Networks and Systems
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Lecture – 48
State-variable methods (4)
Evaluation of e at
Time domain solution

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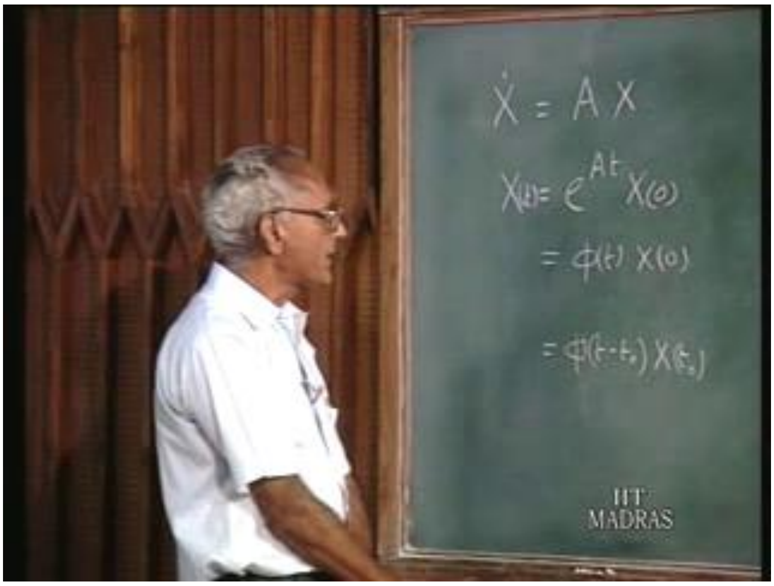

In the last lecture, we were considering the solution of the homogeneous matrix equation of the form X dot equals AX where X and X dot are n vectors column of n entries A X square matrix of order n by n. And we said that e to the power of at is itself a matrix of order square matrix of order n and e to the power of at can be put in a more compact form as pie t X 0: X 0 is the initial state.

Xt is the state at any point of time t and pie t or e to the power of at they are called this state transition matrices this is another way of indicating e to the power of at this state transition matrix or simply the transition matrix. And this first thing we have to do, today this lecture is to find out how to evaluate pie of t for a given a. We have also, said that if the condition such a state is not given at t equals 0, but at t at the point at t equals t not then, we can also write this as pie of pie minus t not times x at a point t not.

This is also another way, or xt can be derived from this initial state x t not instead of at t equals 0. Now, the way to calculate e to the power of at there is several ways of doing this so, we will start some methods of evaluation of e to the power of at you may recall that, in the Laplace transform domain of an equation like this

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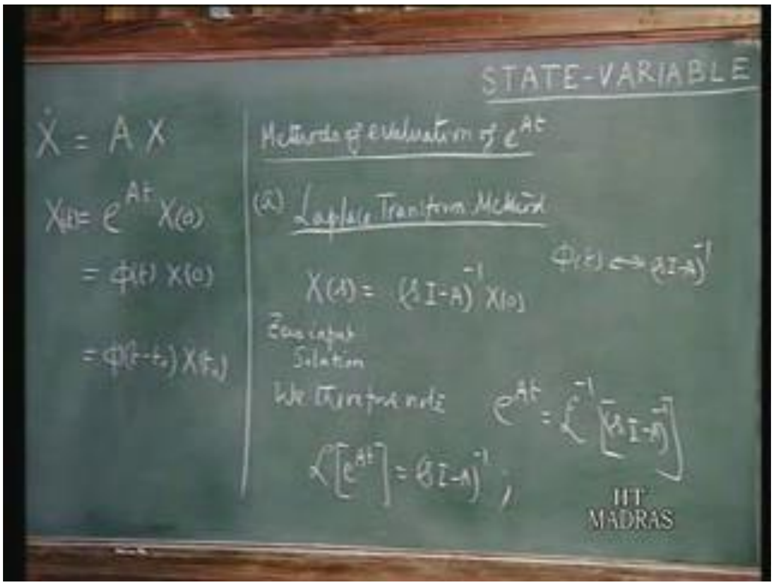

we said the Laplace transform solution of the state equation the Laplace transform of the state Xs recall has been given as S I minus A inverse X0. This is the 0 state 0 input solution, the 0 input solution of the state in Laplace transform domain X of s will be x of s S I minus A inverse X0 this is a constant vector. Therefore, whatever we are having here must be the Laplace transform of this.

Because, we have taken the homogeneous equation is u equals 0 therefore, this is whatever solution you get for this state. If the 0 input solution so, the correspond this is the equation transform domain in time domain the equation is x of t is e to the power of At at 0. So, these 2 corresponds these 2 must be correspond to each other which clearly therefore, if you take the Laplace transform of this x of s.

Find the Laplace transform of this times x of 0 x of 0 is being a constant. Therefore, from these 2 equations we see. We therefore, see that e to the power of At must have a Laplace transform SI minus A inverse. So, e to the power of at Laplace transform of e to the power of AT is SI minus A inverse or you can say SI minus A inverse that is what we said.

Therefore, to get e to the power of AT is inverse Laplace transform of SI minus A inverse. So, 1 way of finding out e to the power of At is to form the matrix SI minus a take it inverse you have the denominator, which is the characteristics polynomial. And each term you take the inverse Laplace transform taking the denominator also, into account and that will form the matrix in time domain which corresponds to e to the power of At or pie t which is the system transition matrix.

Therefore, the Laplace transform domain will see the pie of t and SI minus A inverse form a transform here. This is 1 method of this, but however we would like to work out the solution in time domain and there is not much pointing. Now, shifting the transform domain and coming back to the time domain in the final solution. So, while this is an alternative way of doing this is not a time domain solution.

So, we will just indicate this to show the relation between pie t and SI minus A inverse that will not pursue this at this point of time. The second method is to take the infinite series for e to the power of At.

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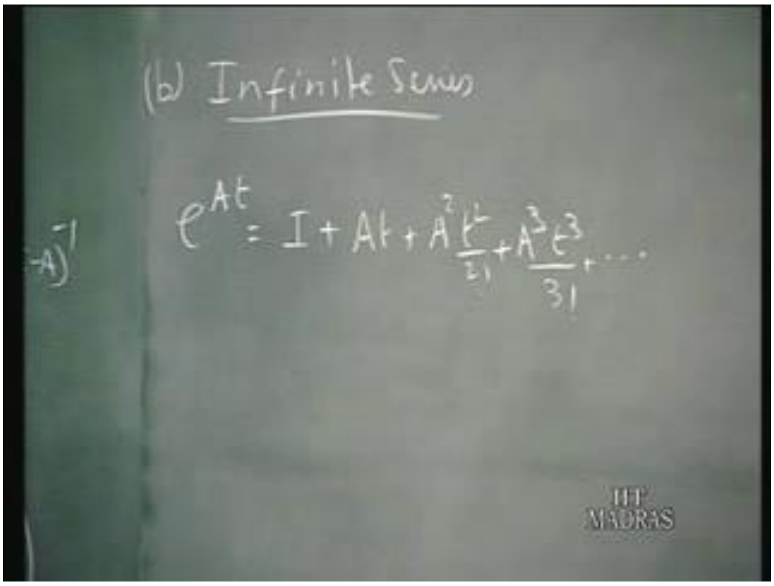

So, we say that e to the power of At is the infinite series I is the unit matrix of dimension n At plus A squared t squared by 2 factorial A cubed t cubed by 3 factorial and so on and so forth. This is the infinite series expansion for e to the power of At therefore, you form A squared A cubed and try to add up this terms. Because, all of them are squared matrices of order n.

There compatible for addition therefore, you can thing of e calculating e to the power of At by taking the infinite series for each 1 of this terms which are present in this matrix, which is present in this matrix. And for the useful systems, where stability is ensured e to the this is the conversion series. Therefore, depending upon the accuracy to need you can go up to a certain number of terms and stop there.

However, this approach will give you the solution perhaps is the very good way for numerical work, but it will not give you a closed form solution and it does not provide you any particular insight into the behavior of e to the power of At. Because, you are getting just the numbers, we can use it for calculation of numbers, but certainly you may not able to put the final solution in the closed form expansion as a function of time.

So, this is 1 time case an infinite series of particular point of time is 1 approach is: a suitable for computer working, for numerical working. But certainly, we will not close form solution we leave at that. Another approach is: To diagonalization of A matrix. It can be shown that, if you pre multiply the A matrix and post multiply by appropriate matrices t is the transformation matrix in order n.

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If you choose a suitable transformation matrix like this. Whatever, a matrix you get can be converted into diagonal form. The diagonal form with the matrix will be you have all the Eigenvalues on the main diagonal and everywhere else the entries are 0. This will be the case when all the Eigenvalues are distinct no 1 is repeated. So, for this distinct Eigen values and then, such is the case it happens the transformation matrix will be a collection of rows each row represents an Eigenvector. So, we have n Eigenvalues and you have n Eigenvectors. And once, you find this out then you can write e to the power of At as T times again a diagonal matrix, where the entries are e to the power of lambda 1t lambda 2t and. so on e to the power of lambda nt is again a diagonal matrix.

Where, the entries of the main diagonal are e raised to the power of lambda 1 lambda i T where lambda is the Eigenvalue times t minus 1. So, that will be the situation when the all the Eigenvalues are distinct. The all the Eigenvalues are distinct and in this case once, you identified the t matrix here you can find calculate e to the power of At by multiplying this diagonal matrix by T in 1 side and T minus 1 in other side and you get this.

Slightly to get slightly, more complicated where some of the Eigenvalues are repeated. But we will not pursue this this particular approach we leave it we will again just mention this, as a possible alternative and we will not pursue this anymore. The particular approach which we will consider is: 1 which using the theorem a linear algebra known as Cayley-Hamilton theorem

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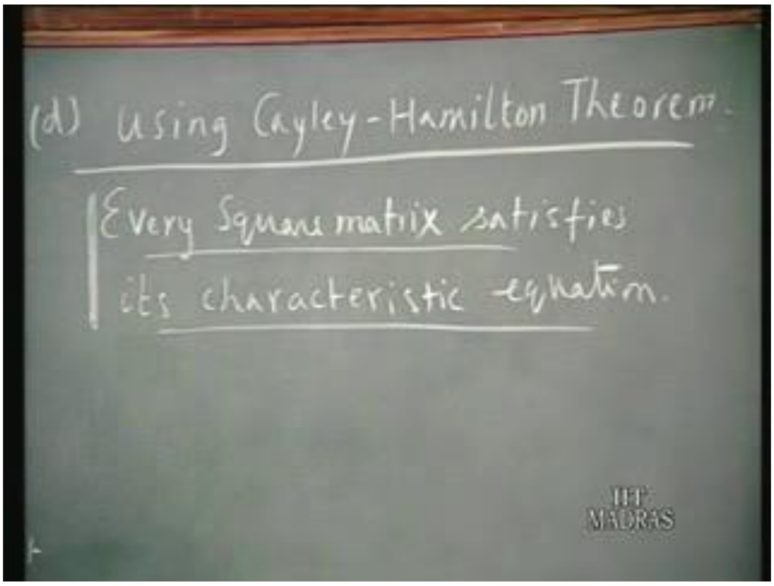

Now, here we make use of this particular theorem which states every square matrix satisfies its characteristics equation this is statement of the theorem. What is that mean?

Suppose you have a matrix A, A squared matrix. Its characteristics equation is of this square matrix is determinant of lambda I minus A. That matrix is equal to 0 which we said you can put it as F lambda equals 0 this is the characteristics equation.

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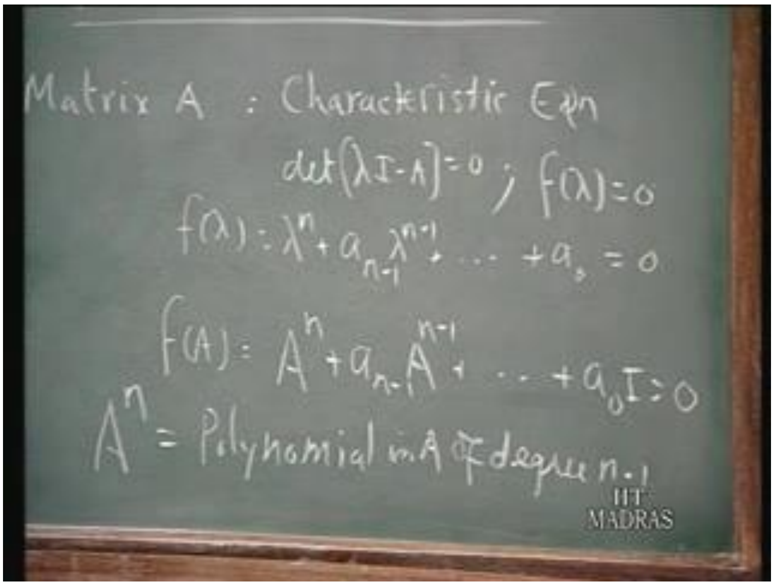

So, the characteristics equation of this matrix A is F lambda equals to 0 where lambda is F lambda is obtained by making the determinant of lambda I minus A. Here i is the unit matrix of order n and if this matrix is order n, this polynomial F lambda is a degree n. So, you have F lambda will be of the form lambda n plus say a n minus 1 lambda n minus 1 on up to a not that is what we are having this is equal to 0 this is the characteristic equation.

Now, what is Cayley-Hamilton theorem? States is if instead of the characteristic equation as a function of lambda if you set up the characteristic equation as a function of the matrix a then, also it is satisfied that means, this theorem states that f as a function of A which means A raised to the power of n plus a n minus 1 A raised to the power of n minus 1 and so on and so forth plus a not I this time equals 0.

So, this is the scalar equation, this is the matrix equation. Here every term it's just a scalar a particular number, here every term it is a matrix a squared matrix of order n A to the power of n a A squared A cubed all this are matrices of order n. And the last term is also a matrix of order n. Because, we are putting the unit matrix and the 0 on the other side is this is also not a scalar, but it is a matrix of 0 a squared matrix of zeros. So, the character a solution of the characteristics equation is lambda a scalar, but instead of lambda a matrix if you substitute A that also satisfies the characteristic equation. That is the purpose of the Cayley-Hamilton theorem.

Now, what is the consequence of this theorem? The consequence of the theorem is that, if you looking at this particular polynomial equal to 0 matrix polynomial A raised to the power of n can be expressed as a polynomial of degree n minus 1. Because, A power n equals minus of a n minus 1 A power n minus 1 and so on and so forth. So that means, A power n can be expressed as a polynomial of polynomial in A of degree n minus 1

So, A power n can be expressed as a polynomial of degree n minus 1. That is not a that is not all suppose, I have a polynomial of higher order. Then, also it can be reduced to a polynomial of degree n minus 1. How it comes about we will see in a moment let us, for a let us consider for instance a polynomial of very high degree a grand polynomial. Let us, say lambda G lambda polynomial of high degree polynomial of large degree.

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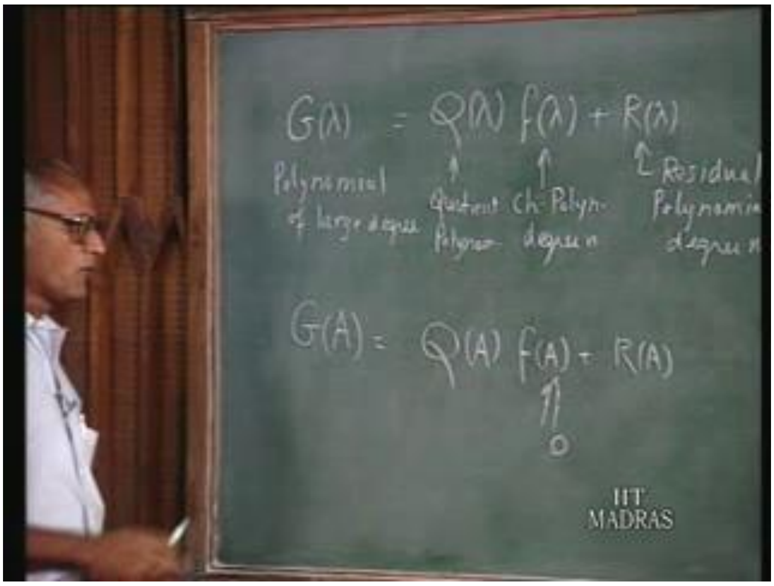

Now, this Polynomial suppose you divide this F lambda, the characteristic Polynomial into G lambda. We can write this as some quotient polynomial times F lambda plus a residual polynomial lambda. Let me, this is a polynomial of very large degree and you divide F lambda into that. Then, you have a quotient a polynomial if this is degree 64 And this is degree 4 then this will be of degree 60. And the residual polynomial is what is that will be if this is of degree n, this will be of degree n minus 1, this is residual polynomial of degree n minus 1 of degree n minus 1.

This is characteristic polynomial degree n. and this we will the quotient polynomial. This is not important for us to know what it is. Now, this is the polynomial in variable scalar variable lambda. Now, the same thing can be deterring in terms of suppose, we have a polynomial same polynomial in A the matrix A. We get similarly, QA FA plus RA where the coefficients in all this polynomials are 1 at the same except that, this is variable lambda you get the matrix A here.

So, whatever you do in the scalar polynomial you can also do in the matrix polynomial. Now, if you look at this form in the equation we know that, every square matrix satisfies its characteristics polynomial, characteristic equation. Therefore, FA happens to be equal to 0 we just now saw if f lambda equals 0 for particular values of lambda FA is 0. Now, the difference is between these 2 is this: F lambda is equal to 0 will be satisfied for particular values of lambda.

So, this particular equation will be satisfied will be satisfied for lambda equals lambda1, lambda2 upto lambda n only for the characteristic values. This is not an identity, this is satisfied for particular values of lambda, the characteristics values or the Eigenvalues. On the other hand, FA equals 0 satisfy its satisfied identically. Once you, plug in the value A into this this will be 0 the matrix A it is equal to 0.

So, whatever a characteristics values satisfies in the in this equation this matrix a satisfies here. So, coming back to this we observed that when you set up GA as in the form QA FA RA this will be identically 0. Which means that, G of A whatever be the degree of this polynomial can be simply written as equivalent to a polynomial of degree n minus 1. You may have G of A a polynomial of degree 1000, but as long as the characteristic polynomial as degree n we can write this as equal to RA which is of degree n minus 1.

Now, how do we? Therefore, whatever be the polynomial we are talking about you can always write this in the form of RA. Where the polynomial it almost degree n minus 1.We can extend this further, we can even write for the exponential this need not be polynomial this accept transfer related function its equivalent to polynomial of an infinite degree even then this is true. Now, how do we calculate therefore, the coefficient in this polynomial RA. Now, for this you look go back to this now this equation if lambda equals lambda1 lambda2. Suppose, you substitute 1 of the characteristic values for lambda then this becomes 0.

That means, G lambda will be equal to R lambda where lambda equals takes any 1 of the characteristics values. Therefore, we can say that G lambda equals R lambda for lambda equals lambda1, lambda2 up to lambda n. Because, F lambda1 is 0, F lambda2 is equal to 0, F lambda3 is equal to 0 this becomes 0 only for values of lambda equals 1 of the characteristics values.

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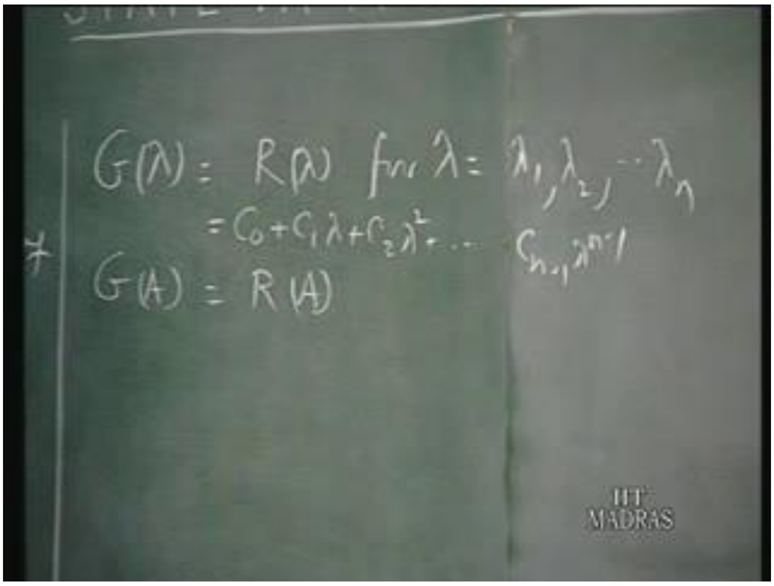

And after all, the form of R lambda is same as form of RA therefore, GA can be written as RA. So, given polynomial G lambda we can substitute lambda equals lambda1, lambda2, lambda3 and evaluate the coefficient of R lambda is a polynomial of degree n minus 1. So, the general form of this will be of the form C not plus C 1 lambda plus C 2 lambda squared up to C n minus 1 lambda n minus 1. So, we have n coefficients the R lambda is defined by n coefficient C not to C n minus 1.

So, given this G lambda you can substitute lambda1 to lambda n and then, you know this function. Therefore, you can evaluate this for different values of lambda you got n equations you can evaluate C1, C2, Cn. And that can be used to form this RA and therefore, GA can be evaluated. That is the approach that we take and whatever, we have done can also be extended to a polynomial at infinite degree in particular if you have e to the power of A. That also can be written as C not plus C1 A plus C2 A squared write Cn minus 1 A n minus 1this is also true.

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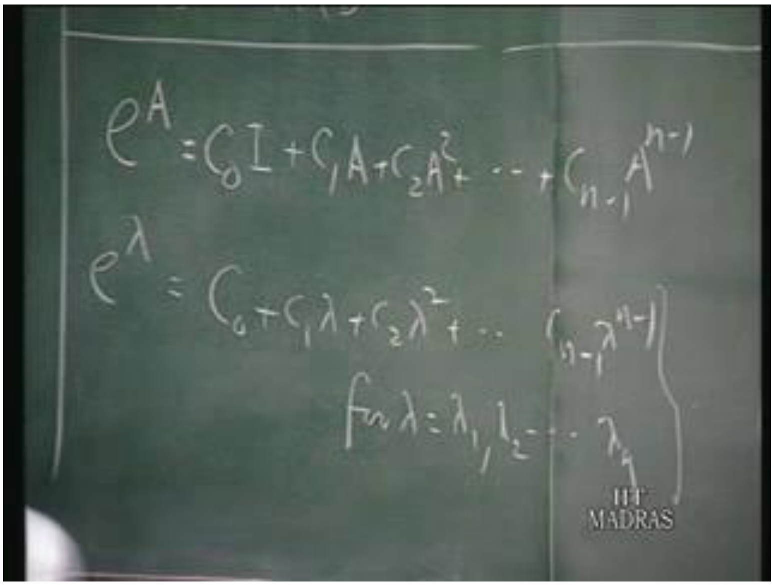

So, we have to evaluate C not C1 C2 Cn minus 1.We do not have to have an infinite series for e to the power of A we do not have to do that. We can express this as a finite series of degree n minus 1. Now, how do we calculate c not C1, C2, Cn? We go back to this if you are taken e to the power of lambda, that will be equal to I am here I must write because this is matrix I must write C not times I.

This is a constant unit matrix so, if you take e to the power of lambda corresponding to G lambda it is equal to C not plus C1 lambda plus C2 lambda squared write up to C n minus 1 lambda n minus 1 for particular with this equation valid for particular values of lambda. Whereas, this equation is valid for identically here, this is valid only for lambda equals lambda1, lambda 2, lambda n.

Because, this equation is valid because only when, F lambda is 0 and that is true only for lambda assumes 1 of this characteristic values. So, the approach is like this: If you want to express e to the power of A by means our finite series like this. You have to evaluate C not C n minus 1 that is A constant you have to evaluate. These constants can be evaluated by going back to the scalar equation substitute in lambda equals lambda1 to lambda n so e to the power of lambda1 equals so much, Lambda2 is equal to so much. You get n equations and you can solve for this n constants and use those constants and this finite series and get this. What we have done is for e to the power of A and e to the power of lambda. But in our system transition matrix, we have e to the power of At so, the only difference is: if you have e to the power of At

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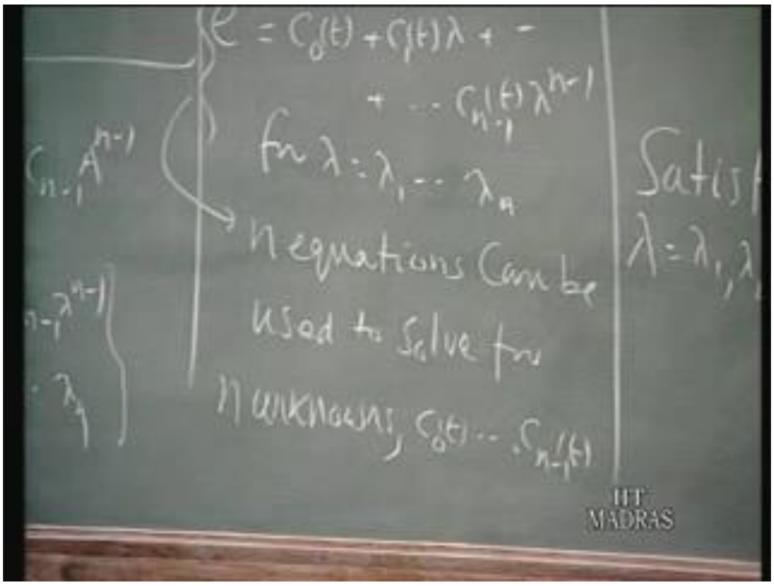

The matrix polynomial that we have to talk about take in to account is C not t I plus C1 tA down the line to C n minus 1 is a function of t A n minus 1.That is the only difference So, instead of being constants here there will be function of t. And to evaluate that, C not to C n minus 1 those values there constant as for A is concerned, but there function of t. So, we take the corresponding scalar equation and write this as equal to C not of t plus C1 t times lambda etcetera C n minus 1 t lambda n minus 1 and this is valid for lambda equals lambda1 to lambda n.

So, we have n equations from this so, this is equivalent to n equations. Because, every time you substitute the particular value of lambda you get 1 equation this is n equations can be used to solve for n unknowns. What are the n unknowns? C not t1 write up to Cn minus 1 t. So, these n unknowns can be used, can be found out from this solution of this equation and they can be used in this expression to evaluate e to the power of At.

In our further work we do not have to explicitly. So, this functional notation C I of t. We know the function of t. Therefore, you may as well simplify or notation write e to the power of At is C not I plus C1 A plus C up to C n minus 1 A n minus 1. Now, an example we will illustrate, the technique that is involved, but the principle. Therefore, is that we make you this Cayley-Hamilton theorem and say that every square matrix is satisfied its own characteristic equation and therefore, any higher power of A even in translate function of A can be essentially reduced to A polynomial in A of degree n minus 1.

And to evaluate the coefficients of that polynomial which is A n coefficients we go back to the scalar equation and that is scalar equation corresponding scalar equation is satisfied for n distinct values of lambda which are the characteristics values. So, you get any equations and used them to solve for C not C1 to C n minus 1. We illustrate, this for this particular situation, but you straight away notice what happens is some of these lambdas are repeated.

Its suppose, lambda1 equals lambda2 then, you do not have n Eigenvalues, but n minus 1 distinct Eigenvalues. How do we tackle those situations? But that we will take up later, but first let us illustrate the this procedure for the case where all the Eigenvalues are distinct. As an example let us, consider the determination of e power At when A is a square matrix minus 1 0 1 minus 2.

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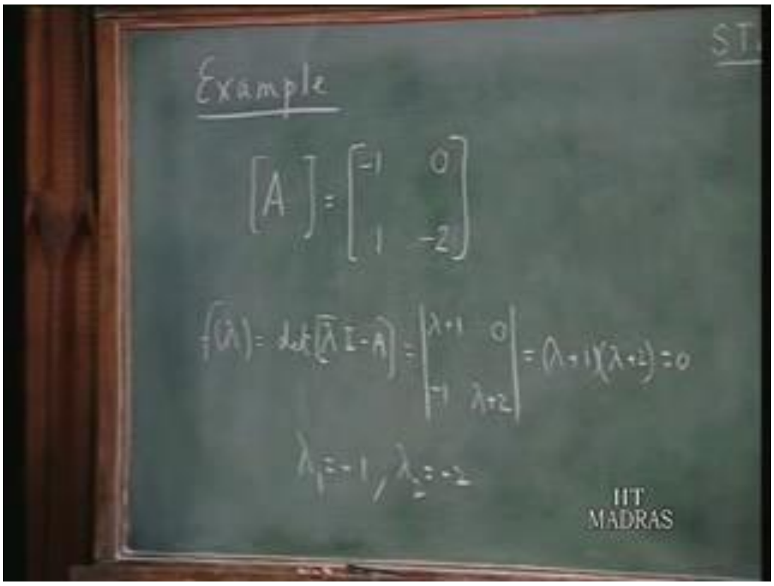

The first find out the Eigenvalues so, to do that we form the characteristic equation. So, F lambda which is equal to the determinant of lambda I minus A which is of course, the determinant form by this matrix is lambda minus A, lambda has diagonal entries lambda and from that you subtract this A matrix. So, this will become lambda plus 1 this is 0 minus 1 lambda plus 2.

So, that is the determinant form by this matrix delta I minus A and that will of course, delta a lambda plus 1 times lambda plus 2 that is equal to 0 that is the characteristic equation. So, the Eigenvalues are 2 Eigenvalues lambda1 is minus 1 and lambda2 equals minus 2. So, there distinct Eigenvalues minus 1 and minus 2 so, we keep that at the back up of our mind.

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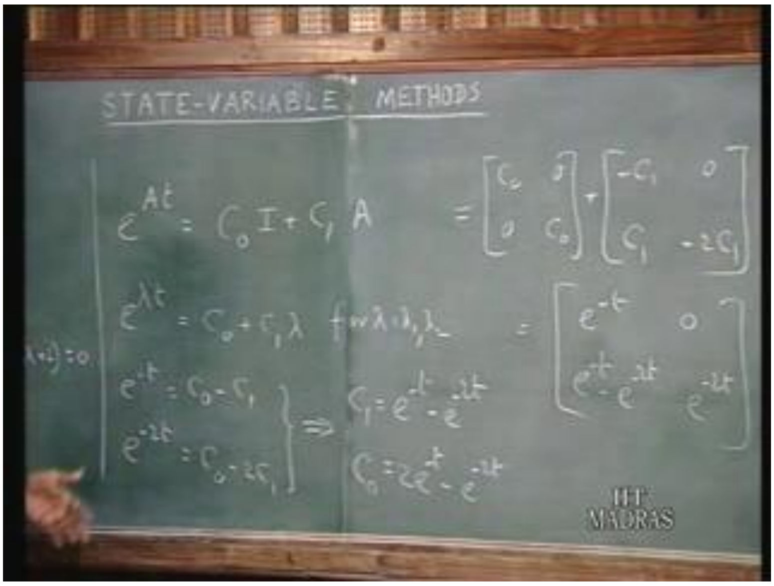

So, we want to Find e to the power of At e to the power of At now it is a squared matrix of order 2 because, A is a square matrix of order. Therefore, the residual polynomial as degree 1 therefore, we write this as C not I plus C1 A we can take only the first degree because, this is the characteristic polynomial of this is of degree 2.

And we know that C not and C1 are function of t we do not specifically expiated in that manner. We know that, C not and C1 are going to be function of t. To evaluate C not and C1 we go back to this scalar equation e power lambda t will be equal to C not plus C1 lambda and this is not an identity, this is an identity. This will be used true only, for valid for lambda equals lambda1 and lambda2.

So, only for lambda has those particular values this equation is true. So, we substitute those values lambda1 is minus 1. Therefore, e power minus t equals C not minus C1and the lambda2 is minus 2. So, e power minus 2t equals C not minus 2C1 these are the 2 equations you get in terms of C not and C1. You can solve this and you get C1 equals e power minus t minus e power minus 2t and C not as 2 e power minus t minus e power minus 2t.

So, those are the 2 values C not and C1. Now, you substitute in this equation. Therefore, you get this as substituting C1 and C not into this. So, you have here really the first matrix is: C not 0 C not, the second matrix is: C1 times A. Therefore, this A matrix every entry is multiplied by a scalar C1. Therefore, minus C1 0 C1 and minus 2C1 and substitute this values in this and you finally, end up with e power minus t 0 e power minus t minus e power minus 2t and e power minus 2t.

So, that is the final solution for e power at is a particular example. So, the steps are straight forward all they have to do is e power At must be expressed as a polynomial in a of degree n minus 1. To evaluate the constant that are involved constant in terms of A. That is what I mean there of course, function of time to go back to this scalar equation and this equation satisfied for characteristic values which we have to find out in the start to start with and once, you substitute this coefficient can be evaluated its substitute and then you get this final result.

The work have done so far was valid when we have n distinct Eigenvalues. So, by substituting each 1 of this in the scalar equation we got n equations and enable just solve for the n coefficients, nc coefficient C not to Cn minus 1. Now, what happens if there are repeated Eigenvalues, suppose out of this n Eigenvalues 2 or 3 particular Eigenvalues repeated 4 times. Then, when you substitute that particular Eigenvalue get the same equation.

Therefore, you do not get 4 equation corresponding to those 4 repeated Eigenvalues. So, what do? How do handle such a situation? Do we run out of equations, do we have enough equations to solve for the n unknowns n unknown coefficients C coefficients. So, we will see this does not create any special difficulty.

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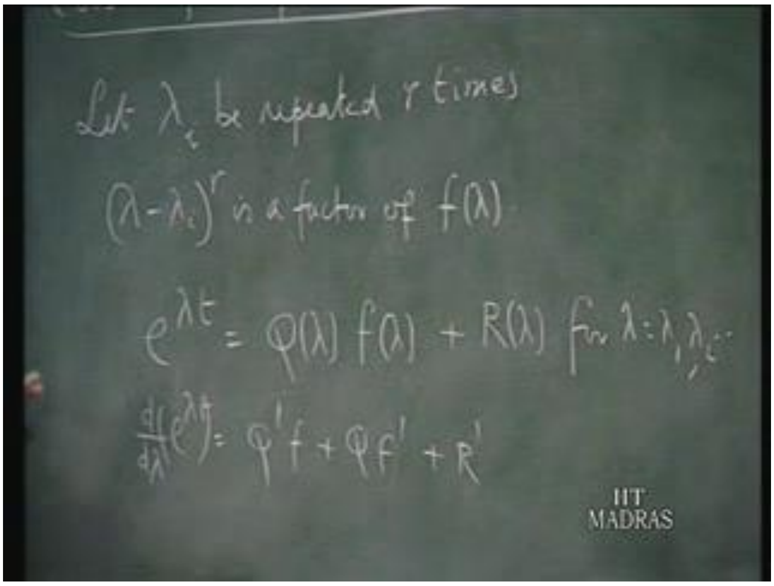

So, let case of repeat Eigenvalues let a particular Eigenvalue lambda i be repeated r times. What does it mean? That lambda minus lambda i the power of r is a factor of f lambda. So that, this particular Eigenvalue is repeated r times therefore, lambda minus lambda i raised to the power of r is a factor of f lambda. So, now when we form this scalar equation e lambda t is a coefficient polynomial times f lambda plus this is the residual polynomial R lambda.

You have f lambda is a factor lambda minus lambda i raised to the power of r. Now, this is satisfied for lambda equals lambda1 lambda 2 etcetera upto lambda n. But some of them are of course, not distinct lambda is repeated n times .So, we will say I will just leave it at that now, when you substitute lambda i then certainly e power lambda t equals r times lambda i because this becomes 0.

But then, what you can see is if f lambda is this particular factor when, you take the derivative of this, when you take the derivative of this say d by d lambda of e power lambda t you take that then, you have Q prime f plus Q times f prime plus R prime. Where, the prime indicate the derivative with respect Polynomials. So, if f lambda has this factor this same factor continues here.

But it also continues here with reduced degree lambda minus lambda i raised to the power of r minus 1 is a factor here. So, this also when you substitute lambda equals lambda i this terms drop out and you have got r prime lambda i must be equal to d by d lambda e to the power of lambda t. That means, when you substitute here not only the you get 1 equation from here, but the if you take the derivative then, also you get a second equation. Like that, you can go on up to higher derivatives if you have the this lambda minus lambda i raised to the power of r is a factor of r here you can go on taking to derivatives up to the r minus 1'th order.

And then, also that will you get an appropriate equation. That will be shown here so, you have got e power lambda i t equals R lambda i. Then, you take the derivative of this d by dt of e power lambda t substitute lambda equals lambda i which means, when you take the derivative of with reference to lambda you get 3 times e power lambda t by substitute lambda equals lambda i effectively right.

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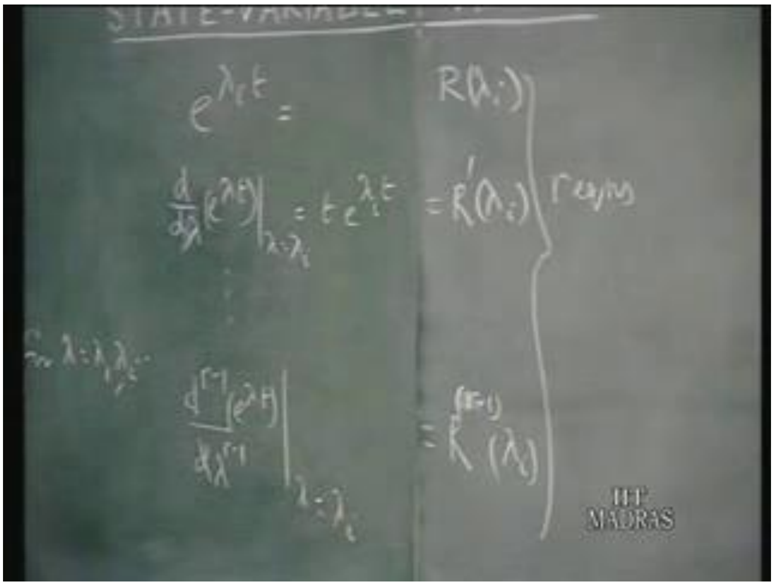

And that will be that means, you are taken the derivative of this and substitute lambda equals lambda i that is what you have here. This will become 0, this will become 0 and this will not become 0. Therefore, you get R prime here like this you can go on you can take d R minus 1 d lambda R minus 1 e power lambda t substitute lambda equals lambda i that also will become r the m minus r minus 1 derivative of this substitute lambda i this 1 equation we get.

Because, even if you take the R minus 1 derivative lambda minus lambda i continuous to be a factor of these 2 of such similar all factors except this last 1. So, we now have collect this you have R equations and this R equations will give you constraints in R the coefficients. Therefore, lambda since factor here is repeated R times corresponding to this R factor R terms in this F lambda you get R equations.

So, we will never be in short supply as per the number of equations are concerned, even if a particular Eigenvalues are repeated. And you form this successive derivatives and substitute lambda i you get required number of equations. Example will illustrate this procedure very clearly. So, let us do that example let us, take the A matrix to be matrix of order 3 0 minus 1 minus 3 minus 6 0 minus 2 5 minus 2 and minus 4.

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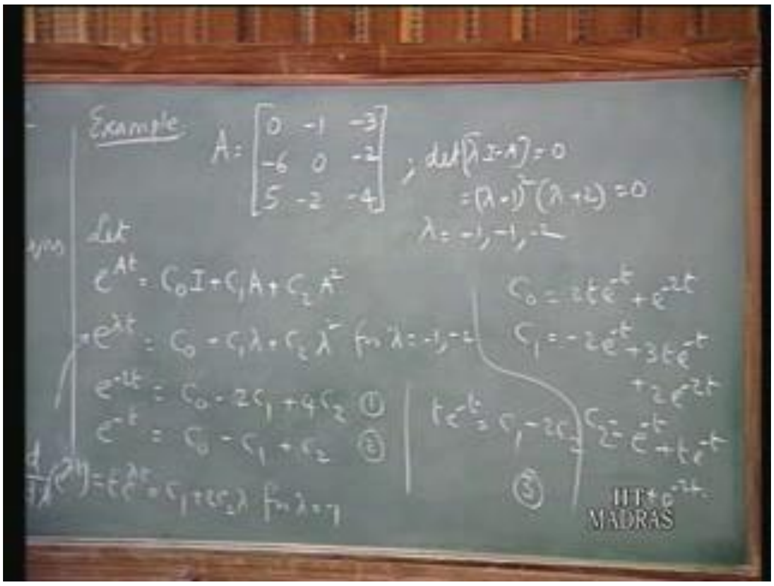

Corresponding to this determinant of lambda i minus A equals 0, it will turn out to be lambda plus 1 whole squared and lambda plus 2 equals 0 that means, you have lambda minus1, minus1, minus2 these are the 3 Eigenvalues. And 1 particular Eigenvalues repeated twice minus1 and minus2. So now, let e power At be equals C not I plus C1 A now since, this is of order 3 we have to go up to the second power of A C2 a squared.

We evaluate C not C1 and C2 we fall back and the scalar equation e power lambda t equals C not plus C1 lambda plus C2 lambda for lambda equals minus 1 and minus 2.

So, let us substitute minus 2 and get it of that to start with so, e power minus 2t equals C not minus 2 C1 this must be squared plus 4C that is 1 equation. And substitute lambda equals minus 1 e power minus t equals C not minus C1 plus C2 that is the second equation. To get the third equation, you take the derivative of this with reference to lambda so, we get d by d lambda e power lambda t which is equal to t times e power lambda t that if take the derivative on the other side you get C1 plus 2 C2 lambda.

And this particular equation is true for lambda equals minus 1 because, that is the 1 that is repeated twice not for 2 lambda equals minus 2 only for lambda equals minus 1.So, when we substitute here you get the third equation which says that is: t e power minus t t e power minus t equals C1 minus 2C that is the third equation. So, you have 2 equations already from the straight forward procedure by taking the derivative you get a third equation t e power minus t equals C1 minus 2C2.

So, using these 3 equations you can solve for C not and C1 and c2.The result will be C not will be 2t e power minus t plus e power minus 2t.C1 will be minus 2 e power minus t plus 3t e power minus t plus 2 e power minus 2t.and C2 will be e power minus t plus t e power minus t plus e power minus 2t. So, those are the 3 constants and you substitute those constants in this equation constant I am saying constant in this sense that they are independent of lambda or independent of A.

So, C not C1 and C2 in this equation and you can evaluate e to the power of At the details I will omit. Notice that since, this particular characteristic value is repeated twice you get t times e power minus t those terms are presented here. So, if it is repeated 3 times, you get t squared terms also will present here t squared e power minus t will be present. This is just like having multiple poles in the case of Laplace transform solution .

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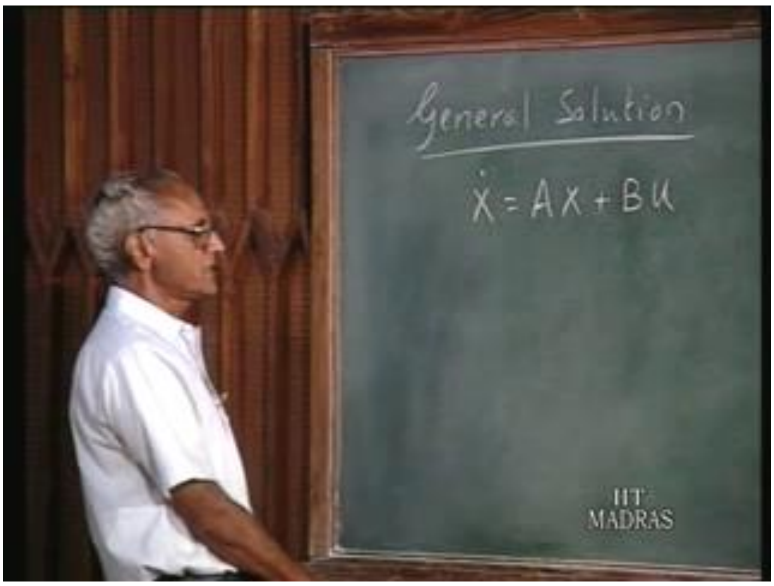

Now, let us look at the with this background let us look at the general solution of the state and output equations. Now let us, look at the solution of the general equation state equations not the homogeneous equations as we consider earlier. To do this let us, multiply by e power minus At write through you have e power minus At times x dot i transfer this to other side e power minus At times A times X equals e power minus At times B u where u is the function of time.

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Now, these 2 terms on the left hand side can be written as d by dt of e power minus at times X because, this is the derivative of product. Therefore, the derivative of e power minus At is e power minus At times minus A. So, times X plus e power minus At times the derivative of X which is this. So, these 2 terms can be put in this form this will be of course, e power minus at times B u of t this u is a function of time.

Since, this is the derivative of this is equal to this e power minus At times X of t this is X of t of course, X of t must be the integral of that. So, you can write e power minus at times X of t is the integral of this. Now, that integral we have to the right left hand side a function of t. Therefore, we can introduce a dummy variable for the integration you can write e power minus A tow B u tow d tow 0 to t that’s what you are having.

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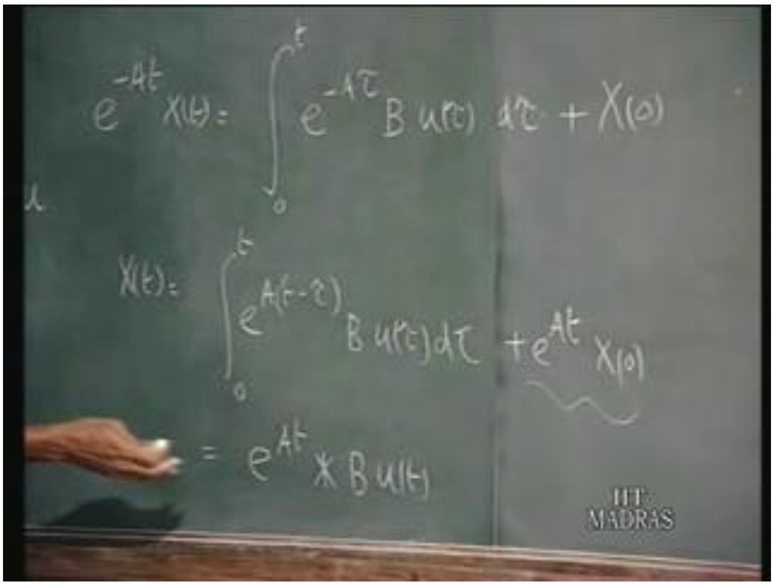

Because, after all the function of time and you have the variable integration should be something other than t i put like this. But we also, have an arbitrary constant because after all you have you are integrated you have the derivative of this is equal to this. So, we are integrating you must put something else and that is a vector e power minus At times X is A vector and that must be equal to suppose, A substitute t equals 0 substitute t equals substitute t equals 0 the upper and lower limits are the same so, therefore this becomes 0.

And here e power minus At which is of course equals to unique matrix. Therefore, this becomes X0 so you evaluate this the arbitrary constant that is involved here X0. So, that takes care of that arbitrary constant that comes in the integral. So, now you can write the solution for X t as 0 to t you can now introduce this insight e power At. So, you have e power At that means, you multiply right through by e power At. So, e power At minus tow B u tow d tow plus e power At times X of 0. So, you observe that this is the 0 input solution and this is the solution that comes because, the presence of the input.

Because, this is something which you have already known that is for Xt. This is the extra term that you that comes about because, of the presence of the forcing function. We can put this in a more compact fashion recognize this to be something similar to your convolution integral except that matrices are involved. Otherwise, is the same so I can write this as e power At convolved with B ut after all in the simple in the form wise it is exactly the same as this.

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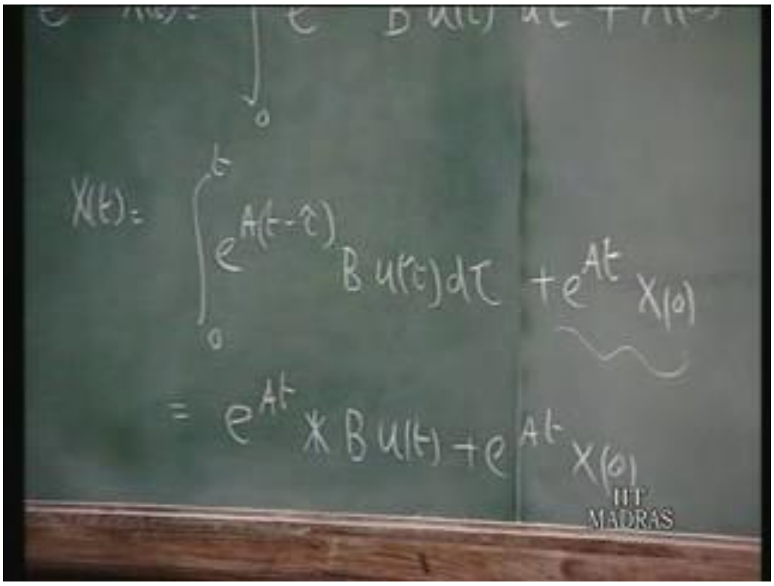

So, i can write this as e power At convolved with B ut plus e power At X 0.So, using this information we can now know the output y is CX plus D u. So, we know the solution for x we substitute this here you get C e power At times X 0 plus C e power At convolved with B ut plus of course, you have got D ut. So, this is the final solution for Y, C e power at X 0 C e power At convolved with B ut D times ut.

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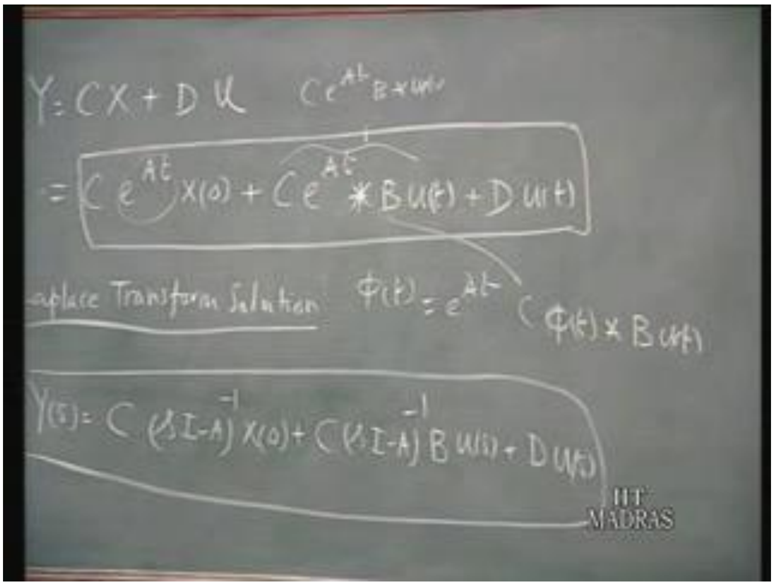

The meaning of this convolution is this syntax now; you compare this with the Laplace transform solution. You notice that, the Laplace transform solution we have Y of s equals C times SI minus A inverse X 0 plus c times SI minus A inverse B us plus D. Since, we already noted that e power At and SI minus A inverse form A Laplace transform pair. So, you can easily see that the Laplace transformation of this is indeed this. Because, this is only function of time is this e power At.

So, its Laplace transform is equal to this, this is a constant and this is a constant of course, constant vector. And here you have got the convolution of 2 time function e power At and ut therefore, you have got the product of the corresponding Laplace transforms B u minus D us. In working out numerical examples you can put this star here either here or here whichever, is convenient for you.

Because, after all this is a constant that can be push either this side or that side .C e power At this is also equivalent to you can write this if you wish C e power At B convolved with ut there is no difference So, this is the solution in the time domain which corresponds to the solution in the Laplace transform domain. So, in fact since we know the solution Laplace transform domain you could have found out the time domain solution by taking the inverse Laplace transform of this we would have ended up with the same thing right.

So, this gives you a very compact way in which you can find out the output quantities. Its usual to write e power At as pie of t is e power At so, you can put this as C pie t convolved with B ut So, in which case this term will be C pie t convolved with this another way of writing the equation. In the next lecture, we would consider working out a numerical example illustrating this various concepts.

So, what we have done today is we started out with the system transition matrix also simply call the transition matrix e power At and we said that e power At can be evaluated in several ways: 1 is using the Laplace transform approach SI minus A inverse happens to be equal to e power At the this is the Laplace transform of e power At transition matrix. And alternative way of doing working out in time domain would be take a finite series approximation for e power At by take in the infinite series and truncated at particular point.

We said this does not recommend itself because, we don't get a closed form expression and it may be suitable for computer working, but not for when you want to do it an analytically and you want arrive at a closed form expression. A third method is to form the diagnolize the A matrix and the diagonal matrix that turns out will be 1 in which you have got the Eigenvalues along the main diagonal. And the transformation matrix t and t prime will be associated with the Eigenvectors.

And once, you have done that i mean different mathematical methods of finding out the transformation matrix needed for this job. And once, you have the transformation matrix the evaluation e power At is very convenient we have given a the form for this, but we have not worked out this particular method in great detail. But the method that we have discussed some detail is the: 1 which is based upon the Cayley-Hamilton theorem which states that, every square matrix satisfies its own characteristic equation.

Therefore, if FA F lambda equals 0 the characteristic equation FA equals 0 is also satisfied for identically for a given squared matrix. Now, this enables to write any transanative function of a any higher degree polynomial of A as equal to polynomial of degree at most n minus 1. So, to find out the coefficient is involved in this deduce order polynomial we go back to this scalar polynomial in terms of lambda and substitute the that particular equation the deduce these scalar polynomial is satisfied for particular values of lambda which are the characteristics values.

We substitute those values and you get enough number of equations to evaluate the various constant that are involved in the expression for the residual polynomial of e power At. We also saw that, even when 1 of the Eigenvalues is repeated several times we can still get a enough equations. We never be short supply as for the number of equations are concerned by taking, the derivative of e power lambda t with reference to lambda.

Then, because a particular factor is repeated several times in F lambda we are taking the successive derivatives that, factors still remains up to the if the original factor was degree R up to the R minus 1 derivative that factor remains at least R minus 1 derivative it will be linear factor not a higher power. And therefore, using that particular property we if there is a factor repeat r type we get additional R equations.

So, with this we were able to get enough equations to solve for the all the coefficients in the residual polynomial of degree n minus 1.And therefore, we can evaluate e power At we worked out a numerical example to show this and then, we once we know how to calculate e power At. We can give, the general solution for the straight vector and from that the output vector and the final solution will be of this form which involves this is the 0 input solution this is 0 input solution, this is the 0 state solution, this part is 0 state solution.

Then, we saw that it corresponds with the solution that we have already obtained in terms of Laplace transformations. That we already discussed in the last lecture we will work out a numerical example of finding out the solution in time domain based on the equation in the next lecture.