Introduction to Nonlinear Dynamics
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Module -04
Lecture-11
1-Dimensional Flows, Bifurcations, Lecture 3

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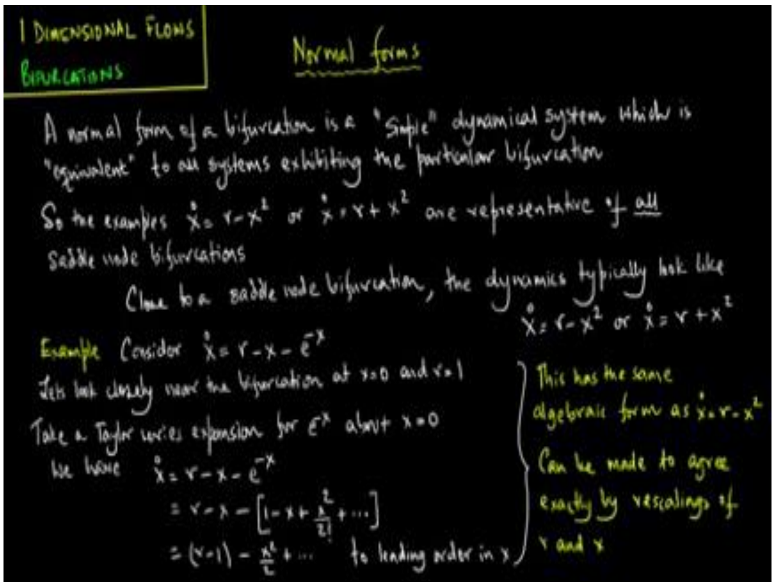

So this lecture is going to be on the topic of normal forms. A normal form of a bifurcation is a simple dynamical system, which is equivalent to all systems exhibiting the particular bifurcation. So the examples x dot = r - x square or x dot = r + x square are actually representative of all saddle node bifurcations. Close to a saddle node bifurcation, the dynamics typically look like x dot = r - x square or x dot = r + x square.

Consider an example, so consider x dot = r - x - e to the -x. And let us look closely near the bifurcation at x = 0 and r = 1. So, we take a Taylor series expansion, for e to the -x about x = 0. So, we have x dot = r - x - e to the -x, = r - x - expanding e to the -x. We get 1- x + x square on two factorial plus higher order terms, which is equal to r -1 -x square on two plus higher order term, which we ignore so this equation is to leading order in x.

Note that this actually has the same algebraic form as x dot = r -x square and in fact can be made to agree exactly by rescaling's of r and x.

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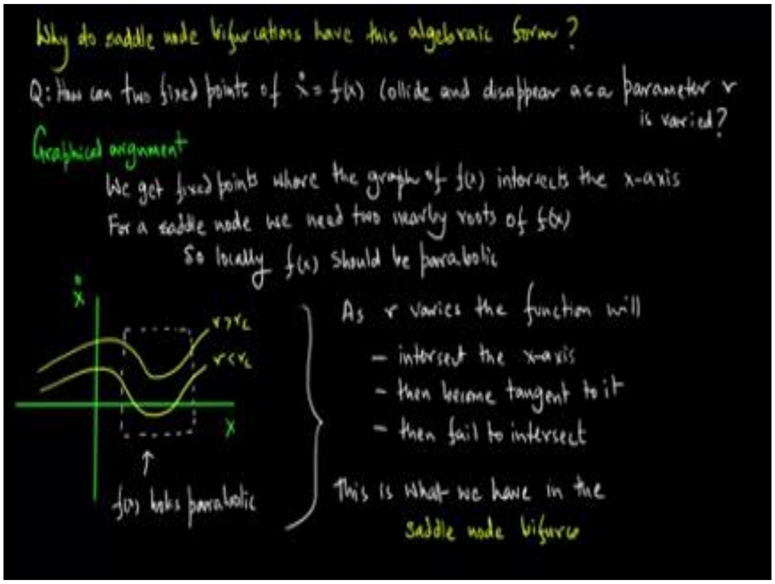

So why do saddle node bifurcations actually have this algebraic form. So, we pose some concrete question, How can two fixed points of x dot = f(x) collide and disappear as a parameter r is varied? We first present a graphical argument. Now we get fixed points where the graph of f(x) intersects the x axis and for a saddle node we need two nearby roots of f(x), so locally f(x) should be parabolic.

So, plotting x dot versus x for a function f(x) that is what we have for r greater than r critical and that is what we get for r less than r critical. So, if you look closely we find that f(x) actually looks parabolic. So, as r varies the function will intersect the x axis then become tangent to it and then fail to intersect. This is exactly what we have in the saddle node bifurcation

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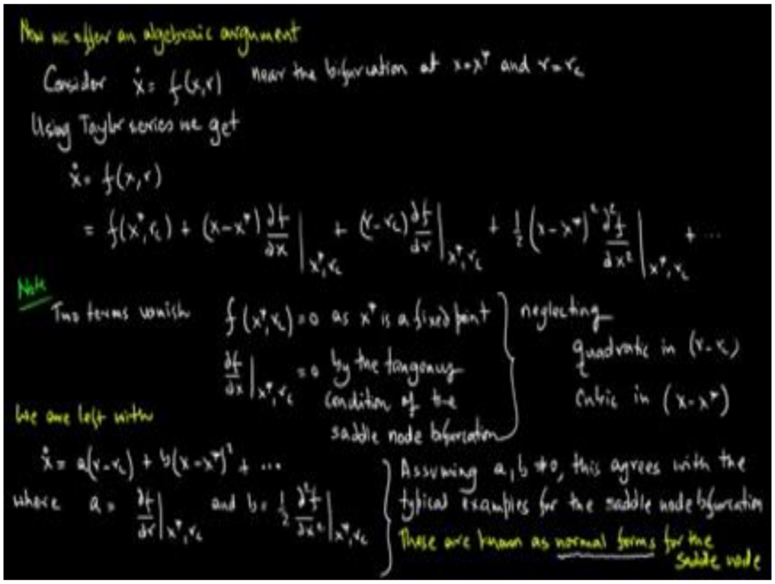

We now offer an algebraic argument. Consider x dot = f(x)r near the bifurcation x = x* and r = r critical. Now using Taylor series, we get x dot = f(x)r = f(x*) r critical +x - x* times df dx evaluated at x* r critical + r - r critical times df dr evaluated at x* r critical + 1/2 of x -x* whole square times d square f dx square evaluated at x* r critical plus higher order terms. Note that we are actually neglecting the quadratic in r - r critical and cubic in x - x*.

Also note that two terms actually vanish f(x*) r critical =0 as x* is a fixed point and df dx evaluated at equilibrium also is equal to zero by the tangency condition of the saddle node bifurcation. We are finally left with x dot = a times r - r critical + b times x - x* whole square plus high order terms, where a = df dr evaluated at x* r critical and b = 1/2 times d square f dx square evaluated at x* r critical.

Assuming a and b are not equal to zero then this actually agrees with the typical examples for the saddle node bifurcation. These are actually known as normal forms for the saddle node bifurcation.

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The normal form of a bifurcation is just a simple dynamical system which is equivalent to all dynamical systems exhibiting that particular type of bifurcation. Now we consider the case of a saddle node. And in the saddle node, we can ask the following question, how can two fixed points of the equation x dot = f(x) collide and then disappear as a parameter in the system is varied ok. And so what we did was we offered two arguments.

We offered the graphical argument and then we offered an algebraic argument to basically deduce the normal form of the saddle node bifurcation. Of course, the saddle not is just one particular type of bifurcation. There are other types of bifurcation as well and so each bifurcation could have its own normal form and we will study other bifurcations later.