Networks and Systems

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Lecture - 08
Fourier Series (2)
Symmetry Condition Examples

We introduced ourselves to the concept of Fourier Series. And to the evaluation of Fourier coefficients a n and b n, which are required to set up the series. In many situations, the waveforms that we have to deal with, exhibit certain types of symmetries, which we can take advantage of. And simplify the work involved, in the evaluation of these various Fourier coefficients.

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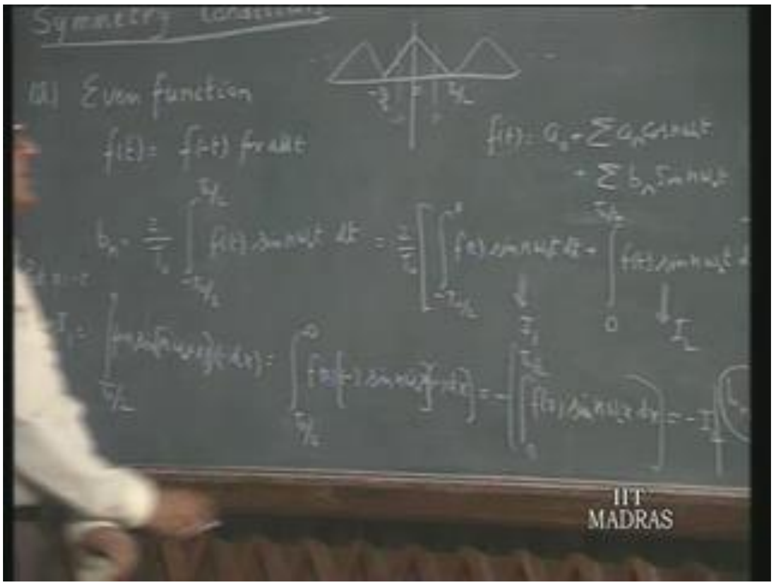

This we will now take up for study, under the heading Symmetric Conditions. We will talk about four kinds of symmetry. And then, see how each individual set gives rise to some advantage in the matter of evaluation of the Fourier coefficients. To start with suppose, we are given a periodic function, which is an even function of time. The definition of an even function is f of t equals f of minus t, for all t.

Example of that would be, suppose I have a function which varies in this manner. For positive T sub 0 to T naught upon 2. In the negative direction, it takes for any particular T at minus T, the values must be the same. So, if this is a periodic waveform, with this as the basic period, it repeats itself like this. And this is an even function of time. You recall that the basic Fourier series, expansion is a naught a n cos n omega naught t plus b n sine n omega naught t. The whole series of terms.

Now, these trigonometric functions here, or this a naught of course, is a constant maintains it is value. So, we will consider that to be an even function. Cos terms are even functions of time, but sine terms are not even functions of time, they are odd. So, if this function is even, we can expect that, we have no place for the sine terms in it is expansion. Because, when you substitute minus t for t, the values of these terms will remain the same.

But, this set of terms will have their value reversed. Consequently f of t can no longer be f of minus t. So, intuitively we can expect that all the b n terms go to 0. Let us prove this a little more vigorously ((Refer Time: 04:12)). So, if you evaluate b of n, you recall the expression for b of n is 2 upon T naught 0 to T naught. Or I can also for the sake of convenience in this regard. I can take this from minus T naught upon 2 to plus T naught up on 2. Because, it is more convenient for us in this context.

F of t sine n omega naught t d t. I can break up this integral into two parts. 2 upon T naught minus T naught up on 2 to 0 f of t sine n omega naught t d t plus 0 to T naught upon 2 f of t sine n omega naught t d t. Now, we are going to show presently that, this integral will be exactly the negative of this. Therefore, b n will be 0. To do that let me call this integral I 1 and this I 2. Our goal now is to show that I 1 is the negative of I 2.

So, to prove this let me evaluate I 1 separately. I 1 will be this expression, but now in I 1, I will put x equals minus t. So, I will change the variable of integration instead of t I use x. Therefore, d t will become minus d x. And since, the variable of integration now is x, I must put the appropriate limits. So, when t equals minus T naught up on 2, x will be T naught upon 2. And when T is 0, x of course will be 0. F of t will become f of minus x and then, you have sine n omega naught minus x minus d x.

By virtue of this, even character of this function, we can write this as, this is T naught upon 2 again. F of minus x equals f of x, because of this relation. And then, sine n omega naught minus x will be minus of sine n omega naught x. And then, you have another d x with a negative sign out in front. Now, you see this negative sign and this negative sign, can be cancelled. And if I interchange limits of integration, then I will get another negative sign.

Therefore, ultimately this will be minus of 0 to T naught upon 2, I interchange. That means, you have got three negative signs. One virtue of interchange is integral limits. One because of the negative sign here and one here. I have got f of x sine n omega naught x. Now, this integral here is exactly the same as this, because this is a definite integral. And the variable of integration could be any value, it is a dummy index really. So, this integral f of x sine n omega naught x d x, could as well we written as f of sine n omega naught t d t, which is indeed this.

And because, there is a negative sign out in fact, this can be written as minus I 2. So, the upshot of this analysis is that b n is 0. So, if the function is even, we immediately can conclude that, all sine terms are absent. This is the result that we get, if the function is even.

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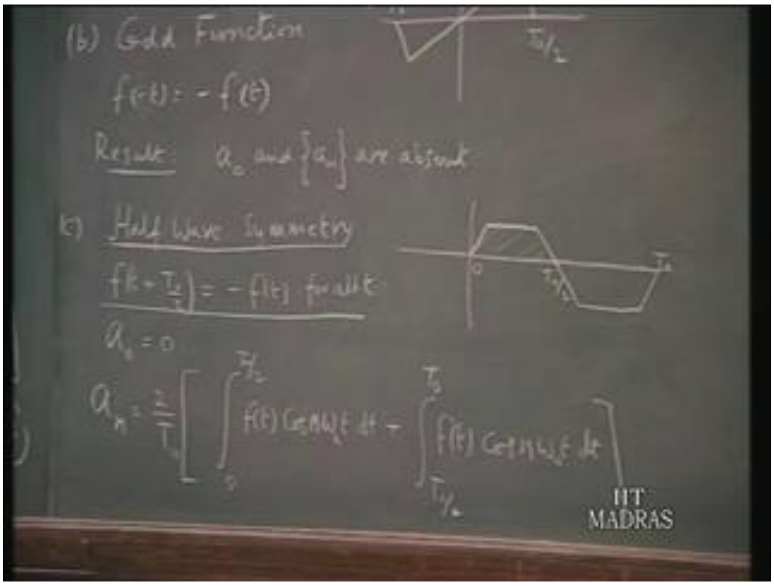

Naturally we would like to ask, what is the corresponding result for an odd function. How does an odd function look like? If I have this as the variation of positive T, the negative T will be like this. So, of I say this is T naught upon 2, this is minus T naught upon 2. So, this f of t over a one period. Whatever sequence of values it takes, in the positive direction of T. The negative direction it will take the negative of those corresponding values. So, f of t, we will say f of minus t equals minus of f of t.

Now, if you again look at the general expression, for the Fourier series. We observe that these are the terms, which give rise to the odd character of the function. And these are the terms which are even. Therefore, if you change t to minus t, we would like the entire function value to reverse it is sign. However, ((Refer Time: 10:10)) these terms a naught and a n terms will have the same value for minus t and plus t.

Consequently they spoil the character of the odd function. And therefore, we expect that these terms be absent, and only b n terms present. And I will leave this as an exercise for you to show, is the similar fashion as here. That in this situation, the result is I will not work this out. I can show it on similar lines, result is a naught and a n terms are absent. So, when we see a periodic function, which is an odd function of time. We do not have to spend time in evaluating a naught and d n.

We immediately know they are 0, so all we have to do is work out the values of the b n coefficients. The third type of symmetry which is quite important, because it comes up in several occasions, is half wave symmetry. Basically what it means is, that if you have a particular variation for half the period. The succeeding half period, it will have the negative of that variation. F of t plus T naught upon 2 is minus f of t for all t.

Such waveforms are quite common, particularly in electrical machinery. When you have a rotor rotating and inducing the waveform in the stator, whatever it is. Then, alternately the conductors come under, the north pole and south pole. And whatever sequence of values for the EMF are induced, when the north pole is in operation. The values will be reversed, when the south pole is under operation. Therefore, the waveform exhibits this kind of symmetry.

The same set of values are reproduced with a negative sign, in this succeeding half cycle. So, what is the off shoot, what is the consequence of this? First of all, we observe immediately that, since the same set of the values are occurring with a negative sign, in the succeeding half cycle. The average of the period, of the average of the function over the complete cycle must be 0, which means the d c value is 0.

What other conclusions can you draw from this. This may not be evident, but let us work them out. Suppose I take a n 2 upon T and this time I break the integral, for convenience 0 to T naught upon 2 f of t cos n omega naught T d t plus T naught upon 2 to T naught f of t cos n omega. By this time you must have got an idea, what we are going to do next.

We would like to see a relate this integral with this. Therefore, for ((Refer Time: 14:15)) this purpose, let us call this I 1, call this I 2. And we would like to compare the values of I 1 and I 2 making use of this result.

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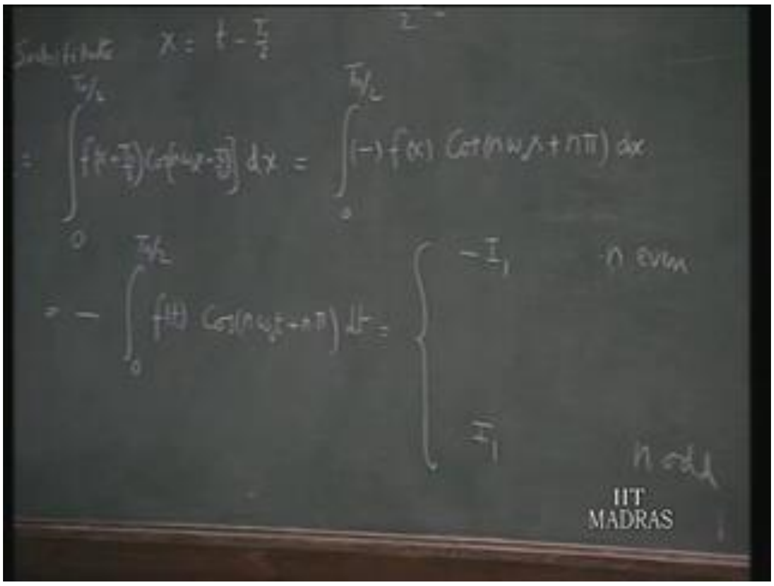

Let me evaluate I 2. In this, let me substitute x equals t plus T naught upon 2; or rather x equals t minus. Then, I 2 will be, so I am going to substitute x for t d t equals d x. Therefore, the variable of integration is x, so I have d x. Then, from small t equals t naught upon 2 x will be 0. When small t equals capital T naught, then x will be T naught by 2. So, you can see now that, the integrals limits are corresponding to this straight away.

F of t will be f of x plus T naught upon 2, because t equals x plus T naught upon 2. And we have in addition cos n omega x plus T naught, that is what it is. Now, this will be 0 to T naught upon 2. And by virtue of this of this relation, that f of t plus T naught upon 2 is minus of f of t. F of x plus T naught upon 2 is minus of f of x. And then, we have cos n omega x plus n omega naught T naught upon 2. Omega naught T naught is 2 pi, therefore ((Refer Time: 17:14)) this is n pi, therefore this will be n pi.

So, this will be, I take the minus sign out in front 0 to T naught upon 2. Now that, I have outlived the residue for use of x; let me go back to t. Because, after all this is a dummy variable, I can go back to t. This will be f of t cos n omega naught t plus n pi d t. Now, we have this n pi, we have to see what this becomes. So, let us take two cases, suppose n is even. If n is even cos n omega naught t plus 2 pi 4 pi and so on, is simply cos n omega naught t.

Therefore, this will be f of t cos n omega naught t d t, which is the same as I 1. Limits of integration is same, the integrand is the same. Therefore, this will be minus I 1. If n is odd cos n omega naught t plus pi, will be minus cos n omega naught t, pre pi also is the same. Therefore, it simply becomes I 1. So, the upshot of this analysis is, that this a n turns out to be 0, for n even and 2 times I 1 for n odd. Therefore, finally the conclusion that we have is, all the a terms with even indices will become 0. Same analysis can be extended for b n terms also.

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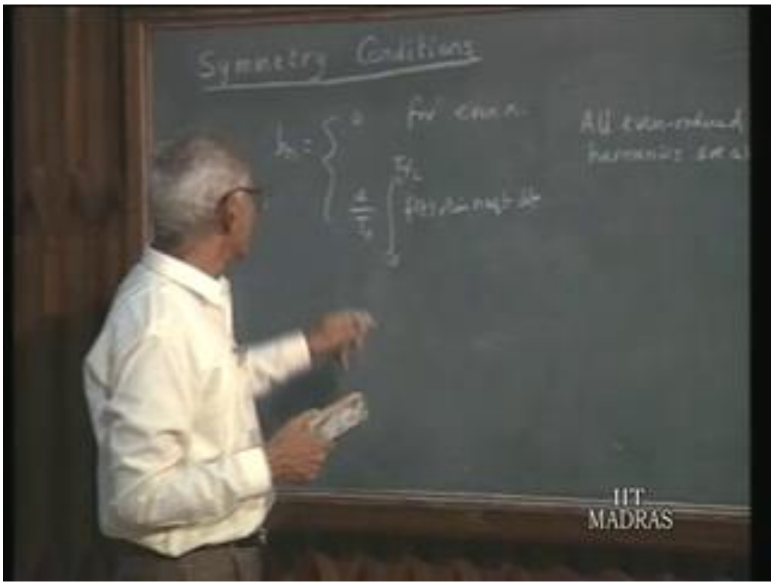

You can show that b n will be 0, for even n and for odd n. You see, a n for odd n turns out to be, 2 i n, I must also write here, because I am equating this ((Refer Time: 19:59)), I must also multiply it by 2 by T naught. So, I will write the complete expression here. So, this 2 by T naught multiplied by 2 I 1. Therefore, that means 4 by T naught, 0 to T naught by 2 f of t cos n omega naught t d t. So, here also you have 4 by T naught 0 to T naught by 2 f of t sine n omega naught t d t. So, that is the result.

We can summarize this by saying that, all even ordered harmonics are absent. So, when you have this kind of symmetry, as you have in the waveforms generated by electrical machines. You will have only the fundamental third harmonic, fifth harmonic, seventh harmonic and so on. You cannot think of having the second harmonic, fourth harmonic, nor even the d c term, because the average of it is going to be 0.

And for the evaluation of the odd harmonic components a n and b n, we have instead of taking the integration over the complete period. It is permissible for us to integrate over half the period. And consequently have 4 by T naught instead of 2 by t naught. That means, you are taking twice the average over half the period, rather than twice the average over the complete period.

So, you can also dispense with lot of integration. And confine yourself to one half cycle, whenever you have this kind of symmetry. Important too that all your harmonics are absent and d c is also considered to be even harmonic, that is also absent.

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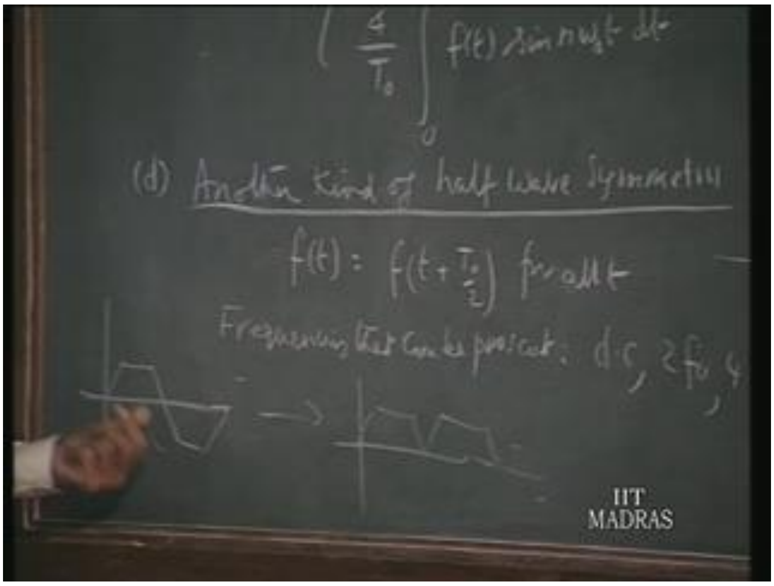

We have another kind of half wave symmetry, where f of t equals f of t plus T naught upon 2, for all t. Previously f of t plus T naught upon 2 is minus f of t. Now, f of t plus T naught by 2 is plus f of t, which would have been the same waveform. If you have for 0 to T naught upon 2, it repeats itself like this. Now, you should look at this closely, if indeed this is the situation that you have.

Strictly speaking, we should have P naught by 2, as the period not T naught. Because, we agreed that the period is the smallest value, which satisfies this type of equation. Therefore, basically T naught by 2 is the period, which means the frequency of this, is twice the frequency. If you have considered this as one period. If you have considered this as one period, it is f naught, if this is the period then it will be 2 f naught. So, consequently the frequencies present are frequencies, that can be present are...

You can of course, d c, 2 f naught, 4 f naught, 6 f naught etcetera, where f naught is 1 over T naught. So, here you have only even, even ordered harmonics are present. All odd ordered harmonics are absent. Now, naturally you question, why do you make a big fuss about this. If T naught is the basic, T naught by 2 is the basic period, why not call that as the basic period, call that as T naught and then, deal with this. The answer to this, often we may have a particular waveform, which we split up into number of component waveforms.

And in one of the component waveforms, you may have this kind of symmetry. But, we have to relate this frequencies to the original waveform, from which these are generated. In particular suppose you have an example, ((Refer Time: 25:06)) from a sine wave you generate, we will have not a sine wave really. Suppose, you have a waveform, something like this and then, you rectify it full wave rectification you get this kind of waveform.

So, we would like to whatever frequencies, present in this waveform. We would like to relate to the fundamental frequency of this. Consequently we still like to have this as the basic period, even when you are dealing with this. Therefore in that context, if you consider persist with it calling T naught, here as the basic period. Then, in this signal, you can have only the even ordered harmonics present. Odd ordered harmonics are present.

The fundamental will not be present naturally, because it is a first harmonic. We will work out an example later, which will illustrate this. So, this is really a matter of definition. And if you call this, as for this you want to deal with this particular wave form on it is own, without reference to any other waveform. Circulate this permissible, for us to call this as the basic period. And then, analyze this in a conventional manner. But, if you call this as the period for this waveform, then we can say that only the even ordered harmonics are present.

These are the four important kinds of symmetry, that one deals with. Also it is interesting to note that, if you have any given function f of t, you can always split this up into it is even part and the part. So, any arbitrary periodic function, may not be even by itself, may not be odd by itself. But, we can generate from it the even part and the odd part.

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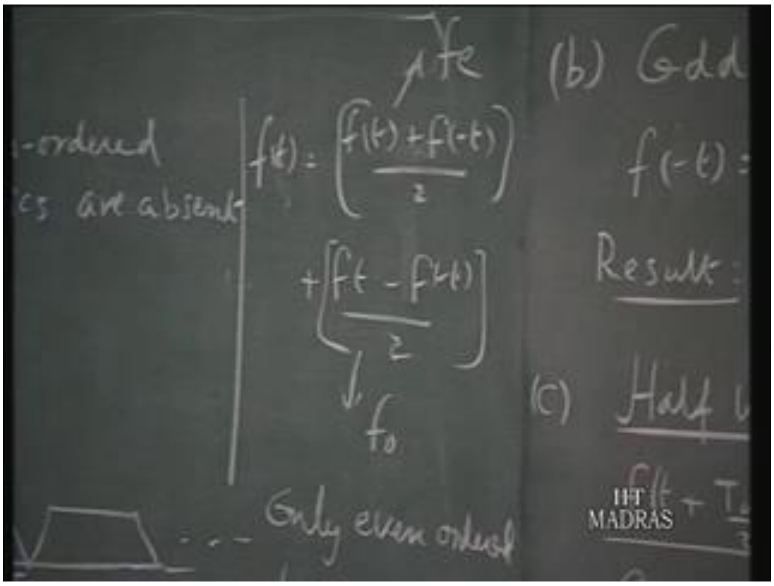

So, if you have f of t, you can call this f of t can be written as f of t plus f of minus t upon 2, plus f of t minus f of minus t upon 2. So, the first part is the even part, the second part is the odd part. So, from any given function, you can generate its even part and the odd part. And naturally, the Fourier series of this will contain only, the d c and cosine terms. And this will contain only the side terms. So, if you have any composite signal the group, a naught plus summation of an cos n omega t, corresponds to this.

And summation of b n sine omega t corresponds to this. And it might be possible for us, in some situations. To break this up into it is even part and odd part. And find some additional symmetries and their respective parts. We will work out a few examples, which illustrate this.

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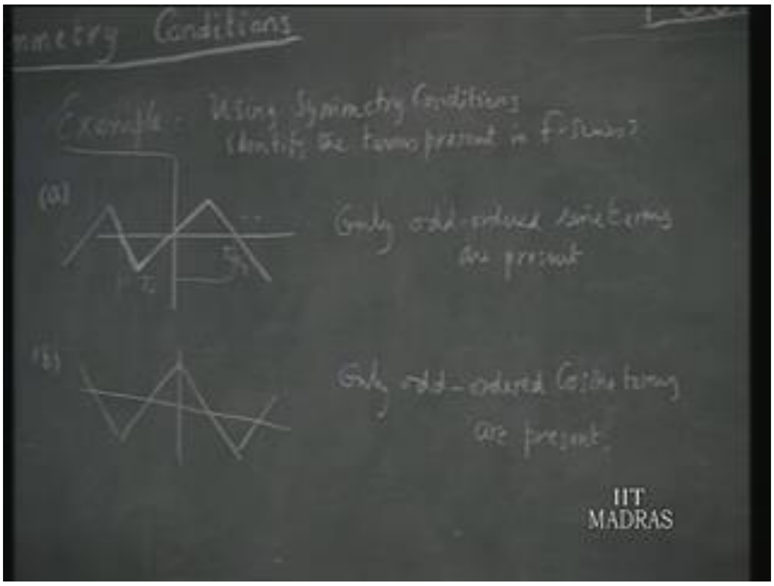

Put down some waveforms. Let us find out first of all, what kind of harmonic components are present. Making use of the various symmetric conditions, that we have discussed. Suppose, we have this kind of waveform. Using the symmetry conditions, identify the terms present in the Fourier series. Or you can say what are the terms that are absent in the Fourier series.

So, if this function is there, first of all you can straight away see, this is an odd function. Because, f of t equals minus f of minus t. Therefore, only sine terms are present and further whatever waveform you are having here, is repeating in the next half cycle with a reverse side. Therefore, this has the half wave symmetry of the type, which we have talked about. So, the result is, that this can have only sine terms, but odd harmonics. So, odd ordered sine terms are present.

You cannot expect cosine terms, you cannot expect second harmonic, fourth harmonica and so on, certainly not d c. Suppose I shift this waveform, you have something like this triangular wave, is symmetrically situated the origin. Then, what do we have, this is an even function of time. Therefore, cosine terms are present. But, that half way symmetry that we talked about still persists. Whatever occurring in this half cycle will repeat itself. Therefore, we can say now, it will have odd ordered cosine terms are only present.

We must carefully distinguish between, the oddness and the evenness of the function. And the oddness and the evenness of the order of the harmonic. When you have cosine terms, you say it is an even function, sine terms will give only an odd function. That relates to the function itself f of t and f of... Whether, it is f of minus t or minus of f of minus t. What we are talking about is the order of the harmonic, the index of the harmonic.

Second harmonic, fourth harmonic or even ordered harmonics. And fundamental third harmonic etcetera, are called odd ordered harmonics.

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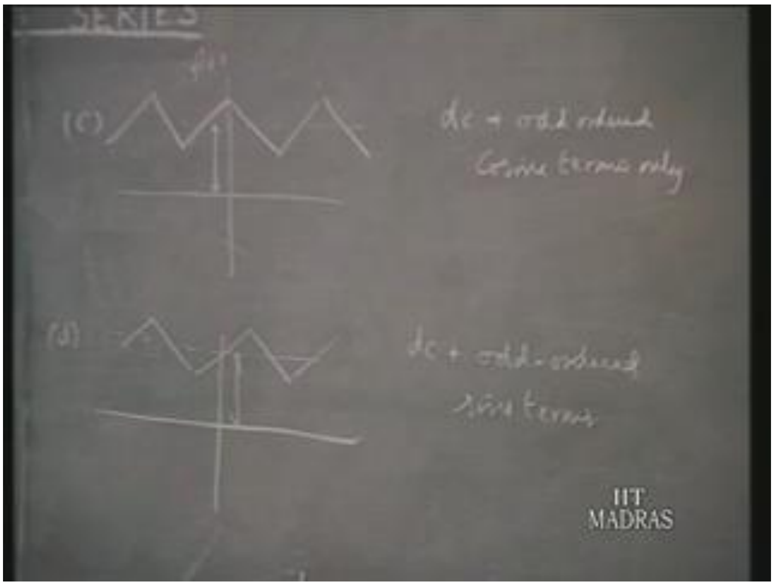

Suppose I have a function like this, which is the same function shifted upwards by a certain amount. So, in other words, if I draw a line here parallel to the x axis. And subtract this value from this f of t, this will be the result. So, what do we conclude from this. It will have all the harmonics that you are having here plus a d c term. So, you have, because if you subtract this value from , then resulting waveform will be exactly this.

Therefore, we have ((Refer Time: 32:26)) d c plus odd ordered cosine terms only. Similarly, if this waveform is shifted up. So, if this is the line, then you have... If you subtract that amount, this value from the waveform you have a waveform, which is given in a. Therefore, you have d c plus odd ordered sine terms. One thing you would notice, from this example is, that a same function if we shift the origin. In one case it may be an even function, other case it may be an odd function.

Therefore, the placing of the horizon plays an important role. But, that does not affect the amplitude of the harmonic, as we will see a little later. And once again it is also possible for us to note that. When you remove the d c term from a given signal, sometimes you find out certain characteristic, which may not be present in the original signal, as these examples show. Now, let us work out another example.

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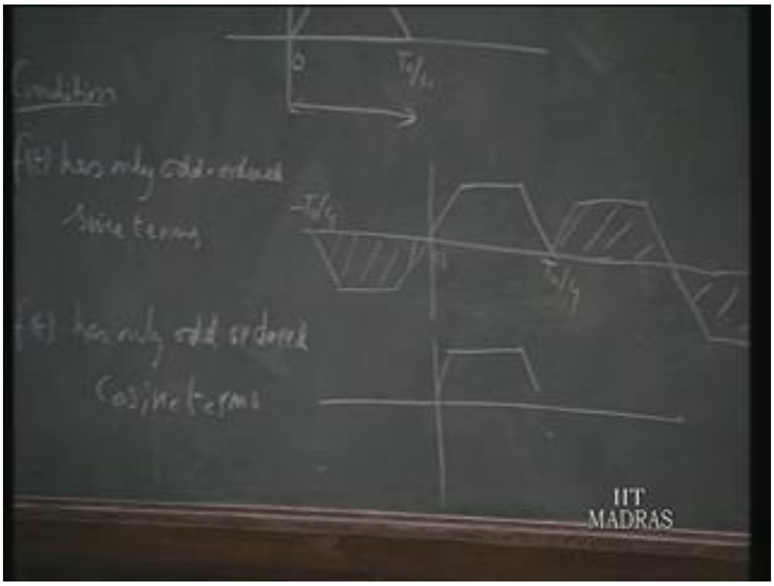

So, a waveform is given only for this interval of time. Now, certain initial conditions are given and we are asked to find out, identify the complete waveform. So, let us say the conditions are f of t, as only odd ordered sine terms. So, if the waveform for the entire cycle is not given, only partial specifications are given, but this condition is given. So, this is the data that is given to us. So, if f of t has only sine terms, which means the function is odd.

Therefore, if 0 to two by T naught upon 4 is prescribed to us, we immediately conclude that this, must be it is image on the negative time axis. So, this portion we can fill up, because only sine terms are present. And then, we are also given that, it has odd ordered harmonics only, which again tells us that whatever sequence of values it takes in 1 half cycle. Is getting repeated with a negative sign in the subsequent half cycle.

So, starting from this, in this half cycle this is the sequence of values. Therefore, the negative of that will be this and therefore, this also is filled. Now, that corresponds to 3 T naught by 4. So, we have completed the waveform a full cycle. So, we can make use of the conditions of symmetry. And then, extend this to satisfy this, the given data and the condition that are given. So, suppose it is said, that f of t has odd ordered sine terms, cosine terms. Then, how does it go?

((Refer Time: 37:15)) This is the waveform, that is given to us. And since it has only cosine terms, it is an even function. Therefore, this would be the situation from minus T naught upon 4 to T naught upon 4. And since it has only odd ordered harmonics, this entire waveform gets repeated with a negative sign. So, this is the extension, that we are having. And that would be the situation for a case, where f of t has only odd ordered cosine terms.

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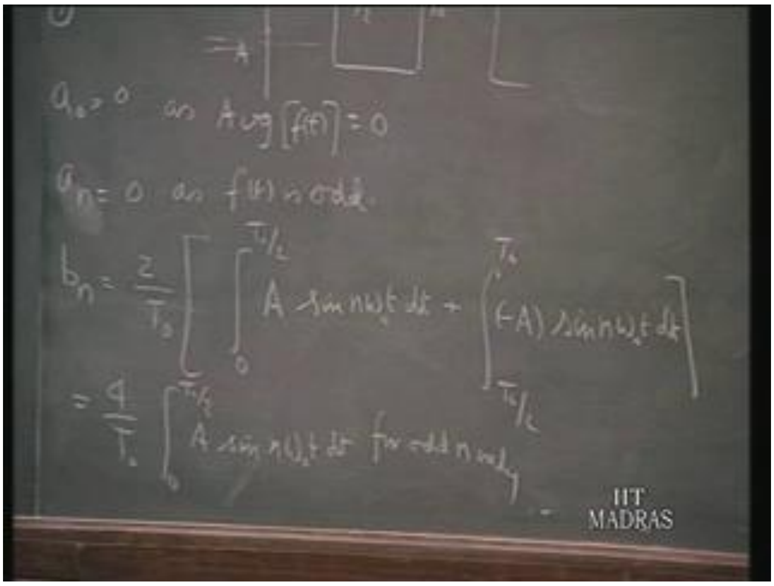

Let us have another example. We will take the square wave, which is a very important waveform. Of course, it repeats itself in the negative time, also like this. And we are asked to find out, it is Fourier series expansion. We can see straight away, that the positive half cycle and negative half cycle have equal areas, that cancel each other out. Therefore, the average of this function f of t over a complete period is 0. Therefore, a naught is 0.

We also can see without further a do, that b n is 0, because is the function is odd, therefore, a n is 0. So, the only coefficient, that can be present are the b coefficients. So, b n would be 2 upon T naught 0 to T naught upon 2, I will break up this integral into two parts. From 0 to T naught upon 2, the value of the function is A sine n omega naught t d t plus T naught upon 2 to T naught. In this interval of time, the value of the function is minus A sine n omega naught t.

You can evaluate this second term, two integrals separately and then, arrive at the result. Alternately we can also make use of one property, which we have discussed already. When you are talking about half wave symmetry conditions. We said which is exactly this situation where, whatever is happening in positive half cycle will occur in the negative sign in the second half cycle. Therefore, at that time if you recall, exactly similar situation, we said this integral is equal to this integral for odd n.

Therefore, we can as well write this as for odd n, this will be 4 by T naught like this. Alternately we can calculate this also independently. So, we can write this as 4 by T naught 0 to T naught upon 2 a sine n omega naught t d t for odd n only. You must be very careful in writing this. You cannot have this and then, evaluate for even n, it will give a result, but that is wrong.

For even n, this term will cancel with this and that becomes 0. So, when you write this expression, you must make sure that, that it is valid for odd n only. This is 0 for even n.

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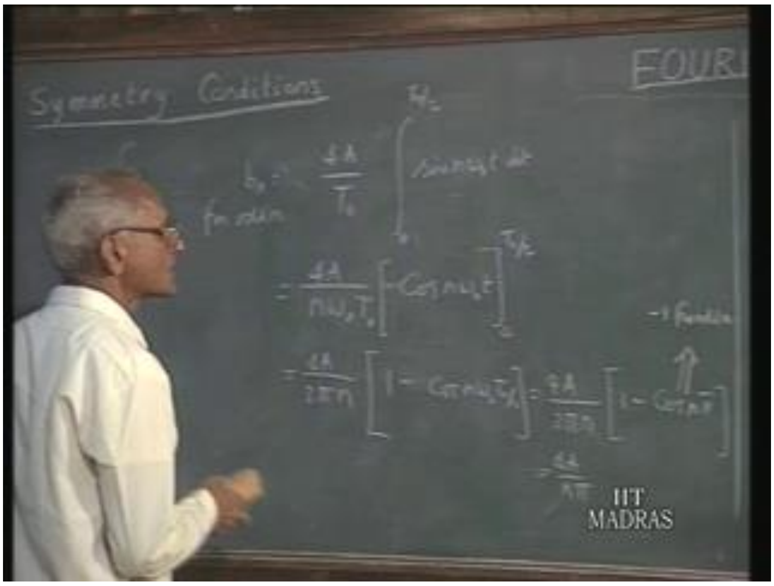

So, you work this out, then it will be b n will be for odd n 4 A by T naught 0 to T naught upon 2 sine n omega naught t d t. So, 4 a by n omega naught t cos n omega naught t 0 to T naught upon 2. Therefore, this will be 4 a by omega naught t is 2 pi. T naught omega naught T naught is 2 pi, so 2 pi n. And here you have 1 minus cos n omega naught T naught by 2. Therefore, is this 4 A by 2 pi n times cos n omega naught T naught by 2 is cos n pi. And we are talking about odd n.

So, cos n pi will be minus 1. And therefore, this will be 4 a by n pi, so only b n terms are present. So, finally, we conclude that the Fourier series expansion, for this would be I will write here.

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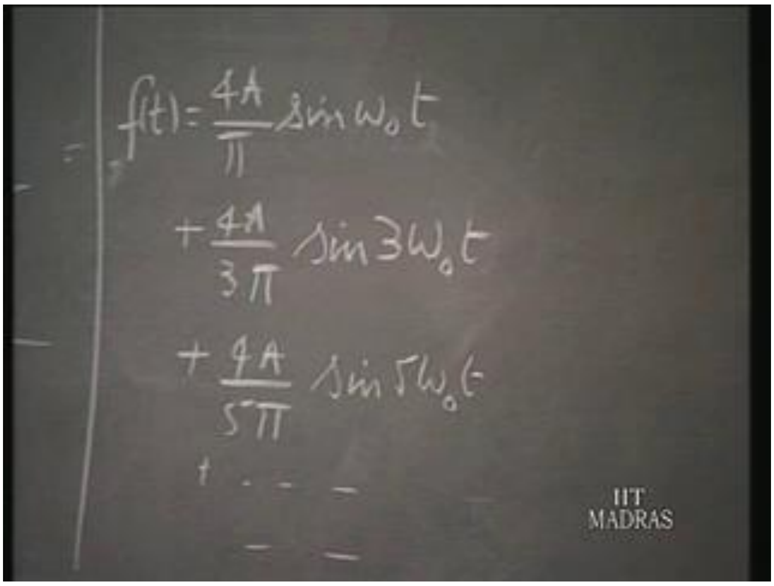

F of t 4 A by pi sine omega naught t plus 4 a by 3 pi sine 3 omega naught t plus 4 A by 5 pi sine 5 omega naught t and so on, and so forth. So, this is a important result, that it would pay us to keep this in mind.

For a square wave like this, with an amplitude A, the fundamental component as an amplitude 4 A by pi. So that means, the fundamental will be something like this ((Refer Time: 44:45)). The third harmonic will be having 1 3rd that amplitude, so it will be. Fifth harmonic will be 1 5th, actually the figure which we have seen in the earlier lecture, correspond to this. So, the amplitudes go down inversely as the order of the harmonic.

We will continue this discussion or to sum up first of all. What we have learnt in this lecture is that, the symmetry conditions on the given functions of time; enable us to calculate the Fourier coefficients with less effort. That it would otherwise be necessary. In particular, we saw if the function is even, then there can be only cosine terms present.

If the function is odd, that is if f of t is minus of f of minus t only sine terms, can be present.

And the third type of symmetry that we talked about is, when whatever waveform you have for 1 half cycle; is repeated for the subsequent half cycle with it is negative sign. Then, we call that half wave symmetry. It is quite common in electrical machines for example. And in that situation, you have only the odd ordered harmonics. You have the fundamental, the next harmonic, the third harmonic and the fifth harmonic and so on.

There is another kind of symmetry, which we mentioned in passing. That is when the function repeats itself identically, for every half cycle. In that case, only the even ordered harmonics are present. That means, the second harmonic, fourth harmonic and so on. This as I said is, we call this function to have a period, twice which we normally expect, or twice of what we expect. Because, this is usually derived from another signal, whose fundamental frequency is already defined.

Using these symmetry conditions. We worked out a few examples, which show how the Fourier coefficients will be evaluated. Making use of the various symmetries, that are present. In the next lecture, we will talk about another method of setting up the Fourier series. Not in terms of trigonometric functions, but in terms of exponential functions, which will give a more compact notation and certain advantage of the competition. This we will take up in the next lecture.

Thank you.