r =0, the unstable fixed approaches the origin and we get a half stable fixed point at the origin and when an r is greater than zero, we again have two fixed points now, except that the origin is now an unstable fixed point. So, if just look at the origin and the origin is first fully stable than half stable and then unstable and all of these happens as the parameter r varies.

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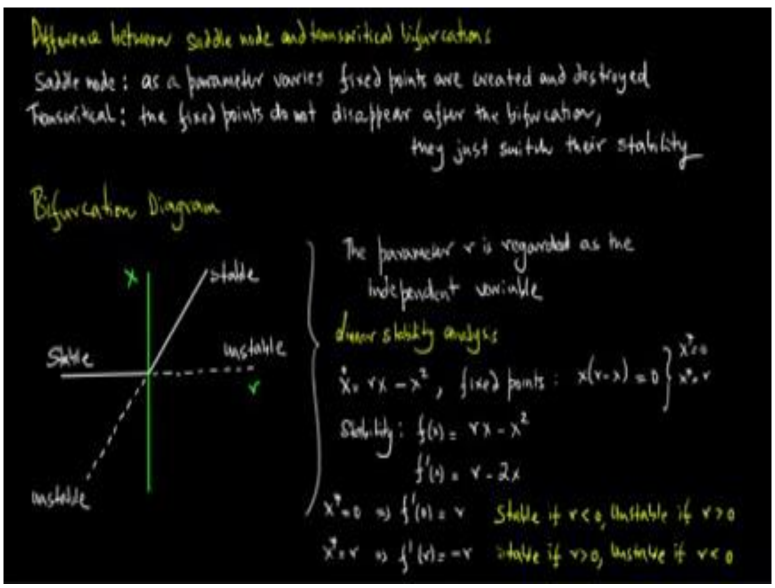

Now let us look at the difference between a saddle node and a transcritical bifurcation. In a saddle node, as a parameter varies fixed points are created and destroyed. In a transcritical bifurcation, the fixed point actually do, not disappear after the bifurcation they just switch their stability. Now let us plot the bifurcation diagram for the transcritical bifurcation. So, we go ahead and plot x versus r and we highlight the stable branches and the dotted lines are the unstable branches. So, the parameter r is actually regarded as the independent variable.

Now let us go ahead and do the quick linear stability analysis for the dynamical system. So, we have x dot = rx -x square and so the fixed points are x times r - x =0, which gives us x*=0 and x*

= r. So now we wish to determine the stability, so f of x = rx - x square, so f prime of x = r - 2x, so x* =0 gives us f prime of zero = r and x* = r gives us f prime of r = - r. So, the origin is stable if r is less than zero and unstable if r is greater than zero and x* = r stable if r is greater than zero and unstable if r is less than zero.

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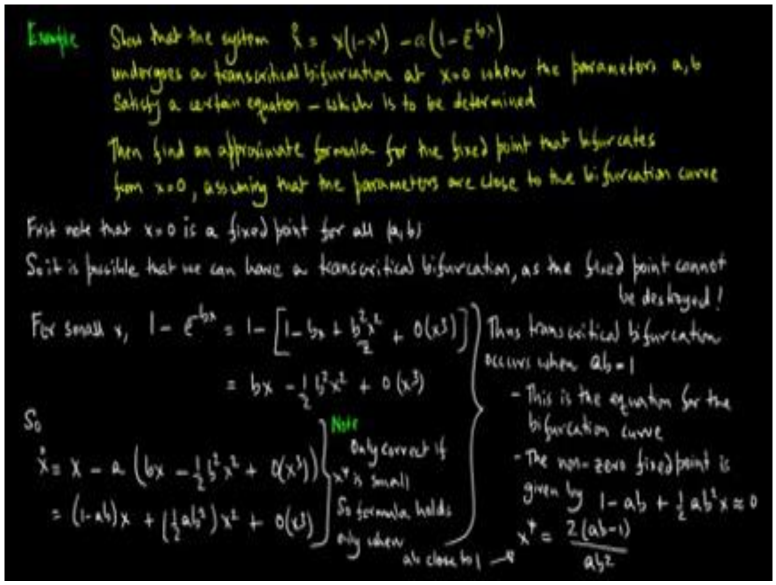

Let us consider an example, show that the system x dot = x time 1 - x square - a times 1 - e to the - dx undergoes a transcritical bifurcation at x =0, when the parameters a b satisfy a certain equation which is yet to be determined. Then find and approximate formula for the fixed point that bifurcates from x =0, assuming that the parameters are close to the bifurcation curve. First note that x =0 is a fixed point for all a b.

So it is possible that we can have a transcritical bifurcation as the fixed point cannot be destroyed. For small x, 1-e to the -bx = 1 - bracket 1- bx + b square x square on 2 + order x cubed terms. Which is equal to bx -1/2 b square x square plus order x cubed terms, so x dot = x - a times bx – 1/2 b square x square + order x cubed terms, which is equal to 1 - ab times x + 1/2 ab square times x square plus order x cube terms.

Thus, the transcritical bifurcation occurs when ab = 1. This is the equation for the bifurcation curve, the non-zero fixed point is given by 1- ab + 1/2 ab square x is approximately is equal to zero. So, x* = 2 times ab -1 divided by ab square. Now we will make a note, this is only correct if x* is small, so the formula holds only when ab is actually close to one.

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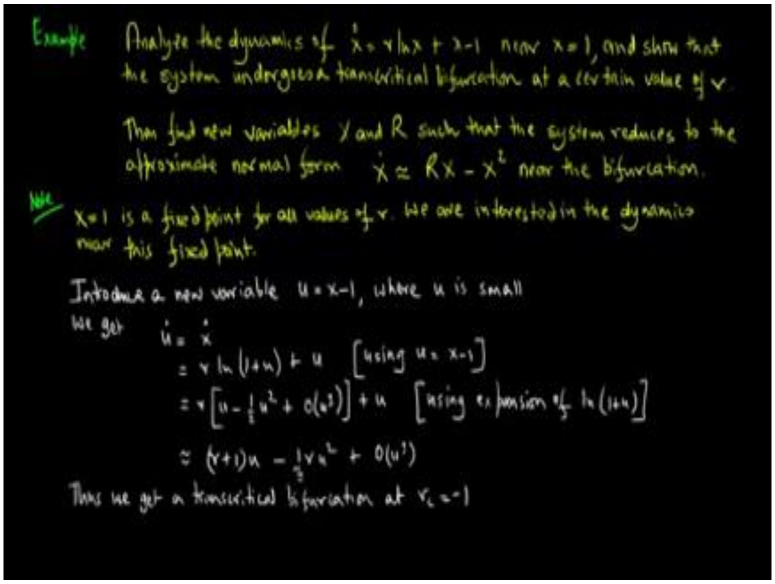

Let us consider another example, analyse the dynamics of x dot = r ln x + x - 1 near x = 1 and show that the system undergoes a transcritical bifurcation at a certain value of r. Then find new variables x and r such that the system reduces to the approximate normal form x dot = rx - x square near the bifurcation. Note x = 1 is a fixed point for all values of r and we are interested in the dynamics near this fixed point.

So we introduce a new variable u = x -1 where u is small. So we get u dot = x dot which is equal to r ln 1+ u + u that is using the fact that u = x -1 which is equal to r times u – 1/2 u square + order u cubed terms +u, that is using the expansion of ln1 + u that is approximately is equal to r +1 times u -1/2 r u square plus order u cubed terms. So, that is, we find that we get a transcritical bifurcation at r critical -1.

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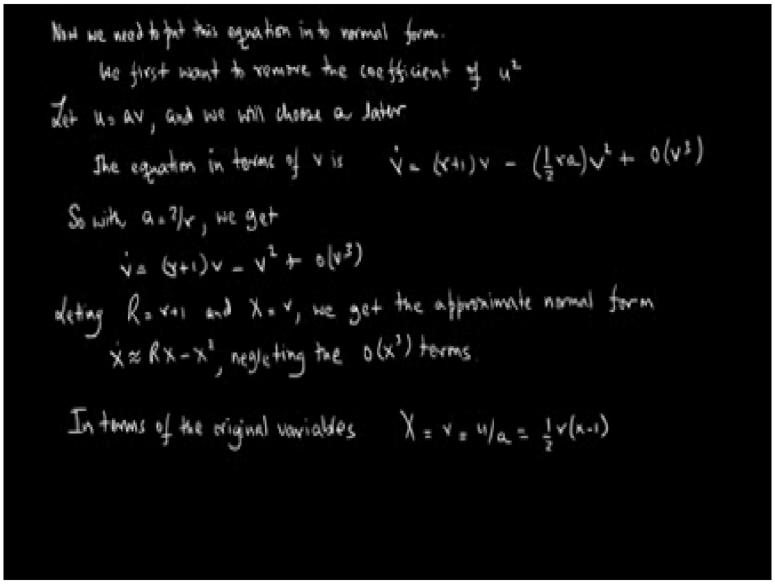

Now we need to put this equation into normal form, so we first want to remove the coefficient of u square. So, let u = av and we will choose a later. The equation in terms of v is v dot = r + 1 times v -1/2 ra times v square plus order v cubed terms. So, with a = 2/r we get v dot = r + 1 times v minus v square plus order v cubed terms. Letting capital R = r + 1 and capital X =v.

We now are in position to get the approximate normal form, which is x dot = rx - x square neglecting the order x cubed terms. So in terms of the original variables X = v = u / a which is equal to 1/2 rx -1.

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In numerous cases of interest, what happens is that irrespective of what values a particular parameter takes, the fixed point of that system has always remain and can never been destroyed. Now what can happen is that as the parameter value changes the stability of the fixed point itself can change. So stable fixed point can become an unstable fixed point or an unstable fixed point can become stable fixed point and this can be fairly dangerous in the real world situations, where just because of parameter changes the stability of the fixed point goes ahead and changes.

The standard way in which this happens is through a transcritical bifurcation. And in this lecture what we did was we introduced the transcritical bifurcation. We look at its normal form, we look at bifurcation diagrams and we looked at few examples, where a transcritical bifurcation can actually take place. Note that this is different as compared to a saddle node. Where when parameters change, fixed point can be created or can be destroyed. For the transcritical is where the fixed points will not destroyed, but the stability of the fixed points can be easily change.