Networks and Systems
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Lecture - 09
Fourier Series (3)
An Example of Application to Network Analysis
Exponential Form of Fourier Series

In the last lecture, we have familiarized ourselves, with some symmetry conditions relating to the Fourier coefficients. We would like to continue this discussion, with an example where we make use of the symmetry conditions. And use them to calculate the various Fourier coefficients, quite effectively.

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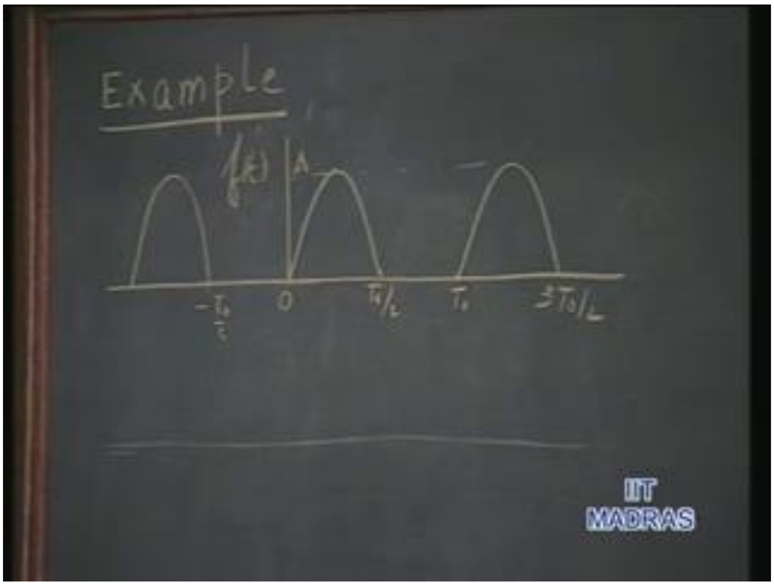

Consider this example, where we are dealing with a f of t, which is a half wave rectified sine wave of amplitude a. And a fundamental period T naught. Now, when you look at this, this waveform does not have either an even symmetry or an odd symmetry. You recall that we had observed earlier, that any given function can be split up, as it is even part and odd part respectively.

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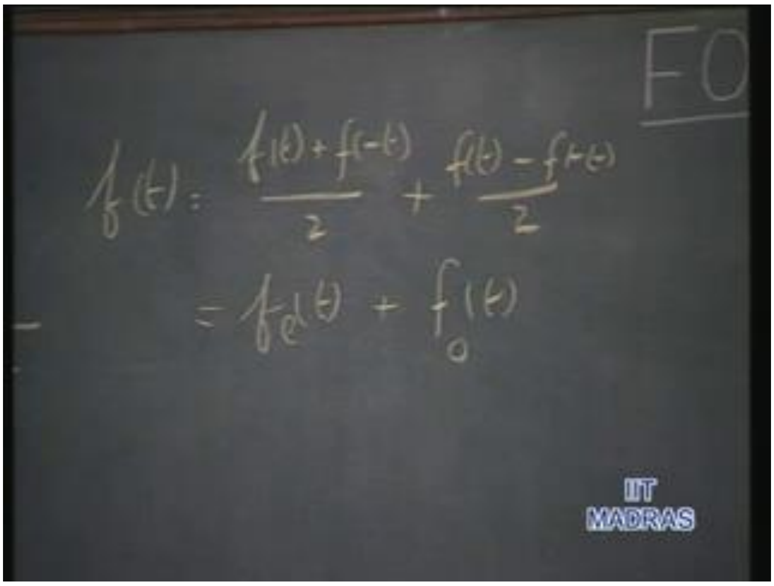

In other words, if you have an f of t, if you break this up as f of t plus f of minus t divided by 2 plus f of t minus of f of minus t upon 2. This becomes the even part and this becomes the odd part. Because, when you change t for minus t, the value of this part does not remains unchanged. But, if you substitute minus t for t, the value of this function gets reversed. Therefore, any f of t can be split up into it is even part and odd part respectively, in this fashion.

And when you do that sometimes, you will find some additional symmetries, which are not present in the original waveform. Let us see how this goes.

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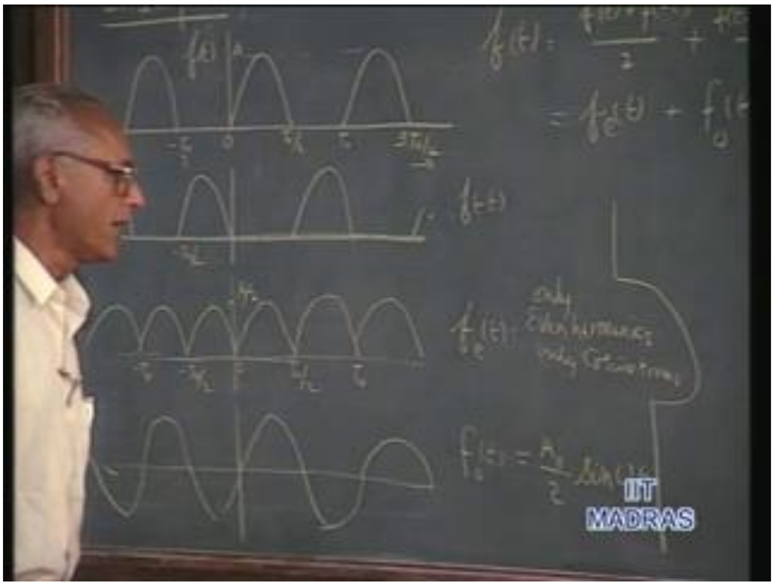

Suppose, this is f of t, then let us construct what f of minus t would be. The sequence of values, which this function takes for positive t, it will assume in the reverse direction. Therefore, you have corresponding to this. And another loop corresponding to this, on the negative side. And corresponding to this, you have a loop like this and it goes on like this. This is f of minus t. To get the even part of the function, then we have to add up these two waves. And then, divide by 2.

So, if you do that, then what you have is whenever this is blank, you have this half cycle of sine wave. Whenever, this is blank this is a half cycle sine wave you have got. Therefore, you have like this ((Refer Time: 04:05)) where the amplitude now, the peak amplitude is A, but you are dividing by 2, this becomes A upon 2. And this is the even part of this. On the other hand, if you subtract f of minus t from f of t and divide by 2.

You get the odd part, f of t minus f of minus t upon 2. So, when you subtract the second waveform from the first, this gets reversed, the sign gets reversed. Therefore, this becomes a negative half cycle. And this also becomes a negative half cycle. And therefore, these negative half cycles, fit in smugly into these blank intervals. And the result is, that you have a waveform like this, which is a pure sine wave.

Because, each half cycle sine wave is reproduced in the proper direction here. And this is f of, the odd part of t f naught. So now, if look at this constituent parts f e t and f naught of your original f of t, you immediately observe that there are symmetries here. For example, f naught of t is certainly a pure sine wave. Therefore, there is no Fourier series expansion necessary for that. That itself constitutes the entire Fourier series, A naught by 2 sine omega.

So, this will be A naught upon 2 sine omega naught t. But, as far as the eve part is concerned you observe that, this has got what we described as a kind of half wave symmetry. The function repeats itself, every half cycle. And you also recall that, we mentioned in the last class, just last lecture. That whenever you are having a waveform like this, we would like to still continue this as the basic period, not as this. Because, we are going to relate this basic period, to the parent waveform from which it is generated.

After all we want to talk about fundamental frequency in relation to this. Therefore, we continue to have the same fundamental frequency, when describes in this waveform as well. Consequently we regard this as the basic period, in which case the function f of t happens to be f of t plus t naught upon 2. Therefore, this will have only even harmonics present. And since the function is even, only sine terms, only cosine terms will be present.

So, if you make the Fourier series expansion of this. And add to this the Fourier series expansion of f naught of t, which is this. Then, you get the Fourier series for the entire function. So, let us do this. F e of t suppose has a d c term.

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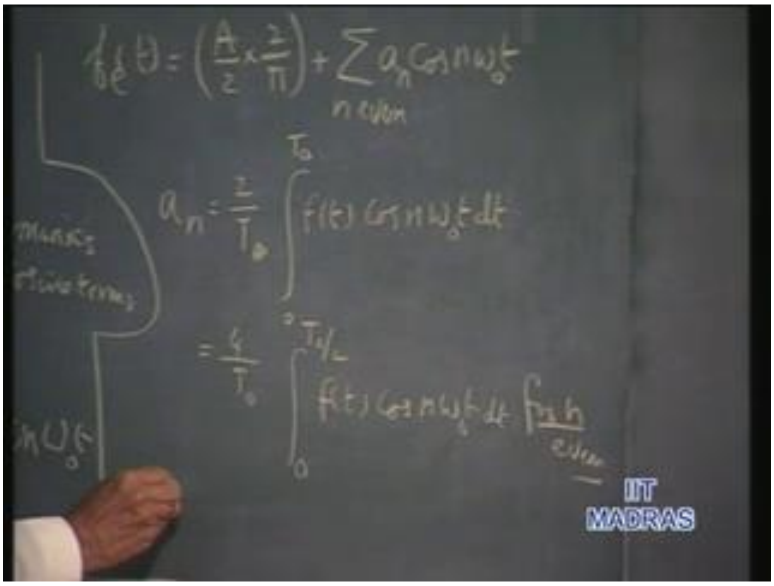

The average value of this, as you know any sinusoid, we are talking about the absolute average value 2 upon 5, that is the d c term. Plus you have A n cos n omega naught t; where you need to have n even only. Because, odd values of n would be absent, because it contains only even harmonics. To calculate A n, you take twice the average of the function 0 to T naught f of t cos n omega naught t d t.

And it can be shown that, the contribution coming from this integral. For 0 to T naught by 2 will be the same, from t naught by 2 to T naught provided n is even. Exactly the same arguments which we used, in discussing the half wave symmetry case, where we have odd harmonics present. Exactly the same way, we can show that this is equal to 4 upon T naught 0 to T naught upon 2. That means, you are taking the average of this function over half cycle.

F of t cos n omega naught t d t for n even only. You see this particular problem, may not be valid for n odd. I will not go to the resulting integration. It can be shown that, ((Refer Time: 09:28)) this will lead to minus 2 A upon pi n square minus 1, that is the value. So, finally, the Fourier series expansion for this can be written. I will write it here.

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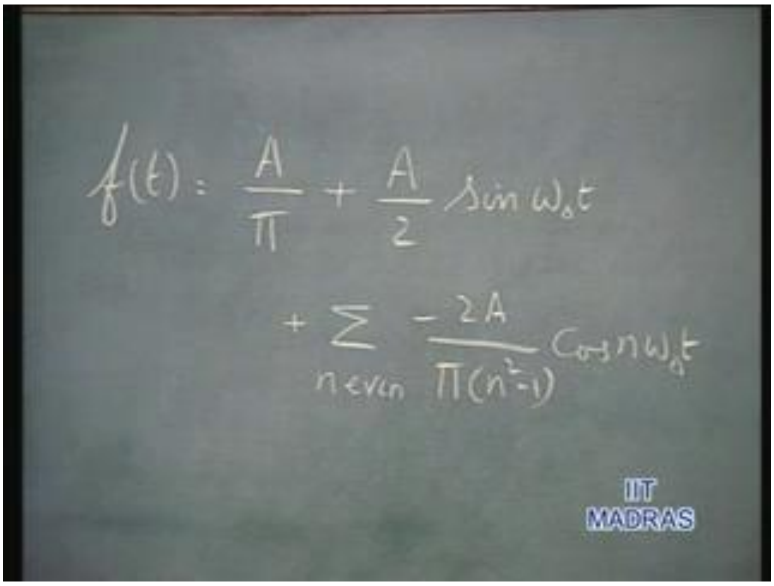

So, the Fourier series for this waveform would be, f of t equals the d c term A upon pi. Plus the fundamental term which comes from the odd part, A upon 2 sine omega naught t. Plus the remaining terms in the Fourier series expansion of the even part of the function, which will be minus 2 A over pi times n square minus 1 cos n omega naught t. For n even starting from n equals to onwards.

So, you observe that, even though this function as such does not appear to have, any of the symmetries that we have talked about. By splitting this up into even and odd parts, you are able to find some symmetry, at least in one of those parts. In the other of course, falls out. It just breaks down into a single term. So, it would be sometimes worthwhile for us. Before, we proceed to get the Fourier series expansion of any waveform. To see if we can produce some symmetries, by resolving this function into it is various constituents.

One of them being the even part and odd part respectively. You can think of other ways of resolving this, but we need to confine our discussion only to this. Now, let us use this, work that we have done here. To work out another example, where such a waveform is applied to an electrical circuit.

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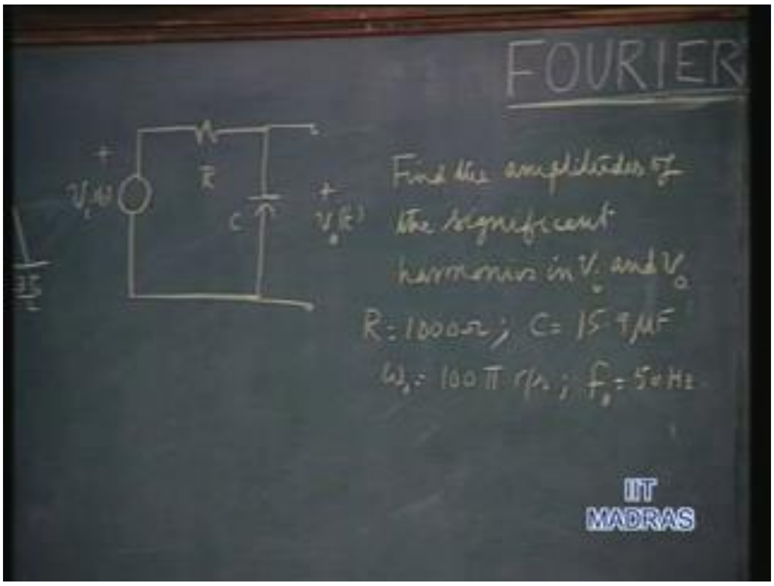

In this example, we will consider that a waveform of a voltage, of this shape, is applied to an R C circuit. And the voltage across the capacitor is taken to be the output voltage. So, we will imagine that the peak value of this half wave rectified, sine wave is 100 volts. Now, this voltage is applied across the R C circuit. And we are interested in finding out, the significant harmonics both the input and the output voltage.

So, the problem is find the amplitudes of the significant harmonics, in v i and the output voltage v naught. The data that is given R, let us say is 1000 ohms, C 15.9 micro ferrets. And omega naught corresponding to this period T naught is 100 pi radiant per second, which means the fundamental frequency is 50 Hertz. Now, since this input wave form is not sinusoidal, we cannot apply straight away the phasor methods, for calculating the output voltage.

On the other hand, the Fourier series tells us that such an input voltage, can be decomposed into a number of sinusoids. And so for each one of this sinusoidal components, we can find out the corresponding output, using the phasor methods. And superpose all the solutions to obtain the output. Or in the problem like this, we are only interested in knowing the magnitudes of the harmonic components, in the output voltage v naught.

So, for each harmonic, we can apply the phasor notation and the phasor algebra.

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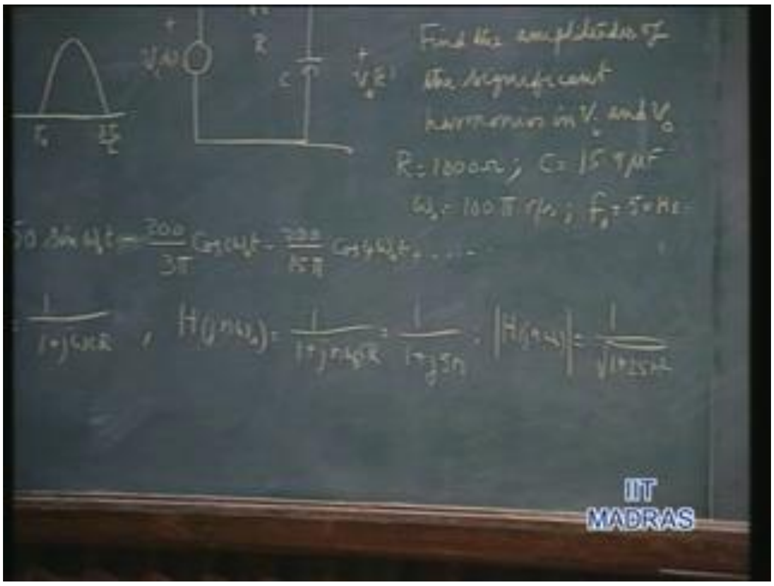

Now, we know that v i, can be expressed by means of Fourier series, using the result that we have obtained. In the previous example, when we have considered just this kind of wave, half wave rectified sine wave. The answer there was, if you substitute the numerical values, it will turn out to be hundred upon pi. That is the d c component, plus 50 sine omega naught t, that is the fundamental. And in addition we have a number of even harmonics.

The first two even harmonics are, the second harmonic and the fourth harmonic. This will be 3 pi and this is 200 by 15 pi cos 4 omega naught t. So, for each one of these, plus other terms which are insignificant, which we will ignore. For each one of these, we would like to find out the corresponding output quantity. And to do that, we must find out, the output voltage to the input voltage ratio, as a function of frequency.

So, the system function in this case H j omega by potential divider action, is 1 over j omega C divided by R plus 1 over j omega C. That will be 1 over 1 plus j omega C R. This is the general system function, as a function of frequency omega. But, we are interested in evaluating this for particular values of omega, which are omega naught, 2 omega naught, 4 omega naught and so on. Consequently we will find out H j n omega naught for a general n.

This will be 1 over 1 plus j n omega naught C R, which when you substitute the numerical values for R C and omega, will turn out to be 1 over 1 plus j 5 n. In particular, we are interested only in the amplitudes of the harmonics. So, we are not really interested in the angle associated with H of j omega naught. So, we would like to know, only the magnitudes in our problem this will be, therefore 1 over square root of 1 plus 25 H square.

So, we know the amplitudes of each one of these harmonic terms. We know the magnitude of the system function. Now, therefore, we can find out the amplitudes of the output voltage. We can do it, we can organize it in this fashion.

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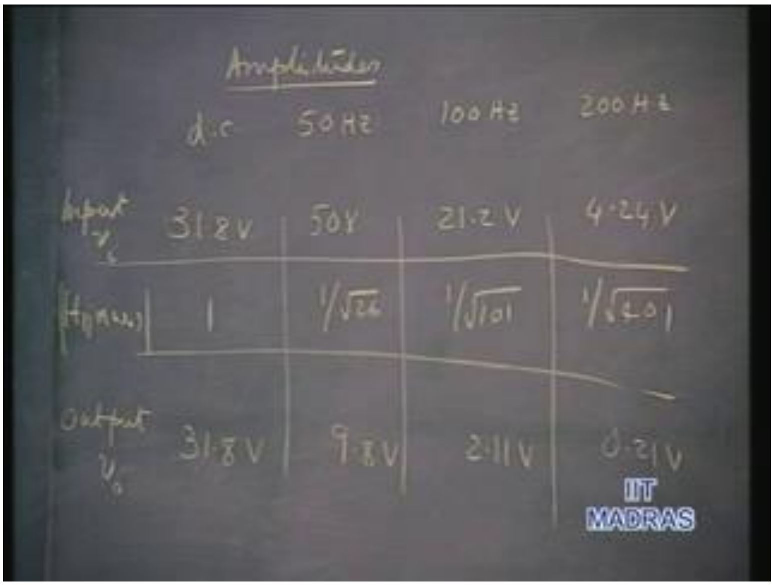

The amplitudes of d c, d c there is only one quantity. We do not have to talk about amplitude. But, we can talk about the amplitude of the fundamental component, the second harmonic and the fourth harmonic. These are the significant harmonics, that are present here. So, the input voltage has a d c component, which is 100 upon pi. That is 31.8 volts. The 50 cycles component is 50 volts.

The 100 cycles component is 200upon 3 pi, that is 21.2 volts. And the 200 Hertz component is 200 upon 15 pi, that turns out to be 4.24 volts. So, these are the amplitudes of the different harmonic components, as far as the input is concerned. And the value of the system function. When you apply a d c input here ((Refer Time: 18:25)), the same d c comes out across here. Because, there is no current passing the circuit and therefore, that is equal to 1. The input output ratio is 1.

In the case of the fundamental n equals 1, therefore this is 1 over square root of 26. In case of second harmonic n equals 2. Therefore, this will be 1 over square root of 101. And in the case of the fourth harmonic n equals 4. So, 16 times 25, 400 is 1 over 400 and 1 square root of 400. So, when you multiply these amplitudes with the corresponding magnitude and system function.

As far as the output is concerned, the various components will turn out to be this multiplied by this, 31.8 volts. The 50 cycles component, turns out to be 9.8 volts, 100 cycles 2.11 volt and this is 0.21 volt. So, this R C circuit here ((Refer Time: 19:44)), essentially acts as the filter. You have an input voltage which is non-sinusoidal, which is a rectified sine wave, half wave rectified sine wave. And we would like to have a filter like this, to swamp out the ripples.

So, all the a c components, should be reduced to the extent possible. And we would like to have the output to be as pure a d c as possible. Now, how good is this filter. Let us see in this case, in the input you have 31.8 volts d c. But, the harmonic components are quite substantial, 50 volts fundamental 21.2. Second harmonic 4.24, fourth harmonic, but as for the output is concerned, the harmonic amplitudes are brought down. Considerably you compare it with the d c.

Therefore, this is a good filter, as far as suppression of the various harmonic components are concerned. We had set up the Fourier series earlier, in terms of trigonometric functions. Now, there is an alternative way of setting up the Fourier series. This will be in terms of exponential functions. Let us proceed to do that.

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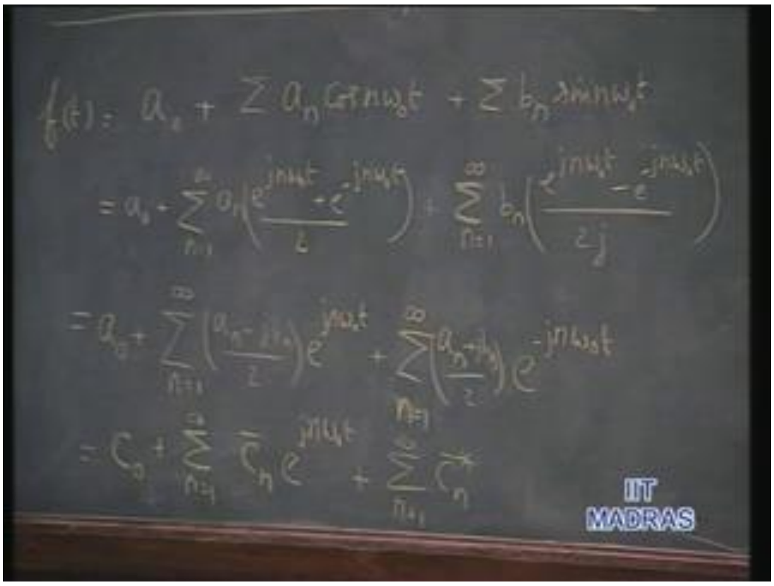

In terms of exponential functions, you recall that f of t. We had written in terms of trigonometric functions as a naught plus the sum of cosine terms plus b n sine n omega naught t sum of sine terms. Now, we can express sine and cosine terms, in terms of exponential functions. So, I can write this as a n e to the power of j n omega naught t plus e to the power of minus j n omega naught t divided by 2. Plus b n e to the power of j n omega naught t minus e to the power of minus j n omega naught t divided by z j n from 1 to infinity.

Now, we have e to the power of j n omega naught t terms here, as well as here. So, let us group them together, so that we write these series, in terms of exponential functions. So, you have a naught plus, what is the coefficient of e to the power of j n omega naught t? A n upon 2 plus b n upon 2 j. So, I can write this as a n minus j b n upon 2. This is the coefficient of e to the power of j n omega naught t n ranging from 1 to infinity. We also have terms like e to the power of minus j n omega naught t.

And what is it is coefficient, a n upon 2. And then, because there is a negative sign here, it is plus j b n upon 2 n from 1 to infinity. So, what we have done is, express f of t not in terms of trigonometric functions. But, in terms of exponential functions of the type a j, e to the power of j n omega naught t. Now, we would like to in the context of expansion, in terms of Fourier, exponential functions. We would like to indicate the coefficients in a different way.

So, we will call this c n, a complex number C n. And this will be C n conjugate. And since we are calling the exponents in the coefficients, giving the symbol C for the various coefficients, we may as well call this C naught. Therefore, I can write this as C naught plus n from 1 to infinity just to indicate that, this is a complex number C n i, put a line on top. That is a complex coefficient. E to the power of j n omega naught t plus n from 1 to infinity.

This is C n conjugate, because this is a n minus j b n upon 2 is C n, this is it is conjugate. The angle imaginary part has it is sign reversed, e to the power of minus j n omega naught t.

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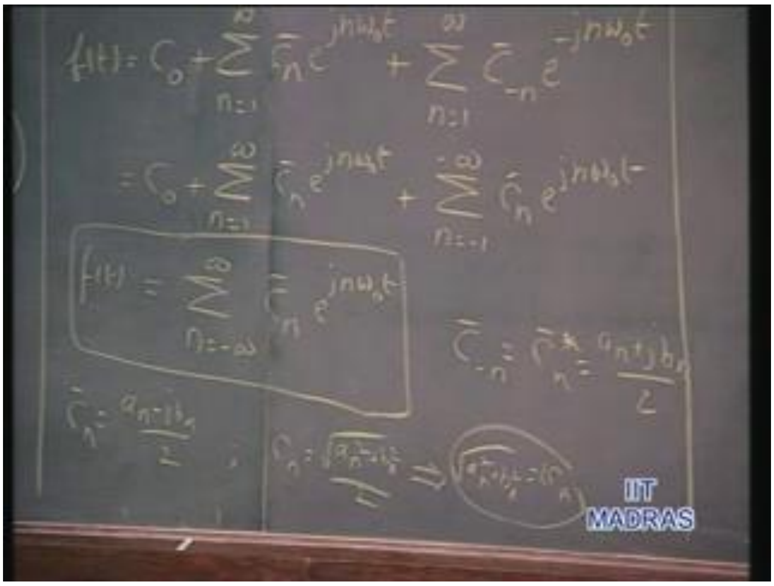

So, let us define C n star as C minus n. Then, we have f of t as C naught plus C n e to the power of j n omega naught t n ranging from 1 to infinity. C minus n e to the power of minus j n omega naught t. Now, we would like to combine these two, summations into 1. That can easily be done by substituting minus sign for n here, in which case, where you substitute minus n for n here. Then, I can write this as C naught n from 1 to infinity C n e to the power of j n omega naught t.

So, when you change the dummy sign of summation minus n by n. Therefore, this n goes from minus 1 to minus infinity, this is C n e to the power of j n omega naught t. Combining these two, I can write n from minus infinity to plus infinity, including this 0. Minus 1 to infinity minus infinity to minus 1 0 and 1 to infinity, C n e to the power of j n omega naught t.

This is f of t, where we observe that c minus n is c n conjugate, which is a n plus j b n upon 2. Because, C n conjugate is a n plus j b n by 2. So, this is a very compact way of representing the Fourier series. You do not have groups of terms like a, coefficients and b. Coefficients you have only to deal with a single set of coefficients C n. The question is, how is the C n related to a n and b n, that we already have seen.

That C n equals a n minus j b n upon 2. And in anticipation of this, suppose I take the magnitude of C n, it is C n square root of a n square plus b n square upon 2. Or in other words, square root of an square plus b n square equals 2 C n. So, in anticipation of this notation only, when we put the Fourier series expansion of a function in trigonometric functions. We said the amplitude of the n th harmonic is 2 C n, rather than C n. It is in anticipation of this formula.

Now, the next question that we would like to ask is, how do we evaluate this C n coefficients? What is the formula for that, in the same way as we have done, for the trigonometric functions. How do we get these complex coefficients C n, directly from f of t.

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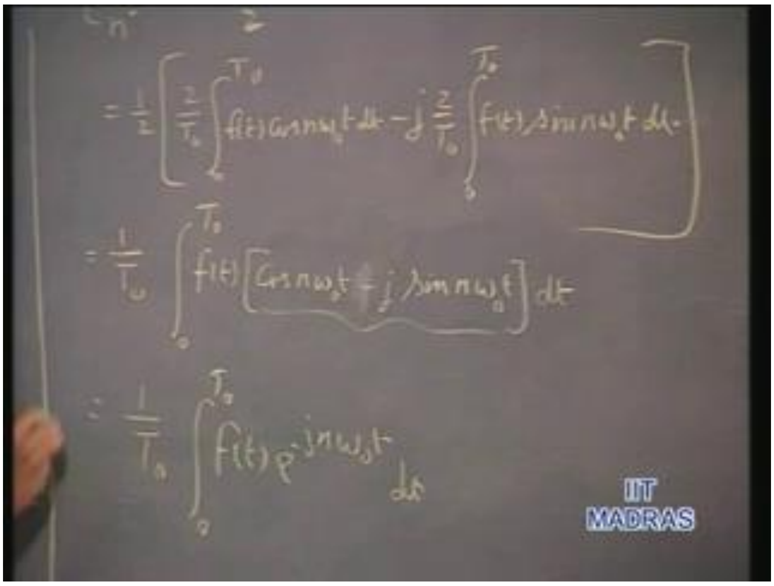

This can easily be derived as follows. C n we know is a n minus j b n upon 2. And we know the formulas for a n and b n, let us substitute that. This is half of a n is 2 upon t naught f of t cos n omega naught t d t over a period. And for d n, I can write 2 upon t naught f of t sine n omega naught t d t. This j term comes from here. This half of course, is conclude here. Now, these two integrals can be combined. First of all, wait a minute, will take 2 by T naught term outside.

So, 1 upon T naught 0 to T naught f of t cos n omega naught t minus j sine n omega naught t. And this we know is e to the power of j n omega naught t. So, the integral that needs to be carried out to evaluate C n is like this. This is really the average of f of t multiplied by e to the power of minus j n omega naught t. What do we observe here and what are the merits of this exponential function, the exponential form of the Fourier series. First of all we observe that, we have only one single formula for evaluating the various Fourier coefficients.

We do not have separate formulas for a naught, a n and b n. And the notation is very compact. More importantly, when we extend this concept of Fourier expansion of periodic functions to a periodic functions. What we will refer to as Fourier integral concept, which we will take up later. There these expressions for C n can be in a more straight forward fashion, extended to the Fourier integral concept, than you had persisted with a n and j b n.

So, we have a single formula valid for all n. The notation is compact and the notation can be extended in the Fourier integral quite conveniently. That is the important thing. So, in the Fourier expansion for this, we note that each term by itself, may not convey to us any physical signal. Because, when you substitute a value real value of time, this does not by each term by itself will not yield a real value of the function.

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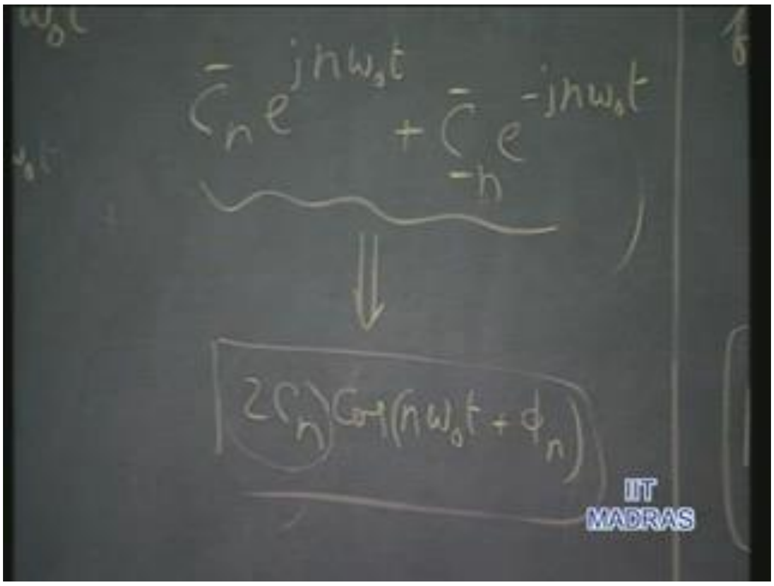

However, we have to bear in mind, that if you take C n e to the power of j n omega naught t. And combine this with C minus n e to the power of minus j n omega naught t. These two together will lead to a real function of time a sinusoid, which will be 2 C n cos n omega naught t plus theta n. You can easily show that. So, the amplitude of the n th harmonic component is 2 C n. And this real function of time comes by combining the two exponential terms, for plus n and minus n respectively.

So, individually it is not a physical signal, but when you combine these two, this is the conjugate of this you will get this. Now, let us work out an example, illustrating the exponential form of Fourier series.

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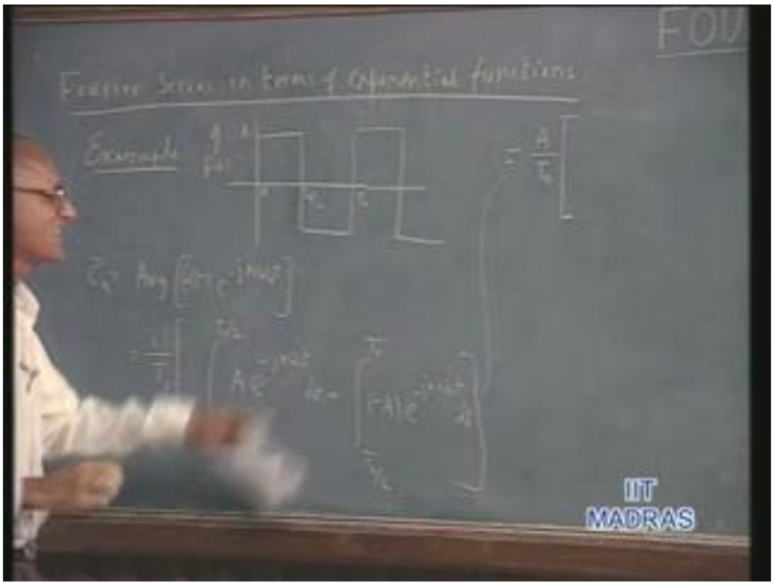

You take the same example, that we have worked out earlier a square wave. Because, we can compare the results, so this is f of t. What we would like to find out is the Fourier series expansion in exponential form. So, C n would be, easy way to remember would be average of f of t multiplied by e to the power of minus j n omega naught t. This is what we have to find out. So, that will be 1 over t naught. And to find out the integral of product, we split this integral into two parts.

One from 0 to T naught upon 2 and other T naught upon 2 to T naught, because the value of the function changes in these two intervals. Therefore, I can write this as 1 over T naught 0 to T naught upon 2. The value of the function is A e to the power of minus j n omega naught t d t plus T naught upon 2 to T naught. And in this interval the value of the function is minus A minus j n omega naught t d t.

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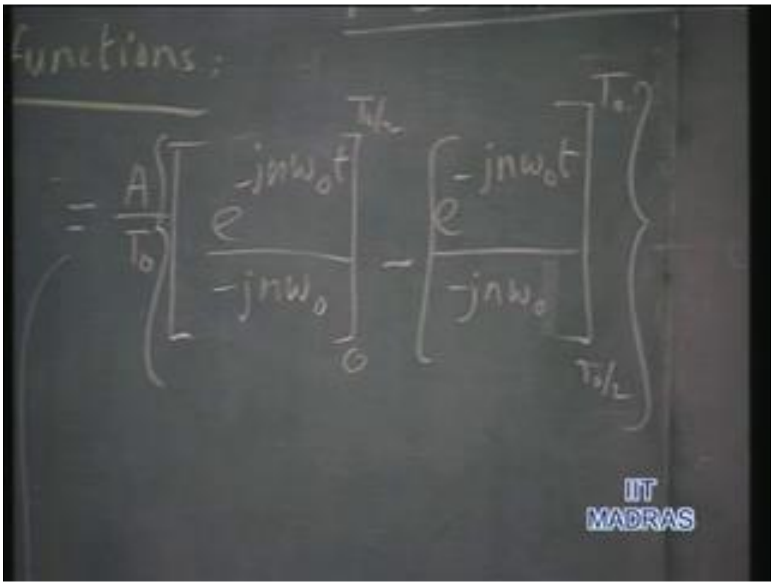

So, I take the A outside, A upon T naught. The first integral will yield e to the power of minus j n omega naught t by minus j n omega naught minus, because of this minus sign here. E to the power of minus j n omega naught t divided by minus j n omega naught. And the first integral is evaluated between 0 and T naught upon 2. The second integral is evaluated between T naught upon 2 and T naught.

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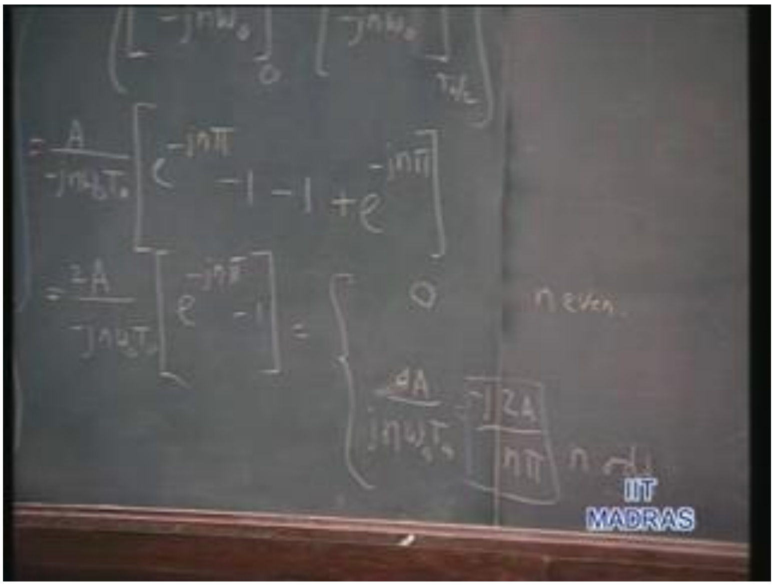

So, you can take out minus j n omega naught outside. And if you evaluate this, it will be e to the power of minus j n at the upper limit, omega naught T naught upon 2 is pi minus j n pi. At the lower limit it is 1 minus 1. And minus at the upper limit, it is minus j n omega naught T naught, it is 2 pi. Therefore, e to the power of j n 2 pi, it is equivalent to 1, because it is an integral multiple of pi.

Therefore, 2 pi minus 1 and the lower limit, because of the minus sign, it becomes plus e to the power of minus j n pi. Because, T naught upon 2 multiplied by omega naught equals pi, leads to pi. Therefore, this can be written as minus j n omega naught T naught, these two can be combined to 2 times e to the power of minus j n pi and minus 2. Therefore, 2 a times e to the power of minus a n pi minus 1.

And e to the power of minus j n pi is either plus 1 or minus 1, depending upon the value of n. Therefore, if n is even this becomes 1. Therefore, this leads to 0 1 minus 1. If n is odd, then e to the power of the angle is an odd multiple of pi. Therefore this minus 1, so you have minus 2 minus 1 minus 2. Therefore, it will become this j can be taken out. Therefore, it will become minus 4 A divided by n omega naught T naught and a j in front. And omega naught T naught is 2 pi.

Therefore, this will be minus j 2 A by n pi for n odd. And since, we know that C n equals a n minus j b n by 2. Therefore, what we now see is b n upon 2 is 2 A by n pi. So, from this ((Refer Time: 40:48)) we conclude that b n equals 4 A by n pi, a result which you have already obtained from the trigonometric form of Fourier. So, this ties up with that. C naught of course, is 0 because the average value of this is 0. And when we are carrying out this analysis, to find out the C n, it would always be advisable for us.

To arrive at the C naught value independently, rather than the straight formula of substituting n. Because, sometimes when n equal to 0, it leads to some difficulties some degeneracy, because n may come in the denominator. So, it is always advisable to calculate C naught independently, rather than substituting n equals 0, in the general form. Sometimes it may work, sometimes it may not.

Now, we have discussed symmetry conditions in relation to the a and b coefficients. Now, what are the similar conditions, that are applicable to the C n coefficients. So, let us now discuss that.

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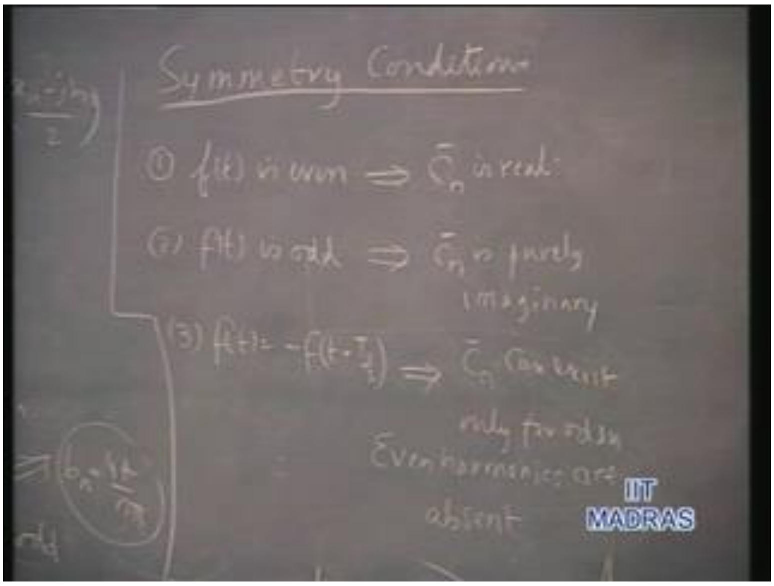

If f of t is even, we said that only the cosine terms are present. That is an terms are present, not the b n terms. Since, we know that C n is a n minus j b n by 2, if f of t is even it means that, C n is real. Because, C n is a n minus j b n upon 2. So, if b n is absent, then C n is purely real. The angle associated with the complex number is 0. If f of t is odd then of course, C n is purely imaginary. We have the b n terms, but no a n terms.

And if f of t exhibits half wave symmetry, then as before we have only the odd harmonics present. C n can exist only for odd n, even harmonics are absent. So, to summarize, what we have done in this lecture. We started with an example in which we have taken a half wave rectified sine wave, found out it is Fourier series, making effective use of the symmetric conditions. We saw that a half wave sinusoid can be expressed, broken up into its even part and odd part.

The odd part was a pure sine wave. The even part was a full wave rectified sine wave. Whatever waveform is there in one half cycle is reproduced. And therefore, we have only the even harmonics present. And we have found out the Fourier series expansion of the even part and odd part separately. And use such a kind of waveform in a practical circuit consisting of a R and C, rectifier circuit. And use that example to illustrate.

How we can make use of Fourier series in analyzing the steady state performance of a simple R C circuit. Then, we took up the question of Fourier series expansion, in terms of exponential functions. So, in terms of exponential functions we expressed f of t, as a summations of various terms. Each term being of the form C n e to the power of j n omega t, C n omega naught t; where C n is a complex coefficient in general, which is related to the a and b coefficients that we already talked about.

As C n equals a n minus j b n upon 2. And we found out that, the expression for calculating C of n is surprisingly simple. It is simply the average of f of t times e to the power of minus j n omega naught t, valid for all values of n. And we said that expansion of this form is useful for us, when we later go to the Fourier integral form, apart from its compactness. And the fact that we have to calculate only one set of coefficients. That is C n coefficients instead of having to calculate a n and j b n separately.

The price we have to pay for that is of course, we have to use complex algebra. Because, here is a complex number, whereas if you are calculating a n and b n, we have to deal with real functions only. So, we have a price for it, but nevertheless it leads to a very compact notation. So, we have two alternative ways of expressing Fourier series. One in terms of the a n and b n coefficients, other in terms of C n and one can always convert, one set of coefficients into the other.

We took up the square wave as an illustration. And showed that the expansion in terms of C n will of course, naturally as we expect, leads to the same results. But, of course, C n now is an imaginary quantity. That means, this is related to b n which is 4 A by n pi which we already found out. And lastly we talked about symmetric conditions, in terms of the Fourier coefficients of the trigonometric expansion, of the exponential expansion. And we said if f of t is even, C n happens to be real.

If f of t is odd, C n is purely imaginary. And the half wave symmetry, which we already discussed with does not yield any new surprising results. Of course, we know that once f of t equals minus of f of t plus T naught upon 2, which means that the wave is reproduced with a negative sign, in the succeeding half cycle. Then, only eve harmonics are present and therefore, C n can exist only for odd n. Some of these, for some odd values of n, C n may not exist also. But, if at all it exists, it can exist only for odd values of n.