

Networks and Systems
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Lecture – 50
State-Variable Methods (6)
Review of solution for discrete-time systems
Example
Exercise N0.9
Concluding Remarks

Let us quickly review the solution for the state and output equation that we have obtained in the discrete time domain that was derived in the last class last lecture. The solution for the state vector X_n would be $A^n X_0$ plus summation on k from 0 to $n-1$ A^{n-1-k} times $B u_k$ if A^n is put in the more compact form $\phi(n)$ this is the state transition matrix. $\phi(n)$ times X_0 plus $\phi(n-1)$ convolved with $B u_n$. We saw that, in the Laplace in the Z transform domain the Z transform of the state vector would have this form $Z I - A$ inverse $Z X_0$ plus $Z I - A$ inverse minus A inverse $B u_Z$.

From this we would see that the state transition matrix $\phi(n)$ and $Z I - A$ inverse Z form. Similarly, if you have $\phi(n-1)$ state transition matrix from the volume of n is decremented by 1. This will be $Z I - A$ inverse a certain point you would note is suppose we have the poles of $Z I - A$ inverse or also the zeros of determinant of $Z I - A$ equal to 0. After all the determinant of $Z I - A$ equals to 0 or the same as the zeros of determinant of $\lambda I - A$ equal to 0.

Therefore, determinant of $Z I - A$ is a characteristic equation in terms of Z instead of λ . So, these are also equal to the characteristic value of the system characteristic values λ_i and this turns out these are also the natural frequencies in the discrete time domain.

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STATE-VARIABLE

$$X(n) = A^n X(0) + \sum_{k=0}^{n-1} A^{n-k-1} B U(k)$$

$$= \Phi(n) X(0) + \Phi(n-1) * B U(n)$$

$$X(z) = (zI - A)^{-1} z X(0) + (zI - A)^{-1} B U(z)$$

$$\Phi(n) \leftrightarrow (zI - A)^{-1} z$$

$$\Phi(n-1) \leftrightarrow (zI - A)^{-1}$$

Poles of $(zI - A)^{-1}$
 = {zeros of $\det(zI - A) = 0$ }
 = characteristic values λ_i
 (Natural modes)

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So we may also write this natural frequencies natural modes at the discrete time system. So, there are all related whether you take the characteristic equation of the determinant of lambda i minus A equals 0 for determinant of Z I minus A equal to 0. They are the natural modes that we get.

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$(zI - A)^{-1} B U(z)$
 Poles of $(zI - A)^{-1}$
 = {zeros of $\det(zI - A) = 0$ }
 = characteristic values λ_i
 (Natural modes)

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Now, going back to the output equation which derives this Y_n equals C times X_n plus D times u_n . So, C times X_n X_n is equal to C times X_n will be this expression after D times u_n which instead of convolution if you put it in summation form this is the what you

would get. And in the transform domain you get $CZ(I - A^{-1})Z^{-1}X(0)$ coming from this $CZ(I - A^{-1})B^{-1}uZ$ coming from this plus DuZ . That is what you have in the transform domain. Of course this would be called the 0 input solution because when u is 0 this is what results. This is the 0 state solution.

So, 0 input solution and 0 state solutions are separately shown like this. Now, we would also like to keep in mind that this summation extends from k equals 0 upto n minus 1. Therefore, this particular term comes into field for n equals 1 onwards there is no meaning for this when n equals 0 then because the upper limit is equal to minus 1. You are starting from 0 and pursuing the positive direction. Therefore, that particular summation or the convolution summation is valid only for n greater than or equal to 1. It is certainly does not apply for n equals 0 same thing here also.

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E-VARIABLE METHODS for $n \geq 1$

$$Y(n) = C \phi(n) X(0) + C \phi(n-1) B u(n) + D u(n)$$

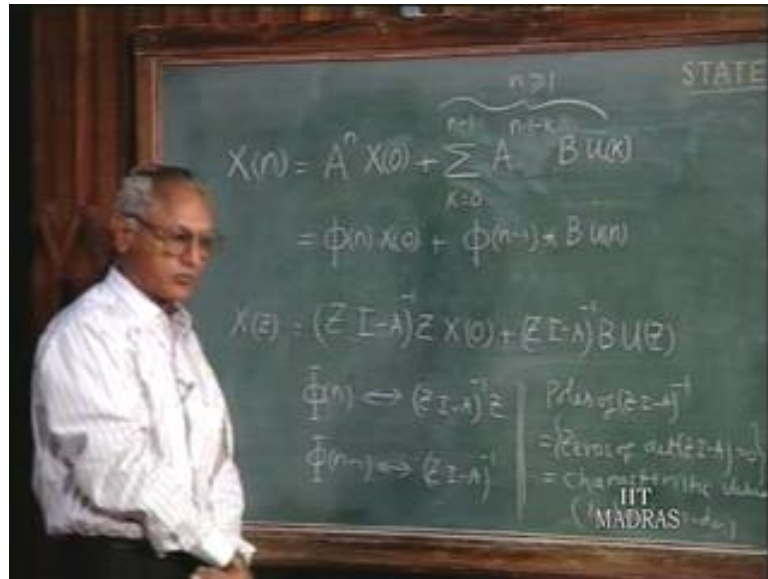
$$= C \phi(n) X(0) + \sum_{k=0}^{n-1} C \phi(n-k-1) B u(k) + D u(n)$$

$$Y(z) = \underbrace{C(ZI - A)^{-1} Z X(0)}_{Y_0(z)} + \underbrace{C(ZI - A)^{-1} B U(z) + D U(z)}_{Y_1(z)}$$

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N is greater than or equal to 1.

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Now, suppose the poles of we saw that the poles of $Z I - A$ inverse or λ . Therefore, coming from this you take the inverse Z transform of this coming from this you get assuming that all poles of $Z I - A$ inverse or simple you have λ i power n coming from this. And here, your poles arrives in not only from $Z I - A$ inverse but, also uZ uZ also have a poles. Therefore, you have $b_i \lambda^i$ inverse n summed n plus c_i suppose the poles of uZ the forcing function of p_i you have p_i i raised to the power of n .

These are poles of u of Z . So, in other words this group of terms these 2 groups of terms will represent the natural response and λ I said to be a natural modes natural frequencies. But better call them natural modes. And this is the forced part of the response this is the forced part of the response. So, that is how we identify the 2 parts in the time domain solution this is natural response that is the forced response.

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VARIABLE METHODS for $n \geq 1$

$$Y(n) = C \phi(n) X(0) + \sum_{k=0}^{n-1} C \phi(n-k-1) B U(k) + D U(n)$$

$$Y(z) = C (zI - A)^{-1} z X(0) + C (zI - A)^{-1} B U(z) + D U(z)$$

$$X(n) = \underbrace{\sum \alpha_i (n_i)^n}_{\text{Natural Response}} + \underbrace{\sum \beta_j (n_j)^n}_{\text{Fixed Response}} + \sum C_i (P_i)^n$$

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Terms coming from the poles of $Z I$ minus A inverse or the natural response terms and those which are coming from the poles of u of Z or the forced response terms. We can also see immediately that the initial conditions are 0 this is total response. And therefore, the Laplace transform the Z transform of the output to the Z transform of the input which is called the transfer function is given by $C Z I$ minus A inverse B plus D .

So, h of Z so transfer function matrix in general if you have n outputs and k inputs it transfer function matrix $H_{ij} Z$ is given by $C Z I$ minus A inverse B plus D that is the transfer function matrix. Where $h_{ij} n$ is the impulse response the inverse Z transform of this will give a matrix in which the ij entry gives the impulse response at the i th output j to impulse present at the j th input.

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Transfer function Matrix

$$[H_{ij}(z)] = C(zI - A)^{-1}B + D$$

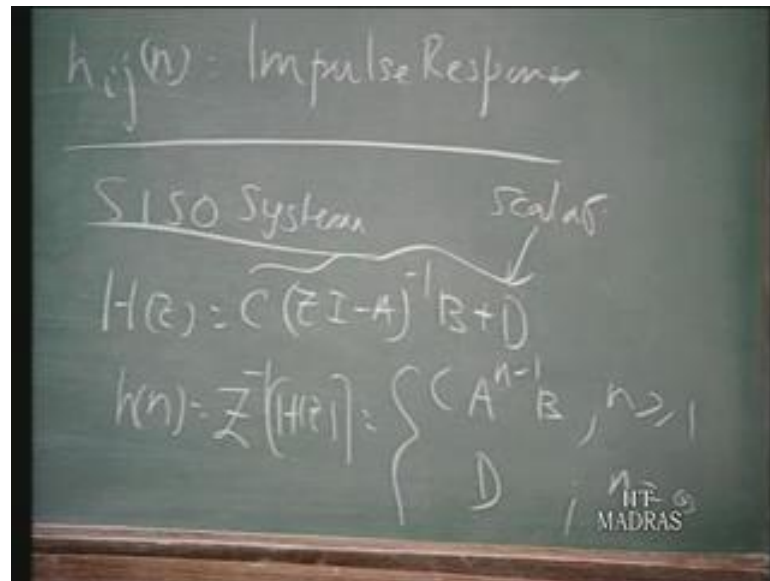
$h_{ij}(n)$: Impulse Response

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So if you have a single inputs single output system if I have a single H of Z itself C Z I minus A inverse B plus D which is a scalar this is whole thing is a scalar its not a matrix. And h_n is the inverse Z transform of H of Z which from this analysis can be showing to be C Aⁿ minus 1 times B for n greater than or equal to 1 and D for n equals 0. So, taking the inverse Z transform of that at n equals 0 this is only term that is present. So, the impulse response means YZ will be C Z I minus A inverse B plus D times 1. Because Z transform of the impulse is equal to 1.

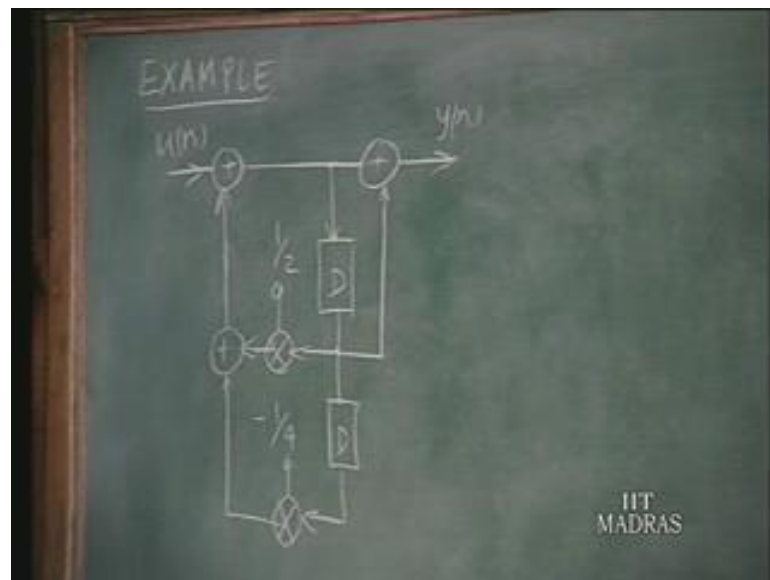
Therefore, the inverse Z transform D will have value at n equals 0 that the value of the D and for values of n larger than 0 it will be the inverse Z transform of this C Aⁿ minus 1 B that is for n greater than or equal to 1. That then is a summary of the result that we have obtained from the time domain analysis as for the Z transform analysis of the discrete time systems. And we would have seen now how the analysis follows exactly in the same lines of the continuous time systems. Except that, we are using now the Z transform in the Laplace instead the Laplace transforms.

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Let us now work out an example. This is the model of discrete time system we like to analyze based on the theory that we have developed.

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So, to set up this state and output equation we will consider the state variables to be those which are at the outputs of the delay units. So, we will call that $x_1[n]$ the signal here $x_2[n]$ will call the signal here $x_2[n]$. So, once you are identified the 2 state variables then the state equations can be written $x_1[n+1] = x_2[n]$ and $x_2[n+1] = \frac{1}{2}x_1[n] - \frac{1}{4}x_2[n]$. We can easily see that $x_2[n+1]$ is $x_1[n]$ of n itself. So, $x_2[n+1]$ is

$x_1(n)$ itself. $1 \cdot 0 \cdot u(n)$ that is the state equation. The output equation $y(n)$ is obtained that $x_1(n)$ plus this signal which is $x_1(n) + 1$.

So, we have an equation per $x_1(n+1)$ to that we have to add $x_1(n)$. Therefore, it turns out to be $\frac{3}{2}x_1(n) - \frac{1}{4}x_2(n) + u(n)$.

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The image shows a chalkboard with two equations. The first equation is the state transition equation:

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n)$$

The second equation is the output equation:

$$y(n) = \begin{bmatrix} \frac{3}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u(n)$$

The chalkboard also has a small diagram at the top left showing a signal $u(n)$ entering a system, and the output $y(n)$ is shown. The IIT Madras logo is visible in the bottom right corner.

That is the this at the state and output equation for the systems. Now, we would like to find out the state transition matrix for this. So, $\lambda I - A$ determinant equals 0 it turns out this will be equal to $\lambda^2 - \frac{3}{2}\lambda + \frac{1}{4} = 0$. This is the characteristic equation. So, the 2 characteristic values λ_1 and λ_2 can be shown to be $\frac{3}{4} \pm j\frac{\sqrt{3}}{4}$.

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The image shows a chalkboard with handwritten mathematical work. At the top, it says "E - VARIABLE". Below that, "Ch. Eqn" is written. The characteristic equation is given as $|\lambda I - A| = \lambda^2 - \frac{\lambda}{2} + \frac{1}{4} = 0$. The eigenvalues are then found as $\lambda_1, \lambda_2 = \frac{1}{2} e^{\pm j\pi/3}$. In the bottom right corner, there is a logo for "IIT MADRAS".

It is the complex Eigen values that we get here. So, we can expect oscillatory behavior in the response. And An the state transition matrix for the discrete time system because the second order system this is equal to polynomial of order 1. So, $C \text{ not } I \text{ plus } C \text{ } 1 \text{ } A$. So, taking the value of matrix A into account this can be put as $C \text{ not plus } C \text{ } 1 \text{ upon } 2 \text{ minus } 1 \text{ quarter } C \text{ } 1 \text{ } C \text{ } 1 \text{ } C \text{ not}$. And we know that the corresponding scalar equation λ^n equals $C \text{ not plus } C \text{ } 1 \text{ } \lambda$ is too far λ equals λ_1 and λ_2 these 2 values that we are getting here.

Solving for this you can show that $C \text{ not}$ equals half raised to the power of n cosine $n \pi$ upon 3 minus 1 by root 3 sin $n \pi$ upon 3 that is $C \text{ not}$. And the $C \text{ } 1$ happens to be 4 upon root 3 half raised to the power of n sin $n \pi$ upon 3. These are the values of $C \text{ not}$ and $C \text{ } 1$.

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Handwritten mathematical derivation on a chalkboard:

$$A^n = C_0 I + C_1 A = \begin{bmatrix} C_0 + \frac{C_1}{2} & -\frac{1}{\sqrt{3}} C_1 \\ C_1 & C_0 \end{bmatrix}$$

$$\lambda^n = C_0 + C_1 \lambda \quad \text{for } \lambda = \lambda_1, \lambda_2$$

$$C_0 = \left(\frac{1}{2}\right)^n \left[\cos \frac{n\pi}{3} - \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} \right]$$

$$C_1 = \frac{4}{\sqrt{3}} \left(\frac{1}{2}\right)^n \sin \frac{n\pi}{3}$$

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Therefore, you substitute C_0 and C_1 into A^n you get the state transition matrix A^n it turns out to be half raised to the power of n $\cos \frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$ plus $\frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$ minus $\frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$ and here $\cos \frac{n\pi}{3}$ by $\frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$. That then is the state transition matrix.

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Handwritten mathematical derivation on a chalkboard:

$$A^n = \left(\frac{1}{2}\right)^n \begin{bmatrix} \cos \frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} & -\frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} \\ \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} & \cos \frac{n\pi}{3} - \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} \end{bmatrix}$$

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Now, suppose I want to find the impulse response h_n . It is as we already observed this is equal to D for n equals 0 and $C A^{n-1} B$ for n greater than or equal to 1. So, using the value of A^n so that means instead of n you must put A^{n-1} substitute over here

and pre multiply by C matrix and post multiply by B matrix. This is the B matrix and this is the C matrix. You carry out the appropriate multiplication and we can show that this is equal to 1 for n equals 0 and half n cos n pi upon 3 plus 5 by root 3 sin n pi upon 3. That then is the impulse response.

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$$\frac{1}{\sqrt{3}} \sin \frac{n\pi}{3} \quad \cos \frac{n\pi}{3} - \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$$

Impulse Response

$$y(n) = \begin{cases} D & n=0 \\ C A^{n-1} / B & n \geq 1 \end{cases} = \begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^n \left[\cos \frac{n\pi}{3} + \frac{5}{\sqrt{3}} \sin \frac{n\pi}{3} \right] & n \geq 1 \end{cases}$$

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Transfer function upon single input single output system so transfer function H of Z C Z I minus A inverse B plus D. And Z I minus A inverse happens to be I will write it here. Because, that is required for to calculate this Z I minus A inverse happens to be Z minus 1 quarter 1 Z minus half that is the matrix derived by Z squared minus Z upon 2 plus 1 fourth. So, this is Z I minus A inverse.

Observe that, the poles of Z I minus A inverse the denominator Z squared minus Z upon 2 plus 1 fourth is the same as what we have earlier written down as the characteristic equation with the substitution lambda for Z.

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LE METHODS

$$(zI - A) = \begin{bmatrix} z & -\frac{1}{4} \\ 1 & z - \frac{1}{2} \end{bmatrix}$$

$$z^2 - \frac{z}{2} + \frac{1}{4}$$

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So, this after simplify you get Z squared plus Z divided by Z squared minus Z by 2 plus 1 fourth. That then is the transfer function you can also obtain this transfer function as an alternative by taking the Z transform of this. The Z transform of the impulse response must be the transfer function. So, hence you know the impulse response you can take the Z transform of that. Because, the same result but, what we have done is we are derived it directly from the formula both are equal.

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Impulse Response

$$h(n) = \begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^n & n \geq 1 \end{cases}$$

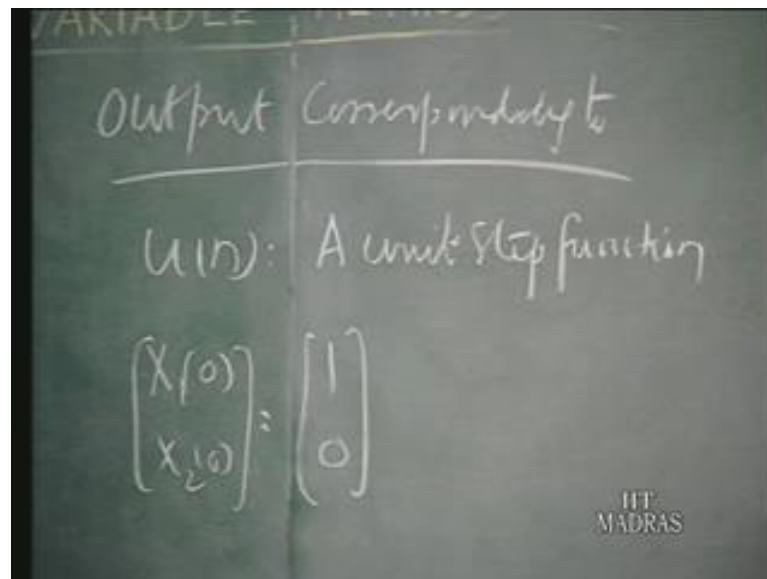
TRANSFER FUNCTION

$$H(z) = C(zI - A)^{-1} B + D = \frac{z^2}{z^2 - \frac{z}{2} + \frac{1}{4}}$$

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Now, let us proceed further; suppose, you are asked find out the output corresponding to specified input in the same problem. So, let us say that you are given a input which is a unit step function that u_n is the a input step function and you are given initially condition regarding the X_1 and X_2 . So, we have to find out output corresponding to the input u_n is a unit step function and the initial values $X_1(0)$ and $X_2(0)$ are given as let us say 1 and 0.

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Then you can calculate the output in 2 steps the 0 state solution and 0 input solution. $Y(0)$ of n equals $C A^n X(0)$ and by working out we have already got the expression for A of n and $X(0)$. Therefore, this turns out to be $\frac{1}{2} e^{-n/3} \cos n \pi / 3$ plus $\frac{1}{2\sqrt{3}} e^{-n/3} \sin n \pi / 3$. That then is the 0 input solutions.

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To get the 0 state solution we have $y_{o_i}(n) = CA^{n-1} B u_k$ plus summation from k equals 0 to n minus 1 of $CA^{n-1-k} B u_k$. Now, since the input is the unit step this is equal to 1. So, consequently we have to see how to get this summation. So, CA^{n-1-k} this should be $n-1-k$. Because the convolution you have $n-1-k$ u_k . So, $C A^{n-1-k} B$ times B from the values that we have got here can be shown to be half of $n-1-k$ times 3 halves of $\cos(n-1-k)$ times π upon 3 plus 1 by $2\sqrt{3}$ $\sin(n-1-k)$ by 3 times π .

Now, we would like to sum of such terms to enable this summation to be effected it is convenient for us to identified to write let e^{a} be equal to half. If you do that then we can say cosine of $n-1-k$ by 3 times π this term times half of $n-1-k$. That can be associated that can be part of as the real part of $e^{(n-1-k) a + j \pi / 3}$.

Because, cosine $n-1-k$ by 3 times π is a real part of $e^{(n-1-k) a + j \pi / 3}$. And this is of course, e^{a} times $e^{j \pi / 3}$ power $n-1-k$ by this identification. So, now this is an exponential term and therefore, it can be simplified it can be summed up. Because this is a geometric series its obtained. And similarly, we have $\sin(n-1-k)$ by 3 times π multiplied by half of $n-1-k$ can be part of as the imaginary part of a similar expression. So, both these now

this terms can now be summed up because after all C A power n minus 1 minus k times B this term here.

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METHODS

$$y_{os}(n) = D u(n) + \sum_{k=0}^{n-1} C A^{n-1-k} B u(k)$$

$$C A^{n-1-k} B = \left(\frac{1}{2}\right)^{n-1-k} \left[\frac{3}{2} \cos\left(\frac{n-1-k}{3}\pi\right) - \frac{1}{2\sqrt{3}} \sin\left(\frac{n-1-k}{3}\pi\right) \right]$$

Let $e^{j\alpha} = \frac{1}{2}$

$$\text{Then } \cos\left(\frac{n-1-k}{3}\pi\right) = \text{Re} \left[e^{j\left(\frac{n-1-k}{3}\pi\right)} \right]$$

$$\sin\left(\frac{n-1-k}{3}\pi\right) = \text{Im} \left[e^{j\left(\frac{n-1-k}{3}\pi\right)} \right]$$

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Therefore, this summation can be effected and finally, you can show that the 0 state solution for n is 8 upon 3 plus half raised to the power of n multiplied by 1 by root 3 sin n pi by 3 minus 5 upon 3 cos n pi upon 3.

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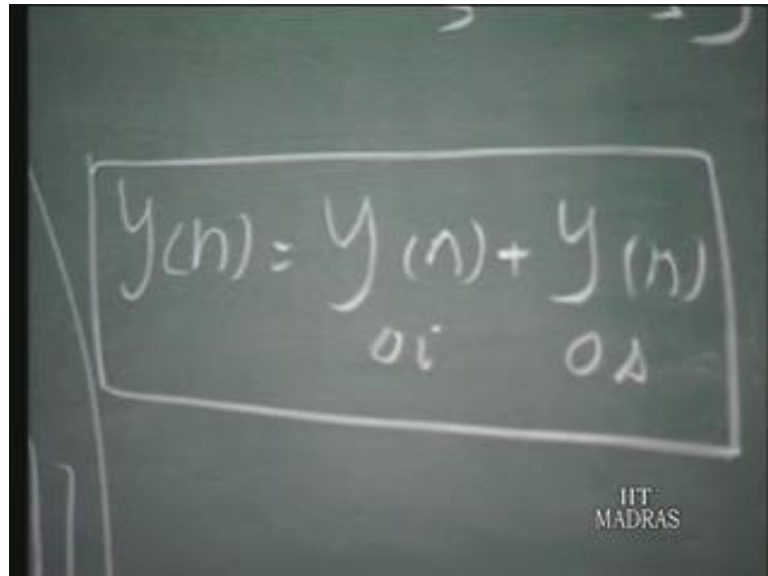
$$y_{os}(n) = \frac{8}{3} + \left(\frac{1}{2}\right)^n \left[\frac{1}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right) - \frac{5}{3} \cos\left(\frac{n\pi}{3}\right) \right]$$

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That is the final solution for the 0 state output 0 state y os n. We already found out the 0 input solution and therefore, the final solution yn would be the sum of y 0 input solution

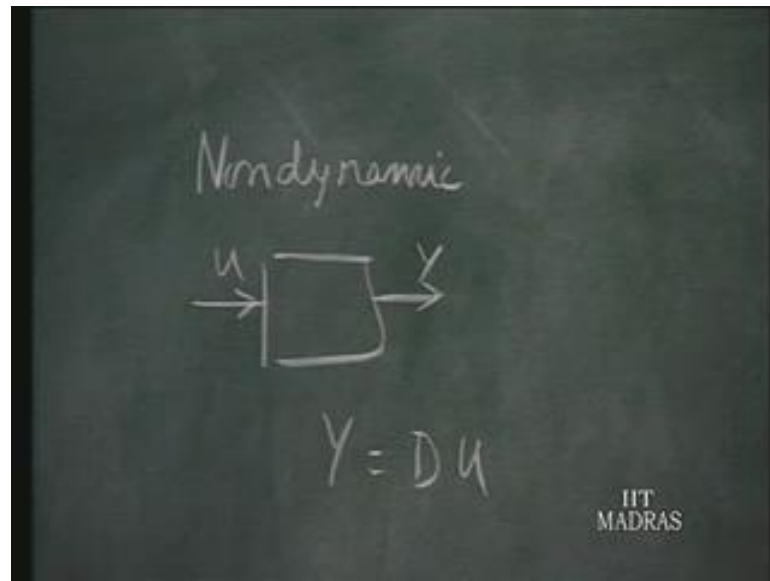
plus the 0 state solution. The 0 state solution what we have here at the 0 input solution is what we have obtained. The sum of this is the sum of this the total solution.

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$$y(n) = y_{0i}(n) + y_{0d}(n)$$

Let us now quickly review what we have done on to the title state variable methods. We define this state of the system has the minimum amount of information relating to the past that we need to know to find out the output corresponding to input prescribed from that point onwards. Now, if you take a non dynamic system if you have a input vector and this is the output Y is related to the input directly without any calculus involved it is a proportionality the matrix Y is related to the matrix u by D.

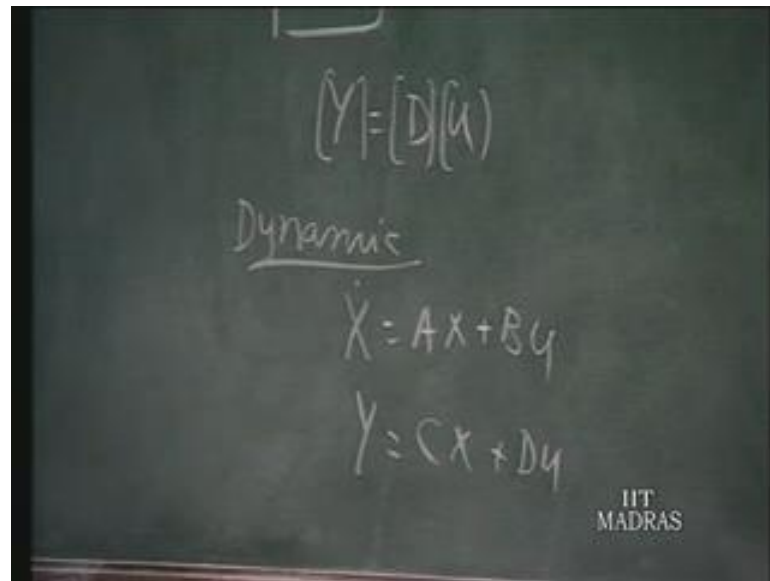
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It is a non dynamic situation. It is resistive network we need to totally to know about the initial conditions and so on and so forth. But, on the other hand if you have a dynamic system you have you have intermediate variable \dot{X} equals AX plus Bu where X is the state and then Y is related to X and u in a non dynamic fashion. So, this can be put in a pictorial way suppose you have this is the input u you have y .

So, you have multiplying by a matrix D you get 1 component of the output Du . And then you have \dot{X} and X this is the state vector this is the derivative of state vector \dot{X} equals AX plus Bu .

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$$Y = [D]U$$

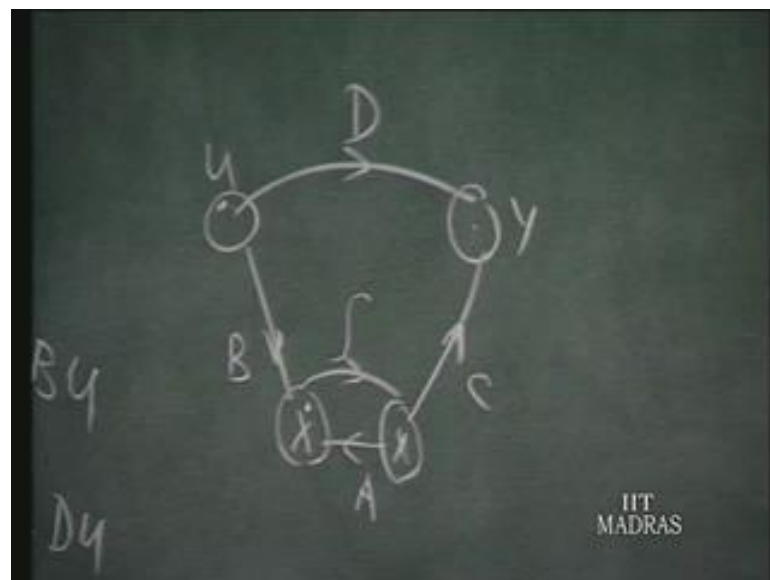
Dynamic

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

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Then X is related to \dot{X} by integral of the state vector and the output is related to X by a matrix C is CX plus Du . This is the representation that you get for a dynamic system. In a non dynamic system this thing is absent you have a direct relation between u and y .

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We saw how the state variable can be chosen in both the continuous time system network and a discrete time network. We have to choose the capacitor voltage and inductor current of the state variables in the continuous time system. And the outputs are delayed units of the state variables in the discrete time system by knowing this is a unique choice

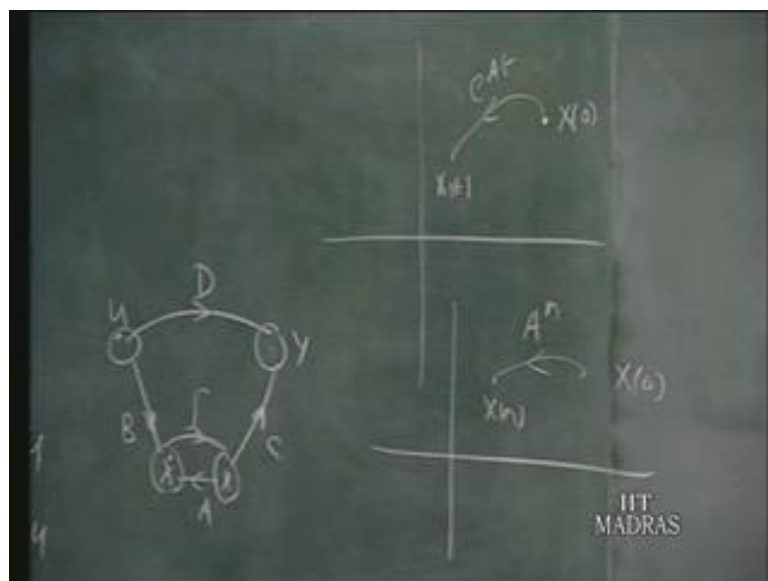
you can have other possible choices but, these are convenient choices. But, whatever the choice may be the number of state variables for a given system will remain the same.

If it is by this choice it is n any other choice also you should have the same number. And then we also talked about state transition matrix which is a very important component to the whole analyses. If you have any state is represented by a point in the n dimensional space suppose it is dimension of the state variable is n . You can think of n dimensional space and the state is represented by a point at each state is represented by a point. Suppose, this is the initial state X_0 and you want to know what is the state at point t then you multiply by e power At in the continuous time domain.

It shows when you multiply X_0 by e to the power of t at different values of t that provides you the locus of the states to be that the state will take in a n dimensional space in a continuous time system. In the discrete time system again X_0 is the initial state if you want to find out the state at X_n . You have to multiply by A power n A power n is that state transition matrix in the discrete time situation.

So, these are the concepts which we pay whiter role in the state phase analysis. And we will notice that the state variable the state transition matrix in each case can be expressed that the sum of n squared matrices n where, that polynomial is obtained by using the Cayley Hamilton theorem.

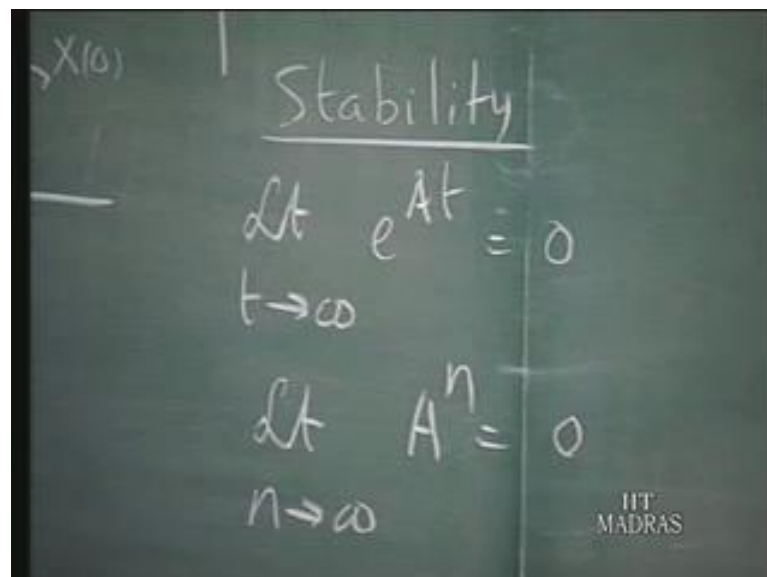
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In each case, if the dimension of the state is n each of them can be expressed that is the sum of n squared matrices. Point up to stability we can say that suppose we have a linear system this is the 0 point in the stable system this is equilibrium point. Suppose, you have a pairs of initial state by giving some energy to the storage elements or whatever it is. And you that of that it must be return back to this point. Suppose, you initial disturb the system for the equilibrium point must return to this point. And therefore, as a consequently of that turns out that stability requirement of stability are limit as t goes to infinity of e power At must go to 0.

Similarly, limit as n goes to infinity of A^n must reverse to 0. The consequence of this is that in the continuous time situation the Eigen values of A must have negative real parts. So, that the transient these natural response the case with time but, the discrete time case the Eigen values of A must have magnitude less than 1. That means, the so that as n goes to infinity the natural response goes down to 0. So, that is the consequent of this you recall that when we talked about discrete time transfer function. We said per stability the poles must be within the units circle and that is the consequent of the fact the Eigen values of A must have magnitude less than 1 for stability to be ensured.

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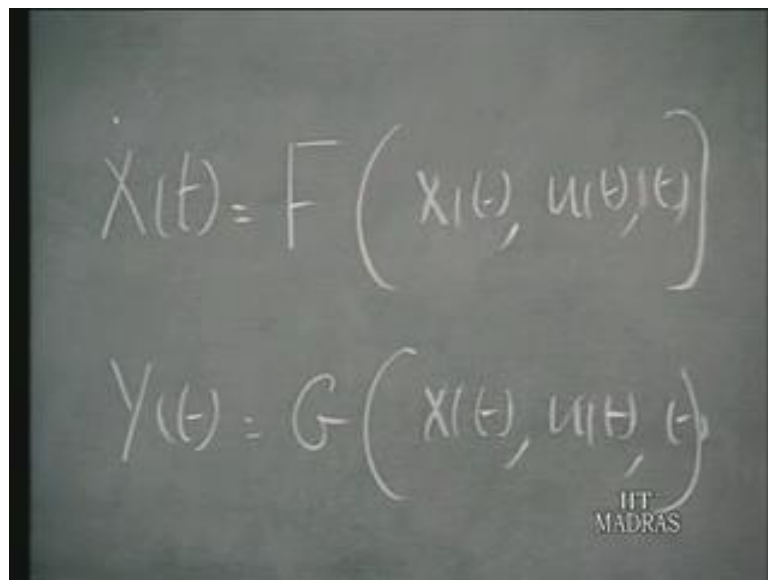
So, in this lecture we in this what discussion that we have studied the we have elementary treatment of state variable techniques what we have studied. We have studied the what is meant by 0 state response and the 0 input response and how to calculate

them. What associated with the state variable theory or concepts of controllability and observability which are important modern control theory. But, this is outside the preview of this course.

The beauty of this state space analysis is that it can be readily extended to non-linear systems. In a non-linear system the form of the equation will be for continuous time situation the F of some function X of t , u of t and t . These are the 3 variables some complicated function that will be the state equation and the output equation Y_t will be some another function of X_t , u_t and t . So, that means you use this to find out the state and once you get the state you substitute here and you get the output.

So, the general framework will remain the same of course, though it will not be linear equation the computer may be necessary to calculate this but, the state variable technique provides you a general framework you studied both non-linear and time varying systems. In time varying systems it turns out the DA matrices are not constants but, function of time.

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The image shows a chalkboard with two equations written in white chalk. The first equation is $\dot{X}(t) = F(X(t), u(t), t)$ and the second equation is $Y(t) = G(X(t), u(t), t)$. In the bottom right corner of the chalkboard, there is a logo for IIT MADRAS.

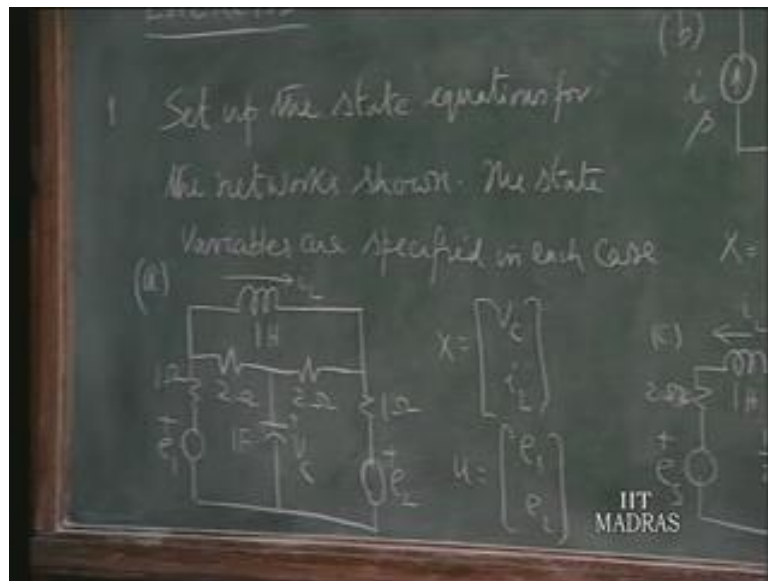
So, in the state variable theory suppose I to say play a very important role is system dynamics and particularly for computer analysis of the dynamic or systems. Because the first order difference equations and first order differential equation are easy to program in the computer. And they provide a framework for study of general systems.

And in control modern control theory the state variables techniques play a very important role and with that we conclude our discussion of the state variable techniques.

Now I will give you a set of examples forming the part of exercise and the topics that you have studied. State the first solve the setup the state equations for the network shown. The state variables are specified in each case. You are given 3 networks ABC this is the first network in which you have 2 sources they constitute the input vector e_1 e_2 . And you have 2 reactive elements the capacitor voltage V_c and the inductor current i_L constitute the state vector X .

So you setup this state equation \dot{X} equals AX plus Bu that is all what you are asked to setup you do not have to worry about the output equation. The values parameter values of the network are given. So you setup a state equation for this network identifying this as the state vector and this as the input vector input vector

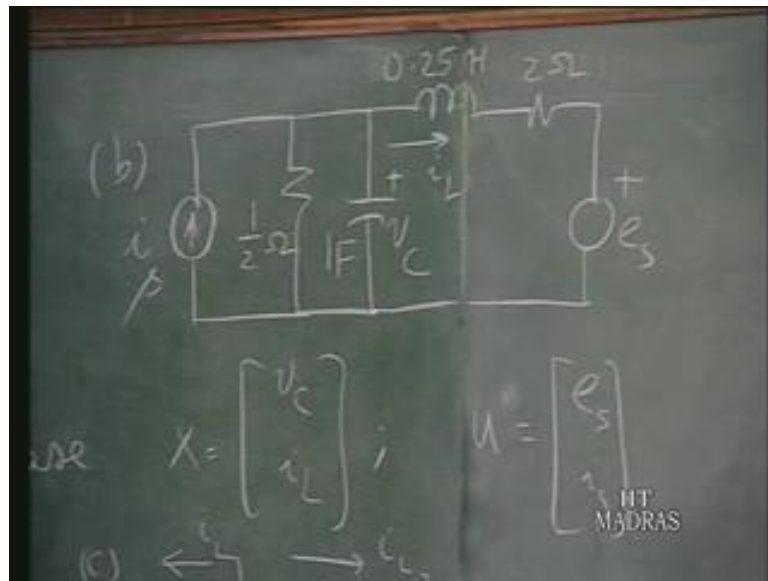
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Then b under the same category we are given another network you have 1 voltage source 1 current source.

They constitute the input vector u and the state variables once again or the capacitor voltage and the inductor current the values of all the elements inside the network are given so you have to find out the state equation in the standard form for this with these the state variables V_c and i_L

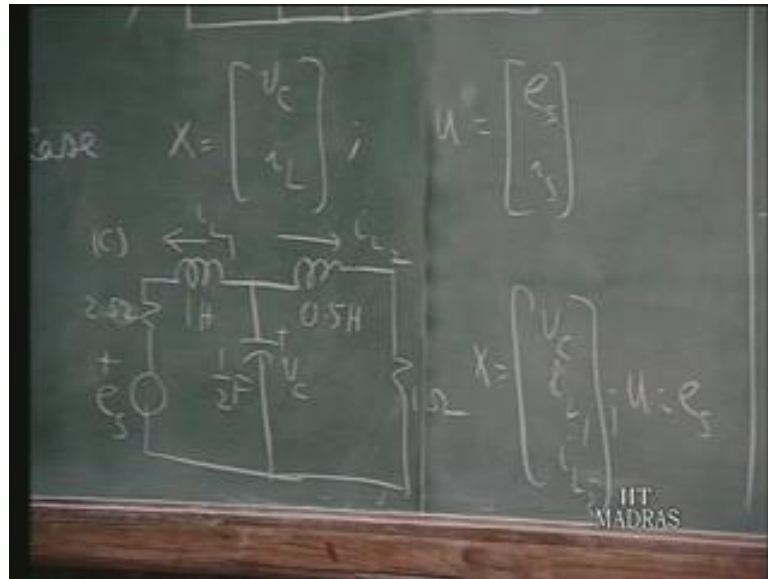
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The third part c again is a network in which there is however there is only 1 source here. Therefore the input is a scalar u equals e_s but, we have 3 reactive elements i_{L1} i_{L2} are the 2 inductor currents and the capacitor voltage V_c these are the state variables V_c i_{L1} and i_{L2} . And the other elements are of course, resistors 2 ohms and 1 ohm.

So once again for this network also you have to setup $\dot{X} = AX + Bu$ the matrix equation relating the state variables the derivatives of the state variables. This is the state variables themselves and the input quantity which happens to be e_s . So this is an exercise in form the state equations you can adopt the standard technique of replace in the inductors by current sources and capacitor by voltage sources and draw the equivalent equation n required equation interms of the state variables.

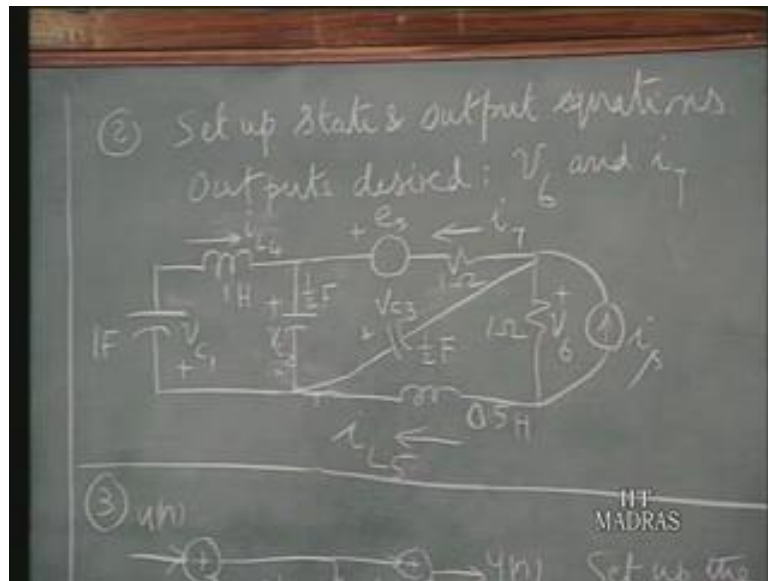
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The second problem setup the state and output equations for the network shown. The output desired are V_6 and i_7 . So you are given slightly more complicated network now. You have e_s and i_s are the input quantities they constitute the input vector u and we have V_{c1} , V_{c2} , V_{c3} , i_{L4} and i_{L5} are the state variables there are five reactive elements

They constitute this state variables and we are asked to find out the output equation taking V_6 and i_7 as the output quantities. The current in that 1 ohm resistance and the voltage across this 1 ohm resistance are the output quantities. So you have to setup $\dot{X} = AX + Bu$ also the output equation $Y = CX + Du$. So this is again a similar X size as before but, here you are asked to find out the output equation as well

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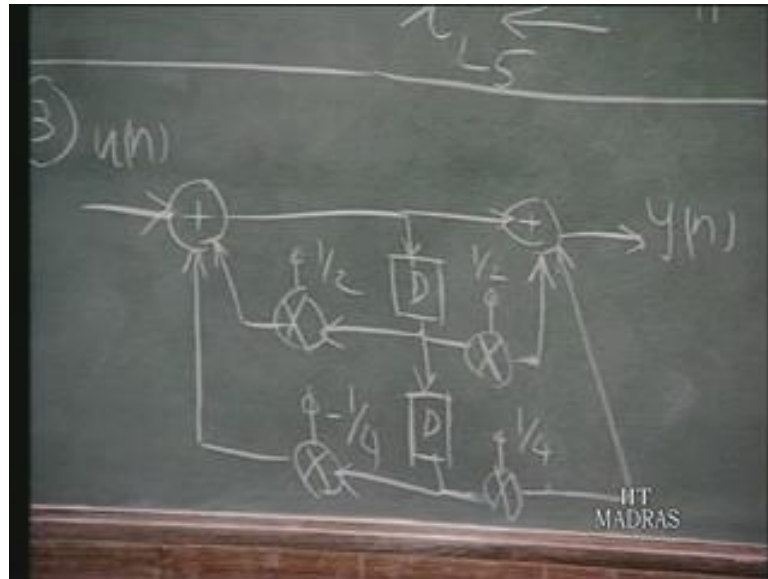
Then go in to discrete time system third problem we are given a discrete time network model in which you have 2 delay elements therefore, this is a second order system and Y_n is the output quantity u_n is the input quantity you are asked to setup the state and output equations for the discrete time system shown on this network.

Now as before we take the output of the delay element this signal here and this signal here as constituting the state vector and once you have got this $X_1(n)$ and $X_2(n)$ $X_1(n) + 1$ is the signal here that can be expressed interms of u_n $X_1(n)$ and $X_2(n)$.

So you got the state equation and the y is the sum of these 3 signals this is of course, $X_1(n)$ multiplied by half this is $X_2(n)$ multiplied by 1 quarter and this signal is a $X_1(n) + 1$ which you have already known.

And summing that up you get an expression for y_n put this n the standard matrix form for both the state equation and the output equation.

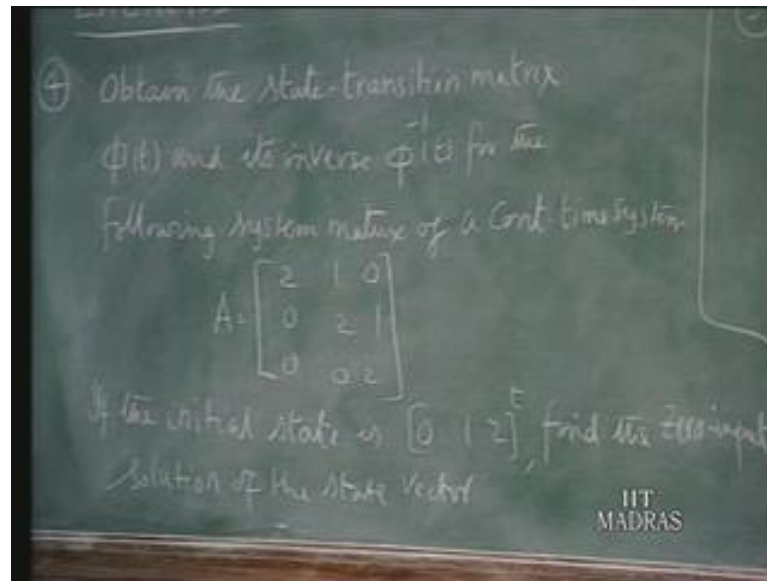
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Let us go on problem number 4 obtain the state transition matrix ϕ of t and its inverse ϕ^{-1} for the following system matrix of a continuous time system. The A matrix is given by 3 by 3 matrix that is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. The A matrix is also referred to the system matrix so this is the system matrix of a continuous time system. And you are asked to find e^{At} which is the state transition matrix.

Another notation for that ϕ of t and also inverse ϕ^{-1} of t . Now if the initial state is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ transports that means instead of writing a column $X \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is $0 \ X \ 1 \ X \ 2 \ 0$ is one. And $X \ 2 \ 0$ is 2 instead of that $X \ 3 \ 0$ is 2 you put this in the form of row and transport that means this is actually a column. That is the initial state find the 0 input solution of the state vector. That means you have to find out e^{At} times $X \ 0$. So you have to find the 0 input solution given the initial state vector and the system matrix here. That is problem number 4.

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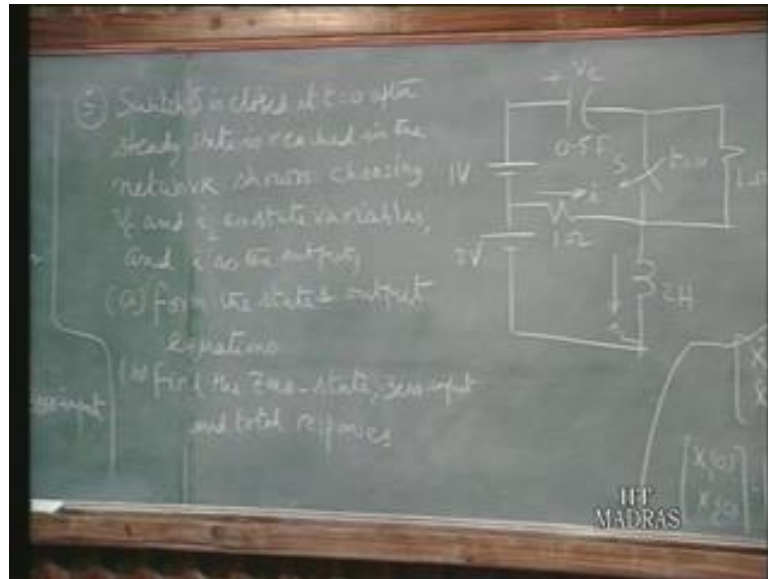


Problem number five you are given a network here to switch s is closed at t equals zero. This is actually closed. After steady state is reached so the switch must kept open for a long time and its closed t equals zero. So switch s is closed at t equals 0 after the steady state is reached in the network shown. Choosing V_c and i_L are state variables and the current i as the output. A form the state and output equations form this equations \dot{X} dot equals AX plus Bu and Y equals CX plus Du .

Find the 0 state 0 input and total responses. So you have to find out the total response we can also find out the 0 state part of the response and 0 input part of the response. So why in this case happens to be this current i this state variables of the capacitor voltage V_c and y the inductor current i_L and the input quantities are the 2 DC sources voltage sources 1 volt and 2 volt respectively.

Now this of course, problem will also involve a second order state matrix A . So you have to find out e power A state transition matrix and from the given conditions of the problem you must find out the initial state vector. That is when the switch is open the final steady state conditions n the circuit will give you an idea V_c 0 and i_L zero. Using that information you must find out the 0 input solution and 0 state solution is in the standard methods.

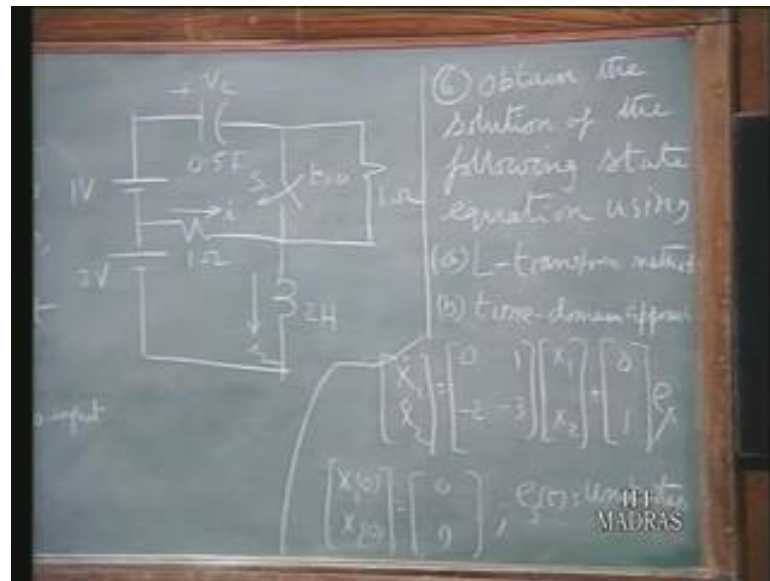
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Last problem sixth problem obtain the solution of the following state equation you are given a state equation using a Laplace transform method being time domain approach. So $\dot{X}_1 = -X_1 + X_2$ and $\dot{X}_2 = X_1 - 2X_2 + 3e^{-t}$ that is the A matrix multiplying X_1 X_2 . The B matrix is $0 \ 1 \ 0 \ 1$ multiplying e^{-t} is a unit step the input in the unit step function.

And the initial state vector is defined by this matrix $X_1(0) = 0$ $X_2(0) = 0$ zero 0 that means it is actually a 0 state solution that we are looking for. You find out the solution by 2 methods. One using the Laplace transform approach then secondly using the time domain approach and compare to solutions and verify that lead to the show in final solution. That is problem number six.

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Question number seven find the transfer function of the continuous time system described by the following equation $\dot{X} = A X + B u$ where $A = \begin{bmatrix} -1 & 1 \\ 2 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $D = 0$. The output equation $Y = C X + D u$.

So this is a second order system it is a continuous time system. And you have a single input and a single output. Therefore you find the transfer function which is a scalar $H(s)$ of the continuous time system which is described by the following equations in a state phase description. This is the A matrix the B matrix this is C matrix D of course, 0

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⑦ Find the transfer function of the Continuous-time system described by the following equations

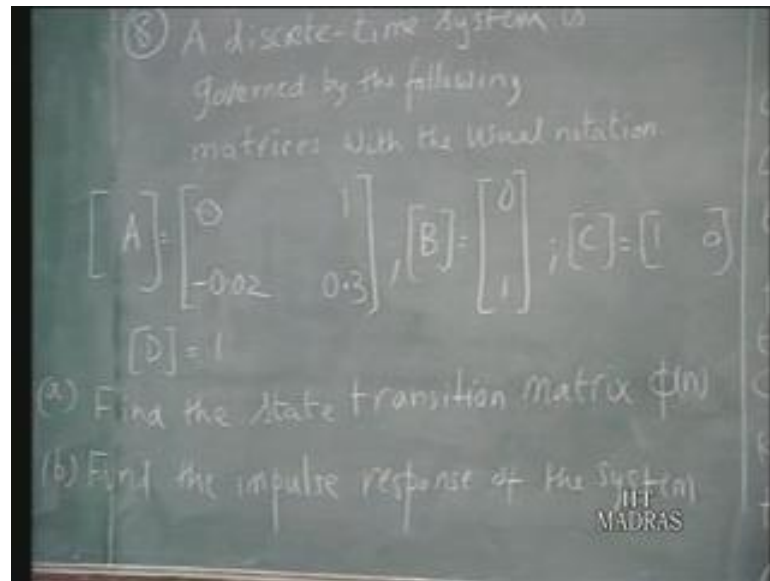
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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So you have to use the straight forward formula C times sI minus A inverse times B and then get this solution for the transfer function. The last problem concerns a discrete time system a discrete time system is a governed by the following matrices with the usual notation in the usual notation. The A matrix is $0 \ 1$ minus “point 0 point 0 two” and plus “point three” that is the A matrix. B matrix is a column with entries 0 at 1 C matrix is a row matrix with entries 1 and zero. D is a scalar 1 find the state transition matrix pie of n now note is that this is the second order system the state vector as a dimension 2 and since Bu is the second term and B has only 1 column that means the input is also single input you are talking about the single input single output system.

Find the impulse response of the system h of n . If it is a multiple input multiple output such situation so get impulse response matrix but, now we are talking about the single impulse response because the single input single output system

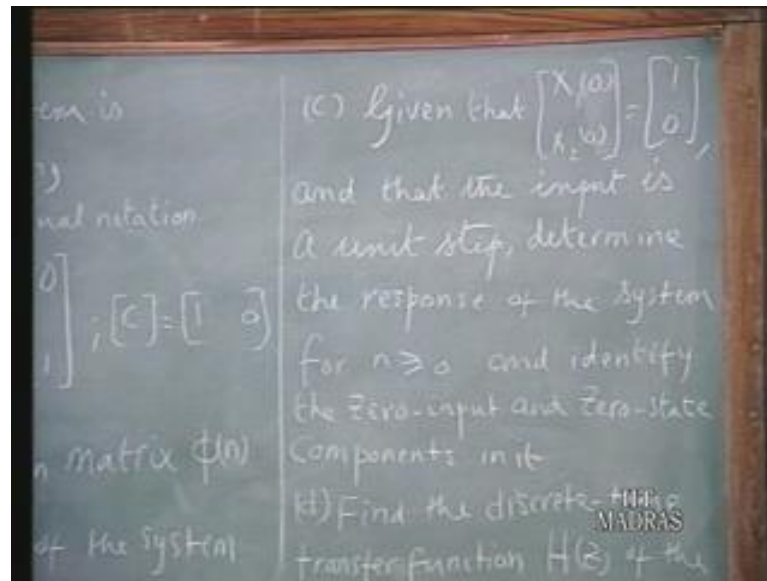
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Part c given that $X(1) = 0$ $X(2) = 0$ that is the initial state and that the input is a unit step. So the input function $u(n)$ equals 1 for all values of n greater than or equal to 0 determine the response of the system for n greater than or equal to 0 and identify the 0 input and the 0 state components in it means in the response. So you have to find out the 0 input response of the system 0 state response of the system add them up you get the total response.

Or if you straight away found the total response find the 2 component separately. You can work this out either in the transform domain or in the time domain it is suggested that you work out in both base and then compare the solutions verify they agree. The last part is find the discrete time transfer function $H(z)$ of Z of this particular system this is a single input single output system its already mentioned. Therefore the transfer function its just scalar in general $H(z)$ would be a transfer function matrix in a general situation that multiple inputs and multiple outputs. In this case it is a single input single output system you are asked to find out the transfer function $H(z)$ in the of the discrete time system.

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And now a final word to all those you stayed with me so far. The material covered under this course entitled network and systems is a very important and useful part of the under that curriculum in a electrical engineering for student specializing is in electrical power or communication engineering. The input and output relations of a system in various descriptions the representation of signals by different means.

The concept of spectrum and harmonic analyses the idea of impulse response the frequency response of a system transfer function or system function as it is otherwise called network function of various kinds the principle of super position and convolution the idea of state the notion of the state and its relation to the input and output. These are all important concepts and we will have a significant presence in the further courses that we take up. So this course will provide you with necessary background to take additional courses in subject like electronic circuits communication theory control theory signal processing in the analog or digital domain.

topics covering aspect of dynamic behavior of power operators and systems instrumentation and the like in all these courses the concept and technique that we have discussed in this present course will be found useful. We have limited our discussion and the main threads of our course is linear time invariant dynamic causal systems threading through all the topics that we have covered here or 2 important principles the super position principle and the concept of convolution.

These are the important concept and we have used them in time and again in our work. When we are discussing the principles of analyses of linear time invariants systems and the techniques for such analysis we have discuss them under the framework of electrical networks There are two reasons for this 1 you will be called upon the analyses of networks is important to its own write.

Because you will be called upon to analyze various kinds of electrical and electronic networks in your further courses. Either because these circuits have been designed and constructed with a specific objective in mind using RLC elements half amp transition and they like control sources that is. Because such circuits or networks are equivalent to representation or equivalent circuits for various electrical operators devices and systems like equivalent circuit of transistor equivalent circuit of transformer whatever it is.

A second occasion for you used the principles of network analyses would be when you are asked to design networks to perform certain functions like filter design design of compensating equalizing networks and so on. These design and synthesis task require as a prerequisite a good graph of the analyses methods.

And therefore, the steady of network analyses methods that we have taken up here would be quite important in all these situations. A certain reason for our studying general system concepts with the framework of networks is because when you studied the general techniques through the familiar and concrete example of a electrical networks it will quick and understanding of the concepts and you will have a formal grip on this techniques.

So that when you want to extend this when you occasion to extend this to other kinds of systems its electro mechanical systems. You will have a greater facility to do this when you want to apply this techniques tool general systems in a wider context. We have discussed continuous time systems and discrete time systems but, you would have noticed that there is a lot of commonality between these two.

Even though the details of different details of different the philosophy of the various techniques the approaches are quite parallel. We talk about impulse response transfer function and so on and so forth almost in the same way it turns of course, of different in 1 case we talk about differential equations and other case we have difference equations.

In 1 case we have functions which have value at every instant of time in the other case we have functions which have values only a discrete instant of time.

There are different of course, but, the the philosophy of the approach to our solutions or essentially the same as would have noticed. Differences of course, arise because 1 is a discrete time situation another is the continuous system among the important differences at you have noticed is instead of convolution integral you have convolution summation in the case of discrete time situation you have the frequency response.

It's a periodic function of the frequency which is not the case in the continuous which is not necessary in the case in the continuous time case. So we find a certain unity in the analyses for that whether you apply them to the continuous time case and discrete time case. The steady of the linear time invariant system that we have taken in this course will be useful to you later on when you take up steady of more advance systems which are neither linear nor time invariant that is when you take up non-linear or time variable systems. Lastly as a followed to this course you would have gained an appreciation of the role of mathematics in the solution of physical problems.

So the principles of linear algebra differential equation and complex variable theory you would have noticed come in very handy n the solution of engineer problems of that type we have taken up. You would also have noticed that to analyze a to solve a particular physical problem there are various approaches are available. You can look at different view points and then take up different approaches to the solution.

Now a mastery of different techniques time to get a mastery of different technique is not a wasteful exercise. Because if you have studied different techniques that will give you a clarity of the behavior of the system and multidimensional sort of depth or inside into the problem. And of course, you can choose the most appropriate effective method in a given context in a given context. This is the feature of the various approaches various method we have studied under the category of network and systems and this is the principle which will find useful in your further work.

Good luck and god speaks to you.