Networks and Systems
Prof V G K Murti
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 41
Discrete – Time Systems (5)
Inverse Z-Transformation.

The last lecture, we looked at a various properties of Z transforms.

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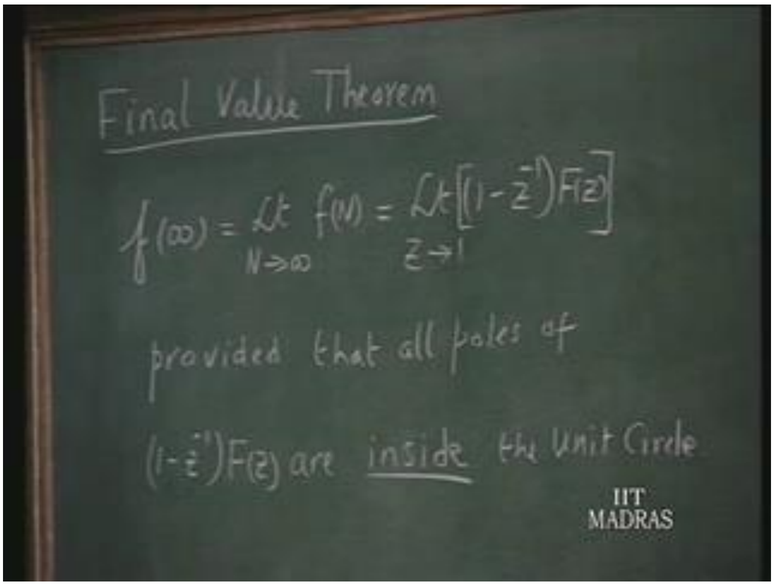

Then the last property we looked at the statement of the final value theorem which in effect states that the discrete time function fn takes the last value the final value of that, which tends to f infinity which limits N tends to infinity of f of N. It can be obtained from the Z transform by making the product 1 minus Z power minus 1 multiplying by F of Z. And taking the limit of that function at Z goes to 1. And this is to provide all the poles of this function 1 minus Z minus 1 multiplying F of Z are inside the unit circle. There cannot be any pole either outside the unit circle or even on the unit circle.

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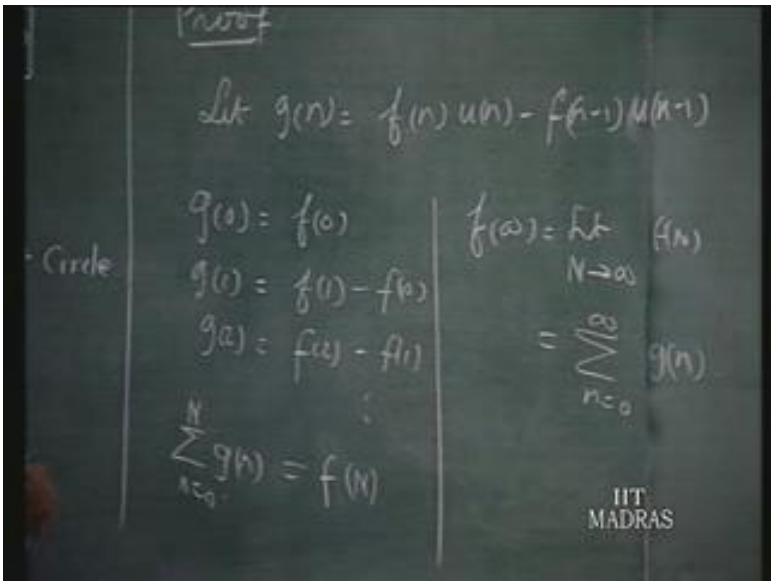

The proof for those can be given in the following manner; suppose, I write gn as fn un minus f n minus 1 multiplied by u n minus 1. Suppose. I saw the new function gn which is fn un minus f n minus 1 u n minus 1. It is obvious that g 0 when n equal 0 this is f 0 u 0 that is a f 0 and because, u n minus 1 is there where n equals 0 that is going to 0.

G 1 equals f 1 times u 1 that is f 1 minus when n equals 1 this is f 0 is of course, u g is also equal to 1 this is f 1 minus f 0. G 2 likewise can be showed to be f 2 minus f 1 and so on and so forth. So, if you take the sum of gn from n equals 0 to some value N then you had a flaw in this function. When you have a g 0 g 1 you will get f 1 from your g 0 g 1 g 2 you end up with f 2. So, it is obvious this is going to be gN.

That is, when we add up gn all the samples gn from n equals small n equals 0 to N this is going to be fn his is going to be fN fN. So, from this analysis you can see that fN for it which is of course, the limit N goes to infinity of f of N is also equal to n from 0 to infinity of gn this function gn. Now, let us see how you get relate f infinity now to the Z transform of f of N. From this definition of g of n its clear that G of Z the Z transform of that will be the Z transform of this minus the Z transform of this.

This is Z transform of this is F of Z and this is a delayed version of f of N delayed by 1 sampling instant 1 unit. Therefore, this is exact power minus 1 times F of Z which is really 1 minus Z power minus 1 multiplied by F of Z that is what it is. G of Z therefore, is 1 minus Z power minus 1 F of Z. But, what is the definition of G of Z? G of Z is gn Z power minus n n from 0 to infinity right. Therefore, from this we can see that only substitute Z equals 1 Z goes to 1 of G of Z is equal to summation when 0 to infinity of gn.

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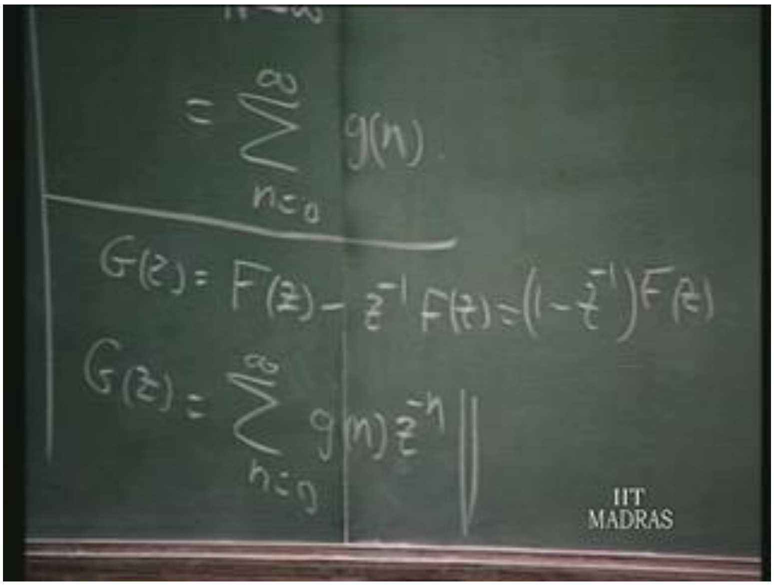

Therefore, limit a Z goes to 1 of G of Z is the summation of the samples in time 0 to infinity of gn which indeed is what we want that is f of infinitive. And what is G of Z.? G of Z is equal to this that means this quantity is also equal to limit Z goes to 1 1 minus Z power minus 1 F of Z. So, the upshot of this is that is very equal so f of infinity equals limit Z goes to 1 1 minus Z power minus 1 F of Z. Now, in yielding this in getting as the result you have put the limit Z goes to 1.

So, this F of Z G of Z whatever we are talking about should be valid for values of Z within the region of convergence.

That mean, Z equals 1 must be within the region of convergence of the transform that we are talking about. In fact 1 minus Z minus 1 to power of F of Z is a transform of G Z. Therefore, G Z should have the region of convergence which includes the point Z equals 1 that means, the unit circle must be within the region of convergence of G of Z. And that is the condition why we said 1 minus Z power minus 1 F of Z all the poles must be within the unit circle. And this is the very important term condition.

Because, if you do not regard to this condition you may get the wrong results; its rule the final value theorem may not apply. But, you still get some value and assume that it is the final value may or may not be true.

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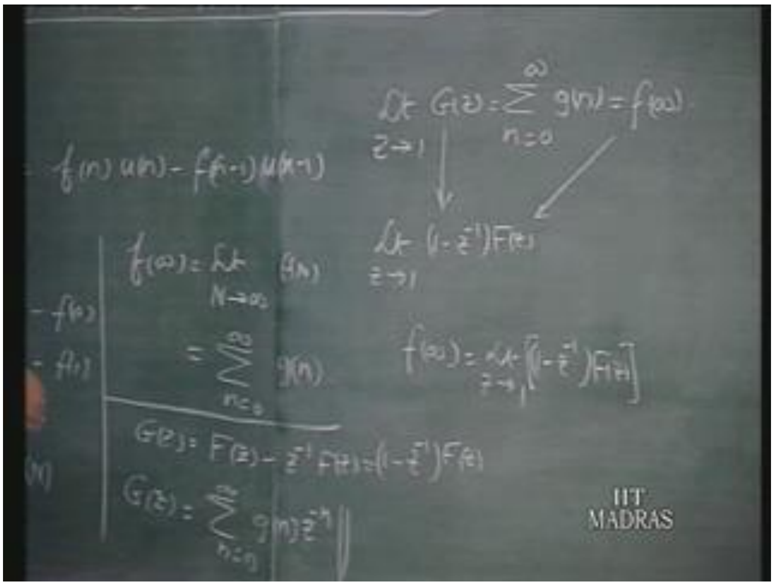

One can consider this proof even alternative at perhaps the simpler fashion by saying after all you are talking about a final value for the sample train fn. One will have a final value after all when all the transient times value out you have steady state term then you can think about a final value. If the steady state terms in f of n or sinusoidal varying or periodic samples then there is no final value because this will be oscillating forever. So, the only way which we can associate a final value with f of n is where it is got some step function as part of it.

There is no other periodic function so if you have step function in this f of n. And you want to find out the partial fraction expansion of F of Z the residue associated with the Z transform of the step Z over Z minus 1. You essentially what you do is if multiply F of Z by 1 minus Z power minus 1 and take the limit Z goes to 1. And that is exactly the value of the step function that is embedded in f of n and that is the final value. If there are sinusoids inside the f of n then obviously this condition be violated.

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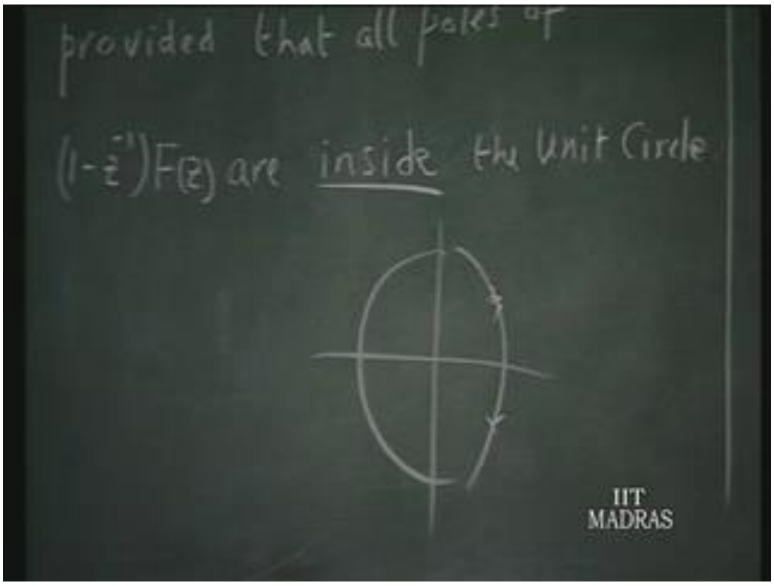

Because, when you multiply 1 minus Z minus 1 F of Z you will have poles there is sinusoid well the poles here and here. So, this condition be violated therefore, the final value will occur only 1 the only steady state function that is embedded in f of n is a step function. And the magnitude as step can be obtained by partial fraction expansion by taking the limit Z goes to 1 of this and that is the final value. So, this is equivalent to this.

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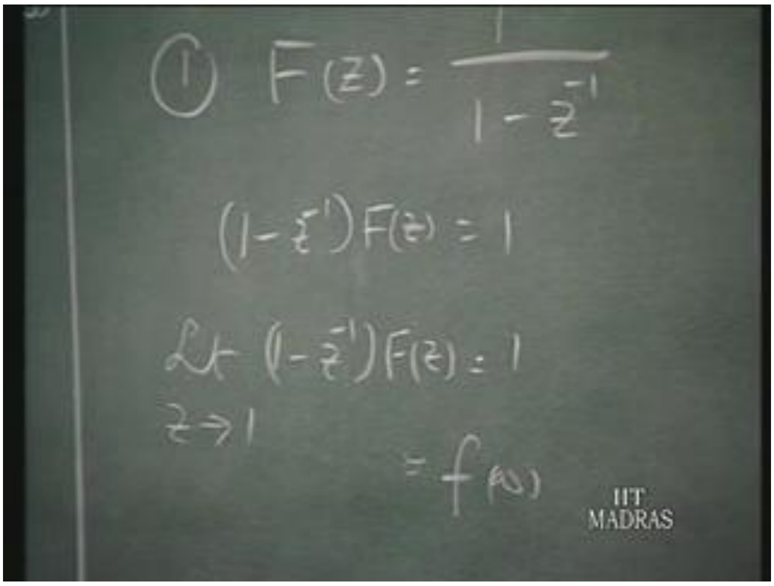

An example that is illustrating 1 example: suppose, we are given F of Z equals 1 over Z minus 1. Then 1 minus Z power 1 times F of Z equals 1 and this is a constant certainly it is annulated it has no poles on the unit circle and outside the unit circle therefore, its quell behaved. So, limit as Z goes to 1 of 1 minus Z power minus 1 F of Z of course, its 1. Since, we know this is the unit step function final value is equal to 1. So, there is a no surface in our result here.

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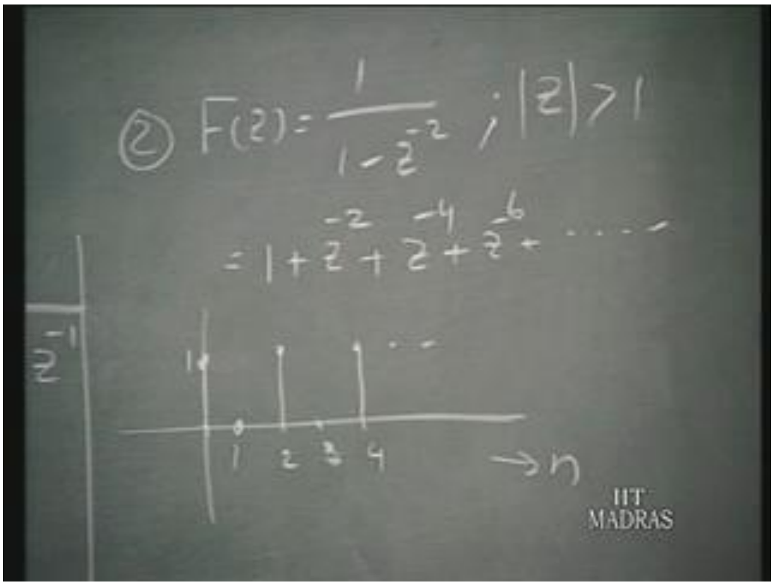

Let us take, the second example suppose I say F of Z equals 1 over 1 minus Z power minus 2 and it is further set that the region of convergence of this Z go to then 1. Now, since the region of convergence of this is the region outside the unit circle I can expand this in towers of Z power minus 1. So, I can write this as 1 plus Z power minus 2 plus Z power minus 4 Z power minus 6 and so on and so forth.

So, that means, this F of Z is associated with a discrete time function which has got this sequence of values at n equals 0. The next sample occurs at 2 because, there is no coabsent of Z power minus 1 0; therefore, at n equals 1 the sample value 0. At n equals 2 once again it is 1. At 3 it is 0. At 4 it is equal to its 1 and so on and so forth. So, over alternate sample is 1 or 0.

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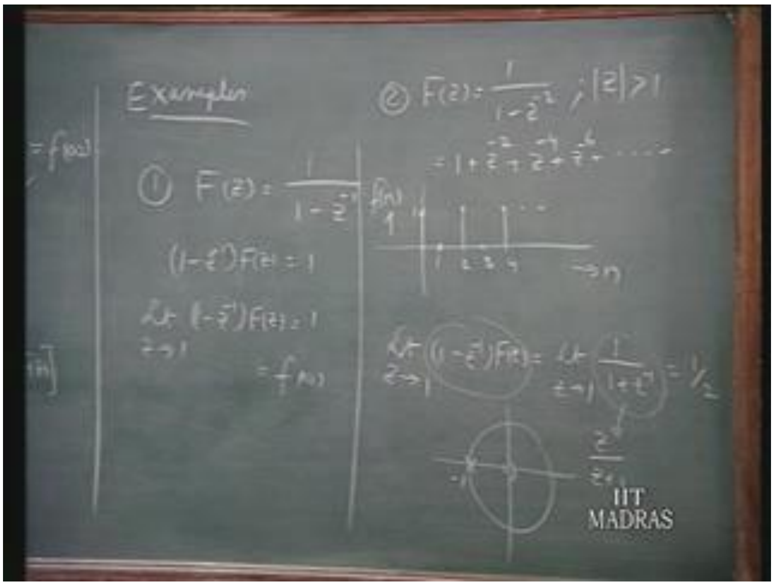

Now, let us see let us see whether the we apply that final value theorem and see if we get in a result. Limit as Z goes to 1 of 1 minus Z power minus 1 times F of Z is obviously limit as Z goes to 1 when you multiply this you get 1 over 1 plus Z power minus 1. And certainly a limit exits for this when you substitute Z equals 1 you get a value half. Right but, then this f of n we are talking about is say 1 or 0 and when n goes to infinity you cannot say what is the final values going to be is an over 0. It's a we can't say it is going to be 0 or 1.

But, the value you get a definite value and as you take the limited pass is equal to half. The answer to the distribution is that the final value theorem does not apply to this case. Because this function now, has a pole at exactly minus 1 after all 1 minus Z power minus 1 F of Z this is equal to Z upon Z plus 1. So, this function has a 0 here and pole at minus 1. So, 1 minus Z power minus 1 multiplying F of Z is not analytic does not have is not analytic and a unit circle.

So, n once should have that particular condition satisfied before apply the final value theorems. Whatever result you got here is not accordance with the actual situation you may get some result that is actually you got an average between 1 and 0. But, that is not the final value associated with this because; nobody can say what the final value of this is going to be. So, after I have considered the various properties of Z transforms. Now, it is time for us to see how will you find the inverse Z transformation?

So far, we are talked about transformation of discrete time function Z transform discrete time function. Now, we should like to see given a Z transform how do you recover the discrete time function from this? And this process called the inverse transformation and let us sees what is the techniques available for us to achieve this inverse Z transformation?

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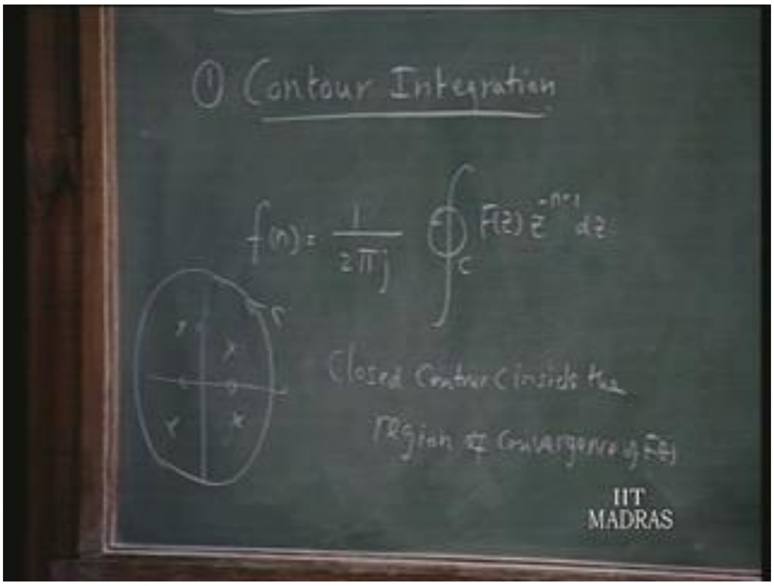

To find the discrete time function, corresponding to a given Z transform the process which is called inverse transformation. The basic this we are going to would be through contour integration. Given F of Z you can recover f of n by using this formula along a closed contour C inside the region of convergence of F of Z. So, Z transforms exits within the certain region of convergence. So, if I have F of Z had you various poles like this zeros and so on. So, you take a closed contour usually a circle go round the contour in the anti clockwise direction and do this integration.

There is you are familiar with the integration complex variable situation complex variables per unit complex variables are involved. You form this function find out the residues of the various poles of this function inside the contour. And since, you have got 2 pi j here the sum of the residues will be your f of n. So, this is the fundamental way of doing this and 1 can also interpret this. Just as you have done in the continuing case as f of actually F of Z suppose you take this contour F of Z. And suppose, you take some angle theta here delta theta, so F of Z times delta theta over 2 pi Z power n. This is the characteristics signal in the discrete time case.

This is coefficient associated with this so F of Z delta theta what 2 pi is the coefficient this can be called the coefficient density. And you can interpret this in a similar way as we have then in the Laplace's transform. But, we will not go deep in to that I will just want to point out there is menology. More or less we will not pursue the contour integration because we rarely applied for our work. Of course, what if we what we like to do is more in the go by the partial fraction expansion which I will talk about little later.

That is most straight forward what we normally applied but, if you have some Z transforms which are not easily recognize as Z transforms of known time functions. Then 1 may have to go to this contour integration in a general case. This is more fundamental and general but, rarely we applied because we have simple methods to achieve the same thing. So we will leave it at that.

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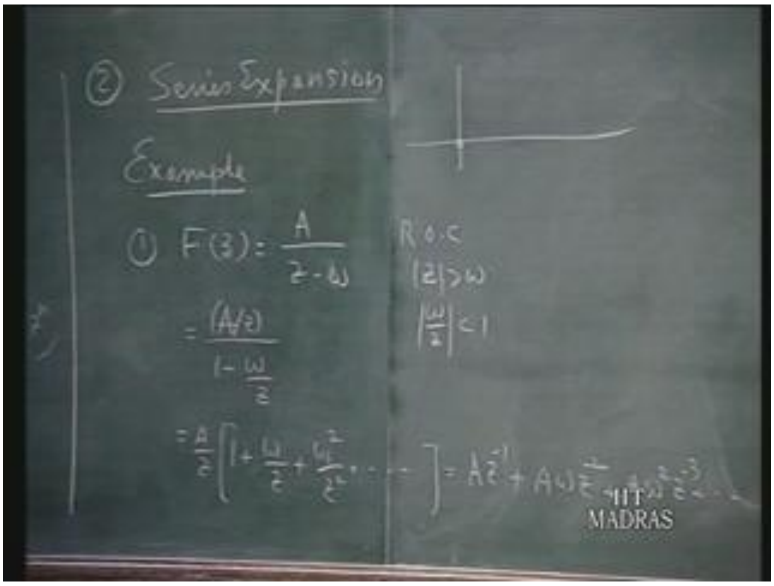

A second find a very simplistic method is the series expansion. Suppose, we have given F of Z and when you expand this in essence Z power minus 1. Then each coefficient of Z power minus 1 Z power minus 1 and so on and so forth defines the value of the sample and appropriate instance of time. Therefore, if all where to therefore is, expand F of Z as a series with increasing power of Z power minus 1. And if you assure that then you know the values of the sample defined different distinct points of time that is what we do.

So, let us take an example. Suppose, I say F of Z equals A upon Z minus a and the region of convergence is I will say omega let a Z greater than omega or omega over Z magnitude is less than 1. Therefore, we would like to expand this in powers of negative exponent negative powers of omega over Z. So, took that for purpose I will write this as A upon 1 minus omega over Z. So, I will put here upon Z A upon Z 1 minus omega upon Z.

So this can be written as A upon Z 1 plus omega upon Z plus omega squared upon Z squared and so on and so forth. And this series come back as to this because omega upon Z magnitude is less than 1. And this is also an accordance with what we have for a single sided Z transform where the F of Z is defined as fn times Z power minus 1. That means, the expansion that we are looking for is incusing power of Z power minus 1. Therefore, this can be written as A Z power minus 1 the first term plus A w Z power minus 2 the second term plus A w squared Z power minus 3 and so on and so forth.

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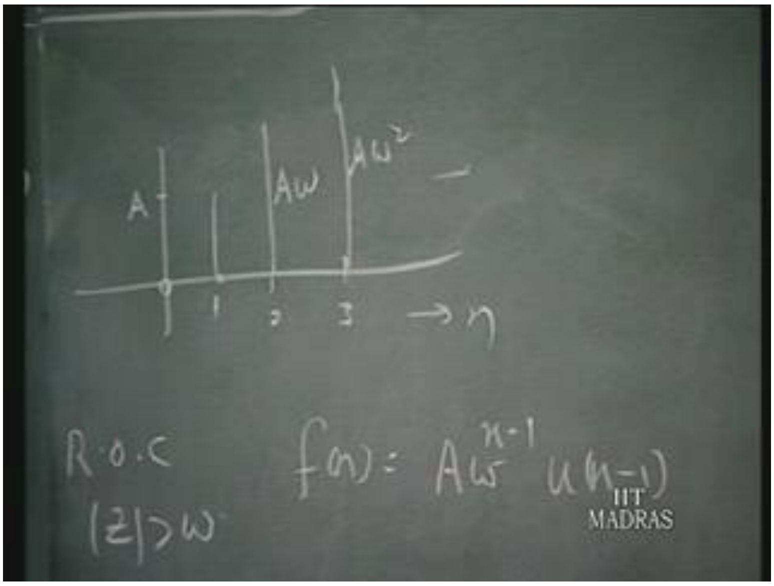

So, you can see that as far as the time function which corresponds to the Z transform is concerned. It must have a sequence like this to first term A power Z minus 1 that means there is a 0 value at n equals 1 n equals 0. At n equals 1 the value of this is A. At n equals 2 the value of this is A omega. At n equals 2 three it is A omega squared and so on and so forth.

So, from this you can conclude that f of n at the nth sample is A w to the power of n minus 1. And this sample sequence starts at n equals 1 it is 0 at n equals 0 therefore, you must multiply this by u of n minus 1 that is what we have got. So, that answers A w power A w A w power n minus 1 u n minus 1 that is your f of n corresponding to this. So, in this case we are observed that we are able to get a closed form expression for your f of n this may or may not be possible all the time.

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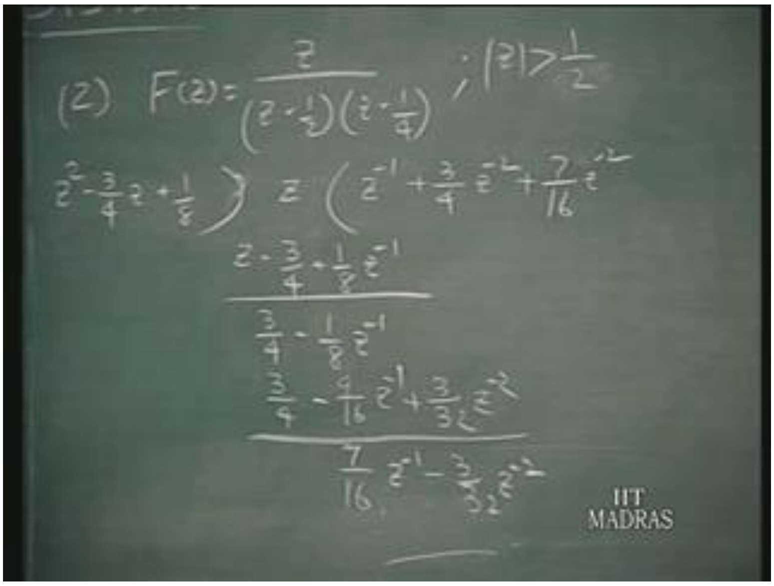

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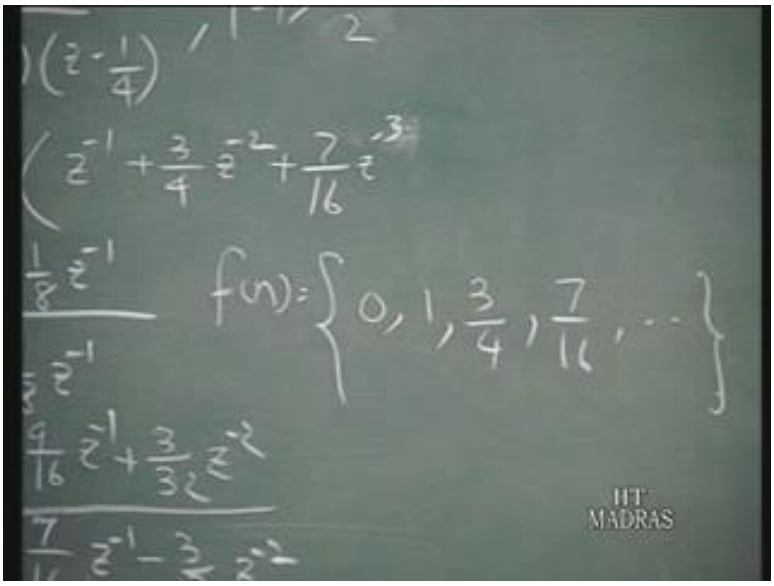

So, let us take a slightly more complicated example F of Z equals Z upon Z minus half times Z minus quarter; so, the region of convergence for this Z, greater than 1 half Z magnitude greater than 1 half. Now, what mean it is we make the long division so the denominator is Z squared minus 3 quarters of Z plus 1 8. So this we divide into Z. So, first term will be Z power minus 1. So, Z power minus 1 multiplied by this will become Z minus 3 quarters plus 1 eighth Z power minus 1.

So, when you subtract from this you have 3 quarters minus 1 eighth Z power minus 1 and that you divide this into this. So, you get 3 quarters Z power minus 2 so that, all the time the first coefficient is knocked out. So, 3 quarters minus nine upon 16 Z power minus 2 times Z therefore, Z power minus 1. And then, you have plus 3 upon 32 Z power minus 2 subtract from this, this gets cancelled out and your 7 upon 16 Z power minus 1 minus 3 upon 32Z power minus 2.

Then, the what again to cancel this out. So, the next term will be 7 upon 16 Z power minus 2 and then that will be cancelled out and so on you can continue like this. So, you observe that when you derived this denominator by this. What you get is a series like this? So, this must be the Z transform this is equivalent to F of Z expanded in ascending power of Z power minus n is equal to this. Therefore, we know identify this, the Z transform of a discrete time function.

So, the first sample occurs at n equals 1 because after all fn times power f 1 times power Z power minus 1 is the term corresponds to f 1. Therefore, f 1 now can be thought of as a sequence of samples there is no constant term. Therefore, n equals 0 the value equals 1. N equals 1 the coefficient is Z power minus 1 is 1 therefore, 1. N equals 2 the coefficient of Z power minus 2 is three quarters. N equals 2 n this must be by the way this must be minus 3. Therefore, n equals 3 because if it is 7 upon 16 and so on and so forth. So, this is how 1 can obtain the inverse Z transform by the long division method like this.

A few complex on this the long division method of course, the simple and straight forward and once we have the quotient series; we can identify the sample values with a coefficient of power of Z as we have seen here. In other simple and straight forward not much involves here it is simple to impudent. And we can carry this out when in search as long as you wish you can expand the series to the desired length you can get the dual point of time. So, you can carry this long division to a length that depends upon your convenience.

The point keep in mind over is that this much be expanding powers of Z power minus 1. Because, we are talking about numerator Z transform numerator Z transform is fn times Z power minus n. N has positive values therefore, you must make sure that you are expanding in a series of power of Z power minus 1. It is very convenient for numerical working as a problem. However, once we have done a series like this and then corresponding f of n you may or may not be able to find out a nice closed form expression for f of n. It depends your ingenuity and some extend to your luck.

So, there is no guarantee that will able to conveniently find out the a nice closed form expression for f of n through this process but, this is convenient for numerical work. So, the last 2 examples have shown how the long division method can be obtained to find out the inverse Z transform of given function of Z. But, as I mentioned earlier it is a partial fraction expansion added which is the most forward method compare with the contour integration method and this long division method. And this is what we will take up next.

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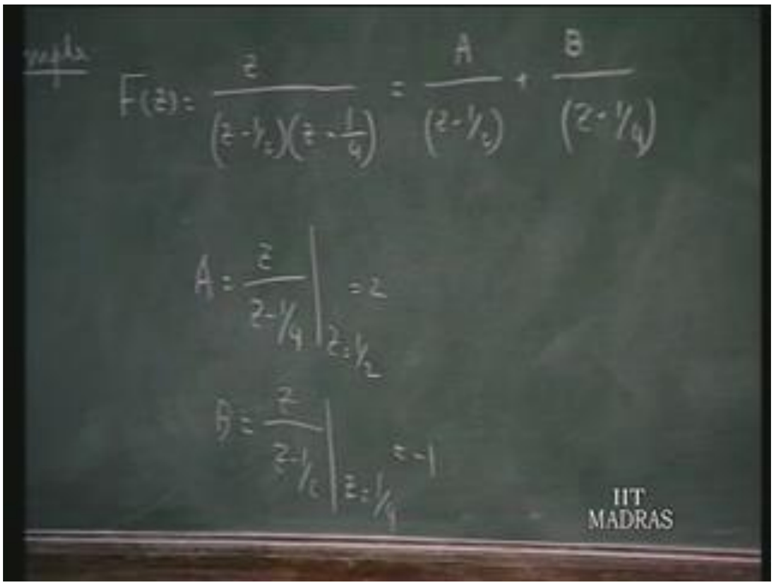

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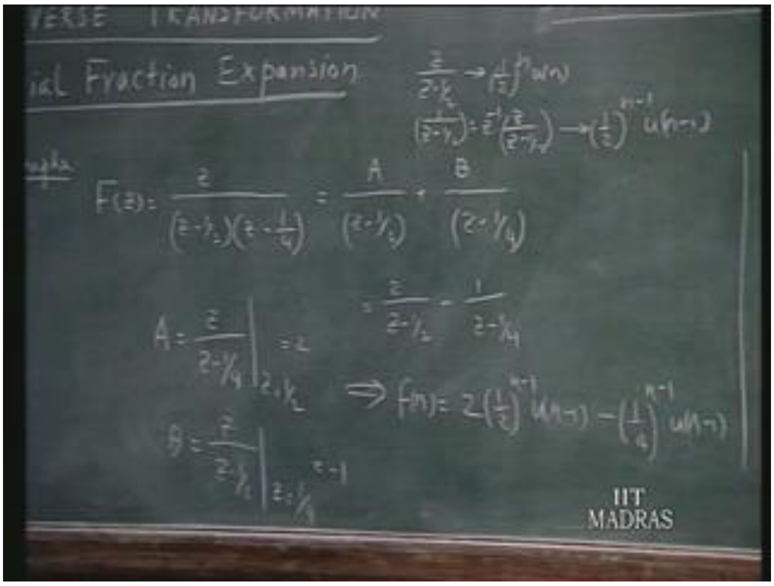

By for the most common method of finding out the inverse Z transformation is the partial fraction expansion technique. Here, as we have done in the continuous time case during the Laplace's transforms. If break up the given F of Z into simple terms each of which is recognized as a Z transform of a known time function discrete time function principle is the same. So, we illustrate this by means of examples because nothing much to described about the technique as such. Suppose, F of Z is given as Z upon Z minus half times of 1 minus 1 fourth it is exactly the same expression that we took up when we are during the long division.

So, let us note the partial fraction expansion of that so you have 2 times Z power minus half and Z power minus 1 fourth. So you call that A and B. A is obtained by multiplying this by Z power minus half. So, once you multiply this expression that by Z power minus half you are left with Z upon Z minus 1 fourth and substitute Z equals half. That is the same way as we did in the case of f of s in the Laplace's transform domain. Same thing we do here also to find the A corresponding to this you multiply this Z power minus half and substitute Z equals half.

Therefore, this is half divided by 1 fourth that is equal to 2. So, at B equalize when is multiplied this term by Z power minus Z minus 1 fourth; so Z upon Z minus half and substitute Z equals 1 fourth. So, this is 1 fourth divided by 1 fourth minus half minus 1 fourth that is equal to minus 1. So, A equals 2 and B equals minus 1. Therefore, this is 2 upon Z minus half minus 1 upon Z minus 1 half. So, these are 2 terms which constitute F of Z. So, if you can find out the inverse Z transform of each of this the difference is equal to f of n.

Now, let us see if I had Z upon Z minus half and I am written this as half n un. Z upon Z minus half is a Z transform of a discrete time function which is half n half restore function of un. But now, instead of Z I have 2 times 1 so 1 upon Z minus half is the same as Z minus 1 multiplying this Z upon Z minus half. After all, if you multiply this by Z minus 1 you get this multiplication by Z minus 1 is a transform domain is equivalent to delaying the time function by 1 unit. Therefore, this corresponds to half of n minus 1 u of n minus 1.

So, using that principle you write this as the inverse trans Z transform of this, f of n equals as for the first function is concerned 2 times half raised to the power of n minus 1 u n minus 1. And using the same arguments here minus 1 half of instead half you have 1 fourth here 1 fourth n minus 1 u n minus 1. Or you can put this as you can leave at that that is. You can put here n minus 1 is a common factor 2 times half n minus 1 minus 1 fourth n minus 1.

So, when we are making this partial fraction expansion here when we have first order times the denominator you have constant terms in the numerator. And therefore, 1 of the constant term that Z minus half Z minus w involved u n minus 1 comes into the picture. We would like to avoid this after all we would like to have a partial fraction expansion terms like this, where we can easily recognize them to the say Z over Z minus w as w to the power of n un.

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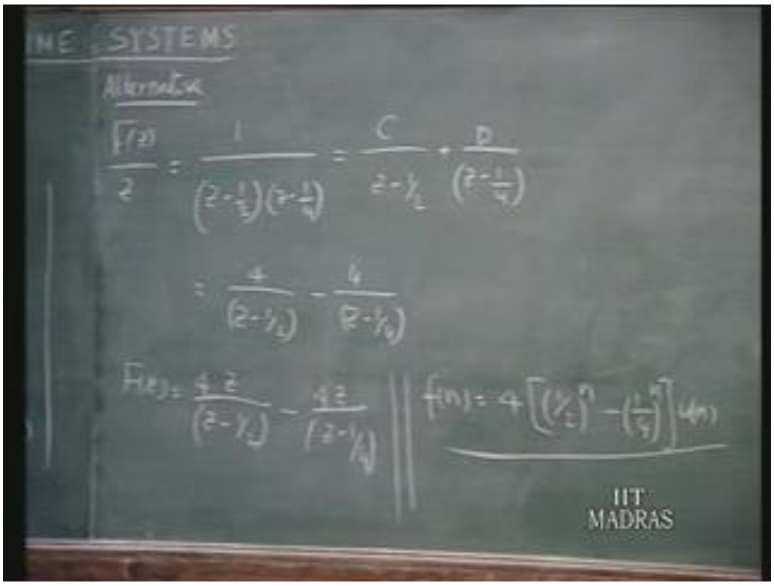

So, we have that partial fraction expansion involving times of this type we have to multiply this by Z somehow have to be that. We its terminate whereas, have to made the partial fraction expansion of F of Z over Z if you make the partial fraction expansion of F of Z over Z. And then multiply Z right, through you get terms like this. Therefore, let us do that it is an alternative which is likely proved at given F of Z you first form the F of Z over Z. In our case this is F of Z therefore, if you divide this by exact you get 1 over Z power minus half multiplied by Z power minus 1 fourth.

Let the partial fraction expansion of this the C upon Z minus half this D upon Z minus 1 fourth. And you can see in of course, multiply this Z power minus half what are Z power Z minus 1 fourth evaluated at Z equals half. And so, you can carry work this out and show this is equal to four upon Z minus half minus four upon Z minus 1 fourth. Therefore, we can now write F of Z equals 4 Z upon Z minus half minus 4 Z upon Z minus 1 fourth.

Now, it is the form which is convenient for us to use so f of n can be written as Z upon Z minus half is a Z transform of half raised to the power of n un. Therefore, this is 4 four is a common factor right through half n. Un will be there and here is minus 1 fourth for minus half 1 fourth raised to the power of n un. So that is your f of n. We observe that, that for the change of F of Z for the same F of Z you got 1 expression for f of n here. Which in terms of n minus 1 u n minus 1 and you got a new expression have f of n in terms of un. It can be shown that both are equivalent of you take the numerical values from n equals 0 onwards we trans out both are equivalent.

Therefore, you got an 1 apparently seeing different answer but, both are expressions are equivalent both are equal to each other probably values of n. And these are the methods which recommend it for most often whatever be even F of Z. We divide by that right through may be partial fraction expansion so that, the Z term the numerator Z term the numerator shows with the first order term is very convenient to handle.

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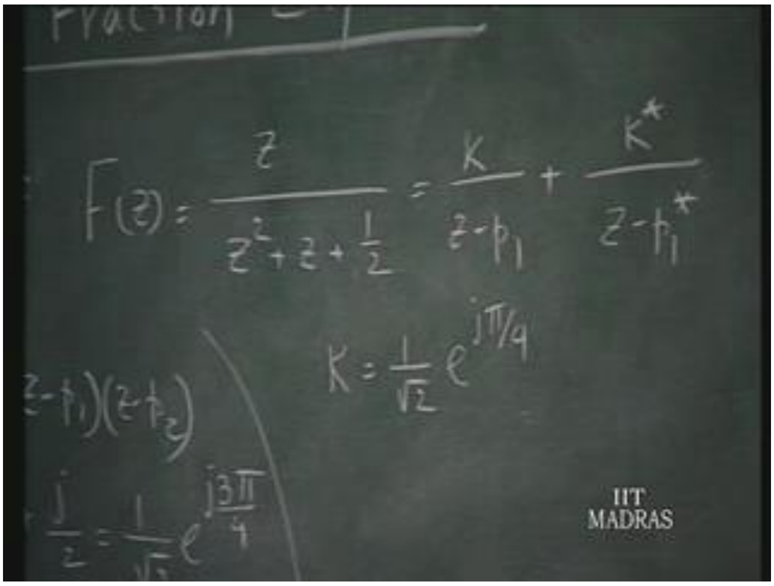

You have to take another example. And the second example: let us take a case when you are coordinated term the denominator Z squared plus Z plus half. The poles associated with this quadratic term in the denominator. Or, so if you take Z squared plus Z plus half has Z minus p 1 times the Z minus p 2, your p 1 and p 2 are the poles. You can show that p 1 equals minus half plus j upon 2; which can be further written as 1 over square root of 2 e to the power of j 3 pi by 4. That is 1 way expressing p 1.

Then p 2 because all the columns are here are real whenever p 1 is pole the complex pole as here p 2. This must be accomplished by complex conjugate therefore, p 2 as got to be conjugate of p 1. Therefore, p 2 turns out to be minus half minus j upon 2 are 1 over root 2 e to the power of minus j 3 pi upon 4. So, you do not have to back carrying the special effect to calculate p 2. So, you can now suppose I write this as k upon Z minus p 1. Then since again all the coefficients are real it involved here are real this will be Z minus p 1 conjugate so you p 2 you do not have to write separately that is p 1 conjugate.

Because, the whole equal poles and 1 is the complex mode it must be accomplished by its conjugate. Therefore, it goes without saying the other pole must be prevent conjugate. Not only that, if there is sequence is k the Z must be associated with the k star. Then only when you have a common denominator and work it out you will have numerator coefficients are real as it is in this case. Therefore, in kindling the partial fraction expansion by this it is some of if you find out k, k conjugate the other residue corresponding to the Z minus p 1 star immediately follows.

You can calculate the value of k I will not work it out it can be showed to be 1 over root 2 e to the power of j pi by 4.

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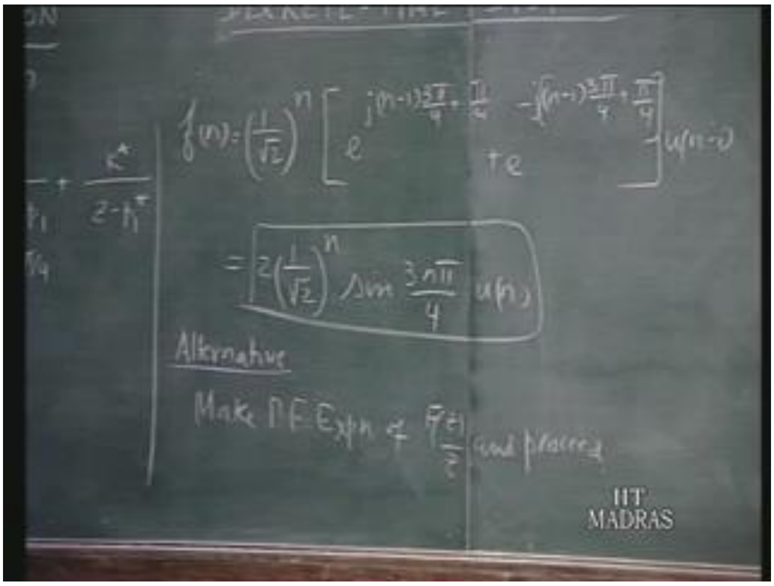

Therefore, you can combine this information with this k and k star and you can write f of n has I will not give the complete step the missing step you should able to fill it out. You can show, that this will be equal to 1 by root 2 to the power of n e to the power of j n minus 1, 3 pi by four plus pi by 4 plus e to the power of minus j n minus 1 3 pi by 4 plus pi by 4 all this times u n minus 1. And again the entire thing can be simplified and can put in the alternative form to times 1 by root 2 to the power of n sin 3 n pi by 4 un. This is the final answer. And alternative is after all I have made the partial fraction expansion of F of Z.

Because of that I got this n minus 1 term here minus 1 all this things are coming into picture. To avoid this make partial fraction expansion partial fraction expansion PFE partial fraction expansion of F of Z over Z and proceed. You get a similar set of answers here. But then, (Refer Slide Time: 30:30) whenever you have quadratic term see up and there the poles are complex conjugate. Then, we know that this is associated with a sinusoid term in the discrete time function.

Therefore, 1 can identify this with what would expect when we have tamped sinusoid and then get at the solution in a much more dilate fashion. And that is what I would like to proceed now this show.

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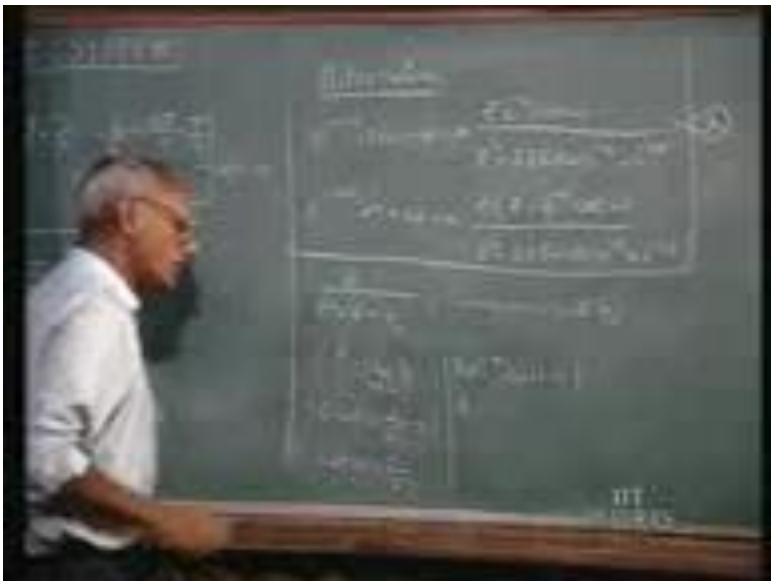

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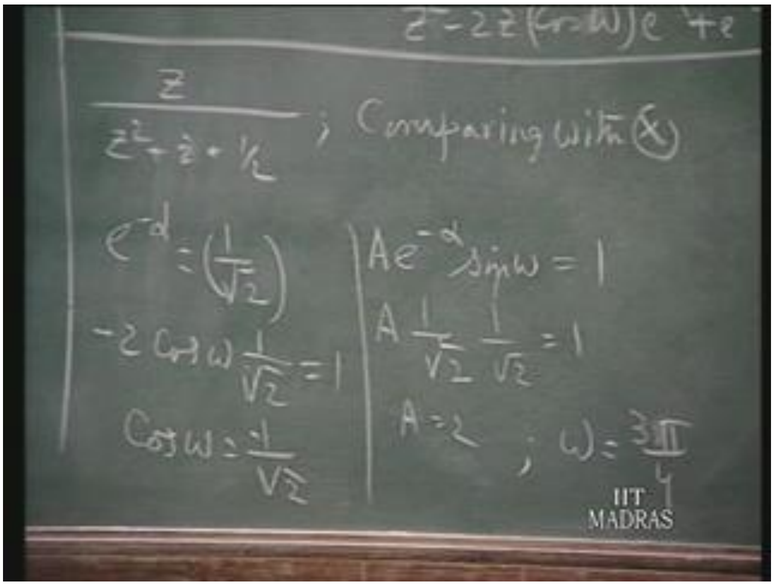

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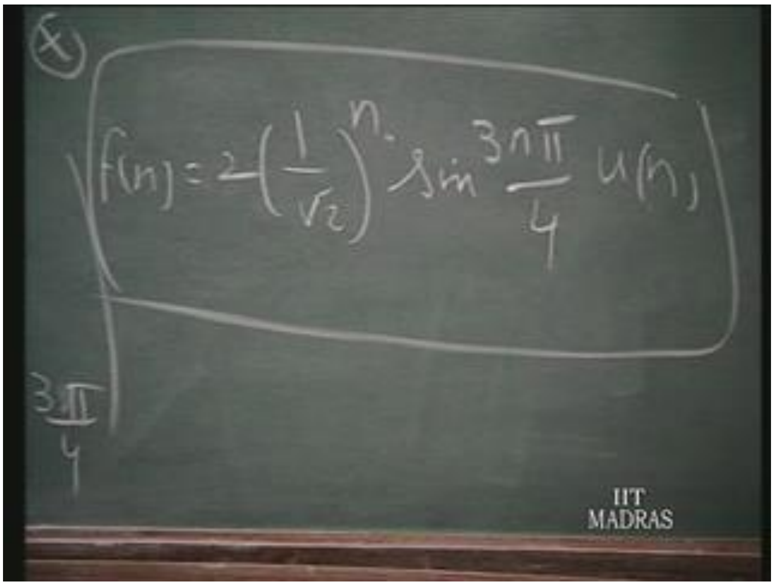

So, as a third alternative we keep at the back up our mind that e to the power of minus n alpha sin n omega has the Z transform Z e to the power of minus alpha sin omega over Z squared minus 2 Z cos omega e to the power of minus alpha plus e to the power of minus 2 alpha. And e to the power of minus n alpha cos n omega has the Z transform Z Z times Z minus e to the power of minus alpha cos omega divided by same denominator. So, this is the background that we have.

Now, in our case we have Z upon Z squared plus Z plus half. So, we have to compare this with either 1 of this expression as a combination of three expressions f. After all, since the numerator has only Z. So, it is we should able to fit this particular expression. But, if you have for example, a Z squared plus Z we will have to think about a combination of these 2. So, comparing this, with this expression X we can see the e to the power of minus 2 alpha is equal to half. Therefore, e to the power of minus alpha equals 1 over root 2.

The denominator you have minus 2 cos omega times e to the power of minus alpha e to the power of which is 1 over root 2. That should be equal to 1 because the coefficient of Z equals 1. Therefore, cos omega equals 1 by root 2 minus 1 by root 2 cos omega is minus 1 by root 2. Let us see the numerator. Numerator is Z e to the power of minus alpha sin omega e to the power of minus alpha sin omega we possibly what we may have is some A times this. So, some let us say A times this A times e to the power of minus alpha sin omega the atoms equals to be 1.

Therefore, A times e to the power of minus alpha is 1 by root 2 and if cos omega is 1 by root 2 the magnitude of sin omega is also 1 by root 2, suppose you take sin omega 1 by root 2 this equal to 1. Therefore, A equals 2. And if sin omega equals 1 by root 2 and cos omega is minus 1 by root 2 with omega will be 3 n pi by right 3 pi by 4. So, that means sin omega is positive 1 by root 2 cos omega is minus 1 by root 2. So, the angle will be 1 35 degrees correspond to three pi by four A is a constant.

So, that means, we have this is A times a quantity like this. Therefore, A where A is 2 therefore, f of n corresponding to this can be written as 2 times e to the power of minus alpha is 1 by 1 by root 2. But, you are talking about e to the power of minus n alpha. Therefore, 1 by root 2 to the power of n sin n omega get sin 3 n pi by 4 and of course, u of n. So, this is the result which we have already obtained earlier. But, we have now obtained this in more direct fashion.

So, whenever you have complex conjugate poles then it will be convenient to fit this in the given function with either 1 of this. Or both sometimes when you have quadratic you may think of a combination of these 2 and try to fit the numerator to show that, it is appropriately linear combination at the Z transform of these 2 functions.

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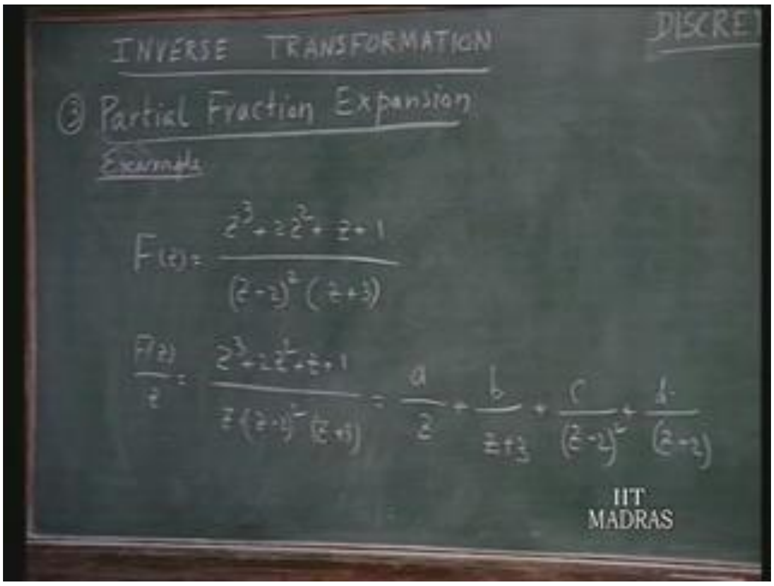

Let us take another example. As a third example let us consider F of Z to be z cubed plus 2 z squared plus z plus 1 divided by z minus 2 whole squared times z plus 3. Obviously here you are having a repeated pole at z equals 2. So, and there is a linear term here. So, let us put F of Z over Z. So, you have z cubed plus 2 z squared plus z plus 1 divided by because we introduce z here z times z minus 2 whole squared times z plus 3.

So, even the partial fraction expansion you have A upon z correspond in to this plus Z plus 3 a term let us say b. And then, corresponding to Z minus 2 whole squared you have c but, see Z minus 2 whole squared is involved. You have 2 terms of Z minus 2 and Z minus 2 whole squared let us call this d.

Now, in this partial fraction expansion here a b c are easier to obtain. To get a you multiply by Z and take the limit remain the function Z goes to 0, to obtain the multiply by Z plus 3, even take the limit as Z goes to minus 3 or value Z equals minus 3. To get c you multiply this by Z minus 2 whole squared and take the value at Z equals 2.

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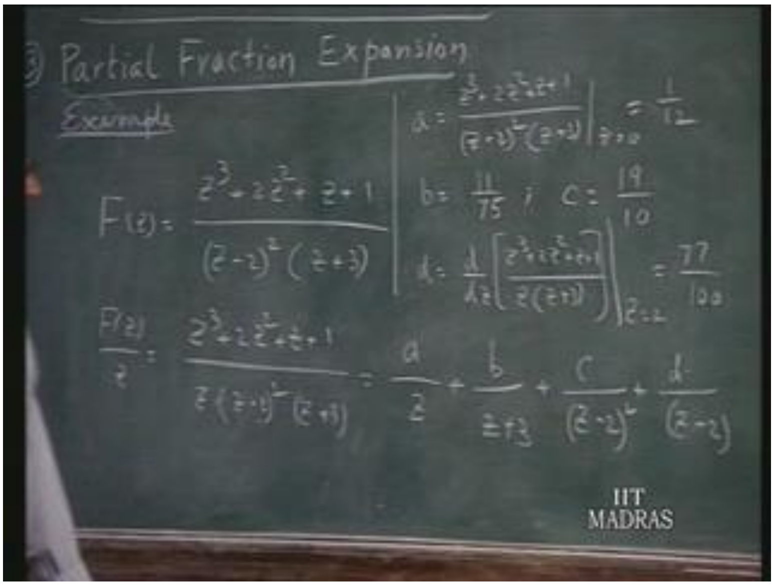

So, let us do that first. So, a equals Z cubed plus 2 Z squared plus Z plus 1 divided by here multiply in this manner. Therefore, this disappears Z minus 2 whole squared times Z plus three by exactly equals to 0. So, that will be turned out to be the value of a will be 1 by 2 12. Likewise, you can find out b in the same fashion take this Z plus 3 out and substitute Z equals minus 3, you will get b as 11 upon 75.

C likewise can be obtained by lifting this portion out and taking Z equals 2. Limit out the function value of the function Z equals 2 that turns out to be 19 upon 10. To obtain d, you multiply this by Z minus 2 whole squared and take the derivative of the remaining function take that Z equals 2 just as we have done in the case we repeated. Poles in the continuous time situation where the laplace transform case whenever we have repeated poles.

We said multiply this by Z minus 2 whole squared and take the limit take the derivative of this with reference Z and take the value of Z equals 2. Much the same thing we do here. Therefore, d will be d by dz of a function that is obtained by Z minus 2 whole squared that is Z cubed plus 2 Z squared plus Z plus 1 divided by Z upon Z plus 3 and evaluate this at Z equals 2. So, I will not do the complete work you can show this to be 77 upon 100.

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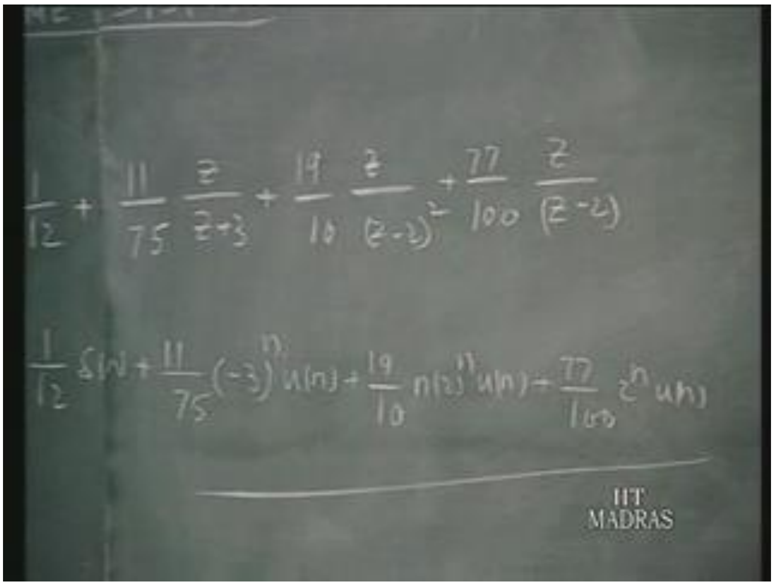

So, we have got all the partial fraction terms now a b c d. Therefore, f of n can be written as corres and now you got this this is partial fraction expansion of F of Z over Z. So, that F of Z can be written as 1 12 plus 11 by 75 Z upon Z plus 3 this quantity plus 19 upon ten Z over Z plus minus 2 whole squared plus 77 upon 100 Z upon Z minus 2.

So, that will be from this you can get f of n 1 by 12 delta n that means there is only 1 sample standing at n equals 0 plus 11 by 75. This will be Z upon Z plus 3 correspond to minus 3 raised to the power of n. Then this will be nineteen upon ten this is Z upon Z minus 2 whole squared. Therefore, you have again n times un n times 2 to the power of n un plus 77 upon 100 2 to the power of n un. So, that is what this.

So that is the final expression for the discrete time function corresponding to the given F of z.

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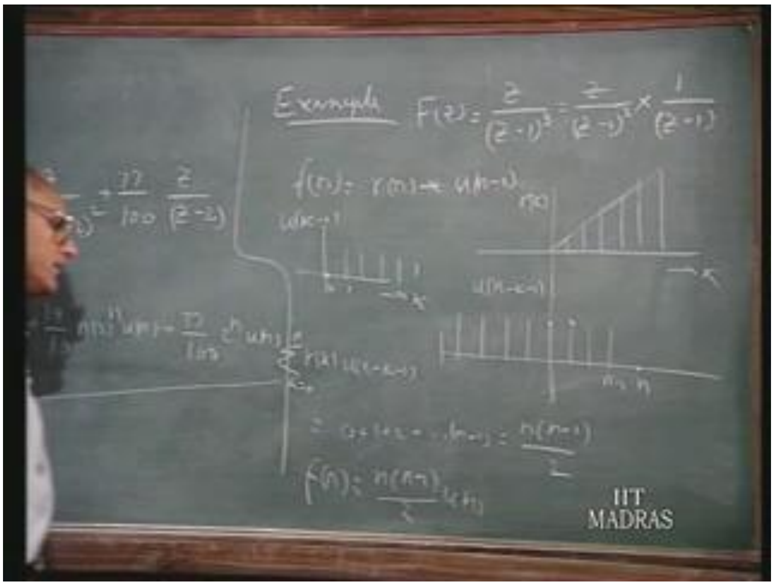

Lastly, let us take tae another example. Let us take F of Z to be Z upon Z minus 1 whole cubed. Now, we knew this as the product of Z upon Z minus 1 whole squared and 1 upon Z minus 1. So, F of Z is the product of 2 functions we know the inverse transform of this that is r of n this is inverse Z transform of this is u of n minus 1. Therefore, f of n is the convolution of the 2 corresponding time functions because; the convolution corresponds to the product in the transform domain.

Therefore, this is rn convolved with u of n minus 1 rn convolved with u n minus 1. So let us see what this implies you have rn so it will be like this. So, this is k rk. Now, this must be convolved with u n minus 1 u n minus 1 u k minus 1 starts like this. This is u k minus 1 with reference to k this is equal to 1 this is equals to 0. Now, you fold it that means the starts with 0 and comes like this and you advance it by n units. Therefore, you have this is u of n minus k minus 1, where the nth sample is here this is n minus 1.

So, what we are being this you are multiplying this different values of n and summing them up. That means, you multiply this sample with this value multiply this sample with this value and so on. Therefore, whatever you are having here this is equals to rk summed u of n minus k minus 1 summation from k to 0 to n. Means; you are multiplying these samples by unit values and then summing them up. So, for the for this particular situation first n minus 1 samples are taken in the rk and then summed up.

That means, you are taking 0 1 2 right up to n minus 1 and that will fetch you the answer n into n minus 1 by 2. So, your f of n now is in this as as a result of this can be written as n into n minus 1 by 2 un. Because, if n is negative then, it will be shifted in this direction and there will be low overlap of this sample that is going to be 0. So, whatever you are having here will be valid only for positive n. So, f of n happens to be n into n minus 1 by 2 times u of n. So, f of n happens to be n into n minus 1 by 2 times u of n.

So, in this lecture, we started with the proof of the final value theorem this statement of which was given in great detailed in full detailed in the last lecture. And we said the final value theorem as in that restrictions it will be valid only if 1 minus Z power minus 1 multiplying by F of Z has no poles either on the unit circle or outside the unit circle. This as 2 examples illustrated in the application in the final value theorem. Then we talked about finding our invest Z transformation.

That is to recover the time function discrete time function corresponding to given F of Z. We saw that three basic ways the contour integration method the long division method and the partial fraction expansion method. The partial fraction expansion method is a most suitable 1 for our needs because; we back up the given function into simpler functions which are easily recognized as a Z transforms of known discrete time functions. This does not require any complex integration the complex plane in the contour integration.

Also we get closed forms are solution which may or may not be available in the straight forward division case. And further more out the partial fraction expansion method which is something which we are familiar with in the common test of Laplace's transformation theory. And therefore, we saw how the partial fraction expansion method is useful to us by working out a series of examples where involving simple functions. And we said that wherever linear terms are involved that is in the denominator has 1 pole in the in the various single poles terms in the partial fraction expansion.

It becomes convenient for us to make the partial fraction expansion of F of Z over Z then F of Z. And when quadratic terms are involved it we may do it by taking the partial fraction corresponding to 2 complex poles or alternatively we can associate this, with the Z transform of a tamped sinusoidal function e to the power of minus n alpha times sin n omega or cos n omega in the case may be.

It is also, sometimes convenient for us when we have got a complicated term to view this as the product of 2 functions and then, used the convolution principle to find out the associated discrete time function as the last example showed. We will in next lecture discuss some other the applications of Z transform theory and find the solution for the difference equations using the Z transform of approach.