Networks and Systems

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Lecture - 15

In this lecture, we will continue with a study of the properties of the Fourier transform. We have been talking about this property of translation in time. You recall but what we studied here is f of t has the Fourier transform f j omega then, the same function shifted in time translated in time by an amount d f t minus d will have the Fourier transform. Which is f j omega multiplied by e to the power of minus j omega d and we observed that, the magnitude spectrum of f of t minus d and f of t remain 1 and the same. Means only the phase spectrum gets shifted. The phase in fact gets smaller by an amount omega d proportional to omega.

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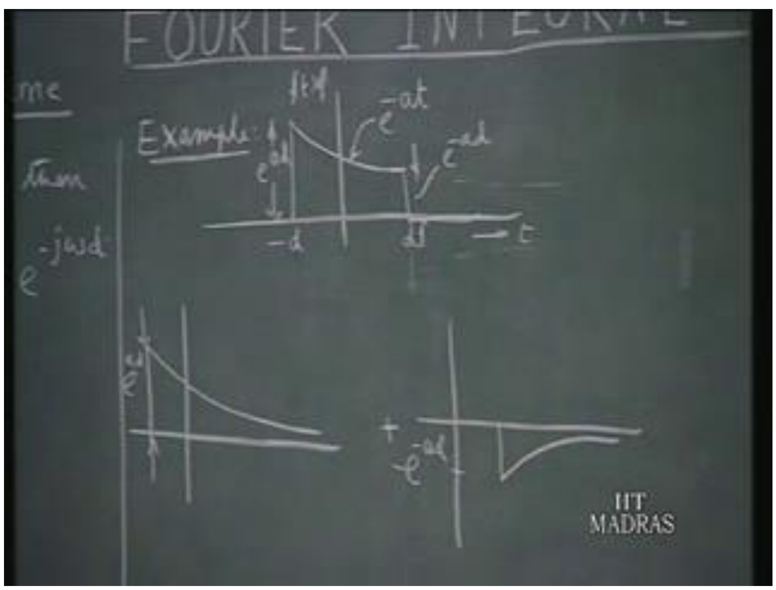

Let us work out an example how this property can be used to find out the Fourier transform of certain signals. Suppose, we take a function which follows the rule e to the power of minus a t for finite duration from minus d to plus d and it is 0 everywhere else. So, that is the function we are talking about. We can find the Fourier transform by integrating this f of t multiplied by e to the power of minus j omega d from minus d to d. Alternatively, we can make use of the properties of the linearity of the Fourier transform as well as the time shift property which we have just been discussed.

What we can do is this f of t can be taken the thought of as composed of 2 functions like this plus another starting here decaying like this. The magnitude of this is of course e to the power of minus a t therefore this will be e to the power of ad. So, that is the amplitude of this and this decays by exponentially as e to the power of minus a t from this point onwards.

So, what we are saying is this composite function can be thought of as the continuous function like this it continuous forever. And then you clip this style by introducing negative exponentially decaying function like this which is given by this. So, now what is the amplitude of this? This must be exactly equal to so that from that point onwards e decays to 0. So this will be e to the power of minus r e to the power of minus ad this is this quantity is e to the power of minus ad.

So, a negative going view form like this will do the trick so we have to find out the Fourier transform of this and the Fourier transform of this and add them up that will give the Fourier transform of the composite wave.

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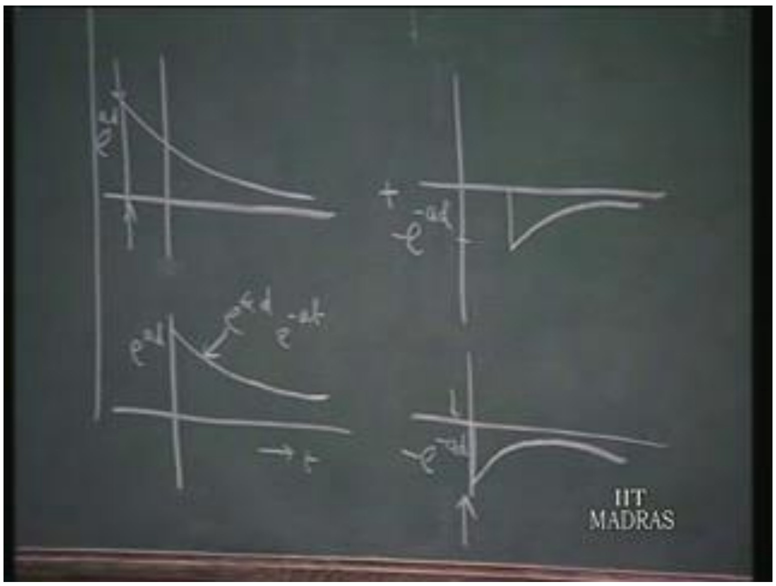

Now, to find the Fourier transform of this what we can do is let us shift this to by an amount d. So that, this is e to the power of ad and this delays e to the power of ad multiplied by e to the power of minus a t and this is exponentially decaying function like this. If you know the Fourier transform of this we can find the Fourier transform of this because, this is translated into time by an amount d.

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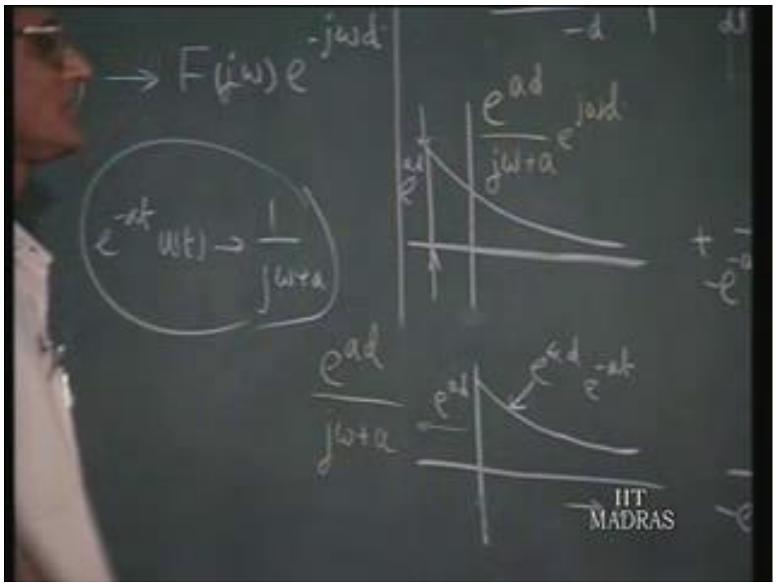

Similarly, as far as this is concerned if you can find the Fourier transform of this with initial amplitude e to the power of minus ad. Then you can find the Fourier transform of this and this is a function for which we already have you recall, that we said e to the power of minus a t u t has the Fourier transform 1 over j omega plus a. This is something which we already found so as far as this is concerned, its Fourier transform would be instead of unit amplitude e to the power minus a t 1 at time t equal to 0 it has an amplitude e to the power of a d. So, this Fourier transform of this would be e to the power of ad.

The Fourier transform of this would be e to the power of ad divided by j omega plus a. 1 over j omega plus would be the Fourier transform of exponential decaying function starting with unit amplitude start with but since, this has the amplitude e to the power of ad start with e to the power of ad by j omega plus a. Now coming from here how do we go to this? This is f of t this is f of t plus d because, it is advanced in time. If this is if you call this is f of t this will be f of t plus d so we go back this rule, this rules applies incidentally whether d is positive or negative. So, this is f of j omega for this as far this wave form is concerned it is e to the power of ad by j omega plus a multiplied by e to the power of j omega d.

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Now, coming to this this is similar to this except that it has the initial value of minus e to the power of minus ad. Therefore, the Fourier transform this would be e to the power of minus ad by j omega plus a with a minus sign because, the amplitude starts with an negative. Now, this function is nothing but this except it is delayed by d units. This is minus d here and this is plus d. Therefore, the Fourier transform for this is minus of e to the power of minus ad divided by j omega plus a multiplied by e to the power of minus j omega.

So, we found out the Fourier transform of these 2. So the Fourier transform of this would be the just sum of these because, Fourier transform follow the linearity property. That is if f 1 of t and f 2 of d a Fourier transforms f 1 j omega and f 2 j omega the Fourier transform of c 1 f 1 t and c 2 f 2 of t f would be c 1 f 1 j omega plus c 2 f 2 j omega irrespective of the values of c 1 and c 2 constants.

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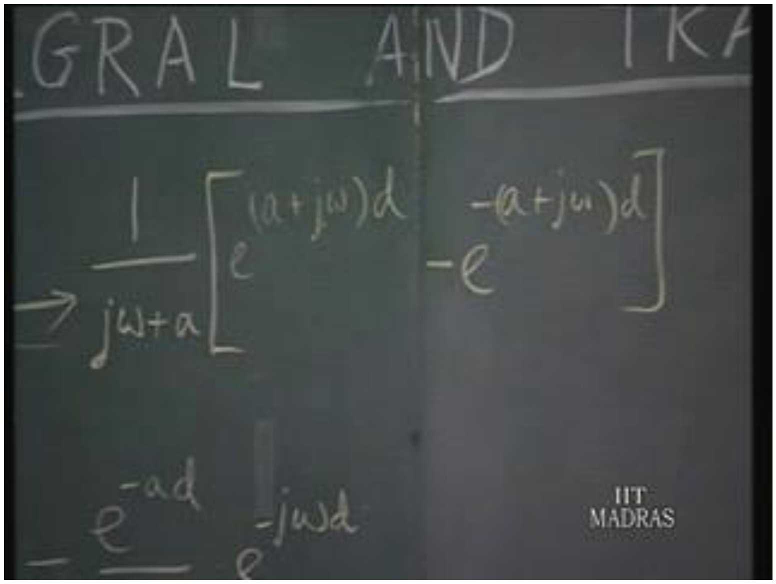

So, the Fourier transform of this now given by the sum of these 2 which of course the common factor j omega plus j. I can keep out side and you have e to the power of a plus j omega d minus e to the power of minus a plus j omega.

So, this example tell shows us that, if you recognize the function as composed of a linear combination of functions whose Fourier transforms you already know. We can intelligently get the Fourier transform of this without having to do the integration starting from scratch.

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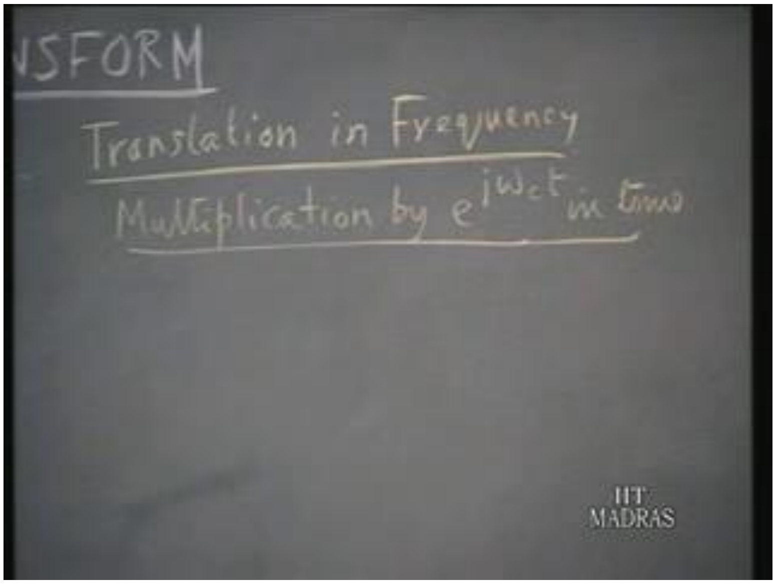

Let us take up the next property just like you have translation in time we can think of translation frequency. Which is, equivalent to multiplication by e to the power of j omega c t in time.

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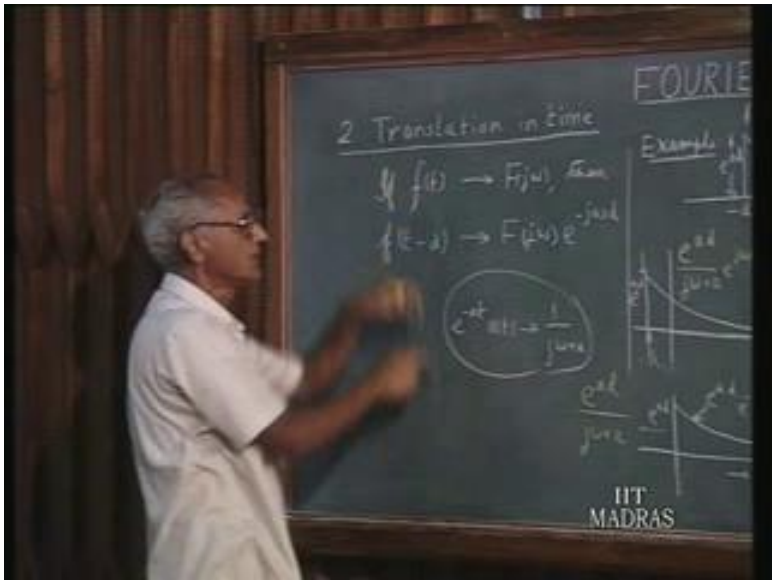

Note what we here, in the property 2 we have translation in time in time axis f of t goes to f of t minus d. In the transform domain f of j omega gets multiplied by the exponential quantity.

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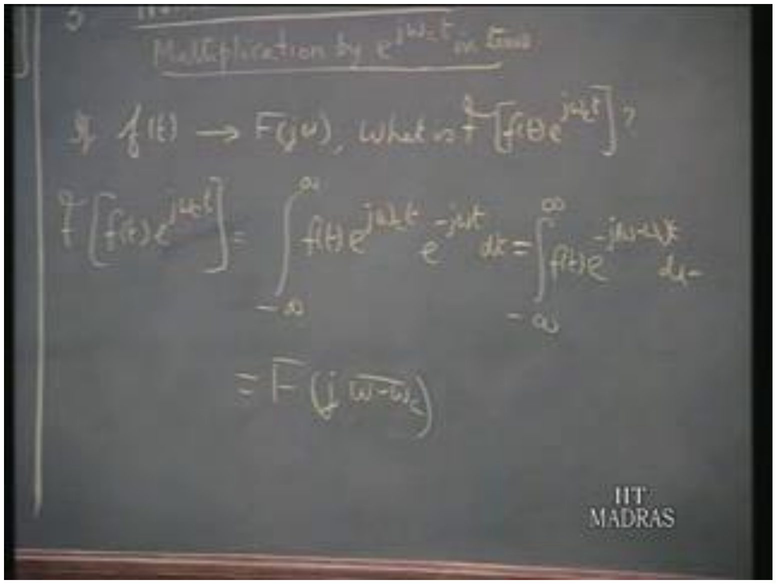

In the similar fashion what we have here is if you multiply in the time e to the power of j omega c t the Fourier transform get shifted in the frequency axis. What happens is this, if f of t has the Fourier transform f j omega. Then what we would like to know is, what is the Fourier transform of f of t multiplied by e to the power of j omega c t.

This is the question which we like to ask I multiply f of t by the exponential function e to the power of j omega c. And would like to know what its Fourier transform is going to be. So, let us do this the Fourier transform of f of t e to the power of j omega c t by definition is minus infinity to plus infinity of f of t e to the power of j omega c t multiplied by the e to the power of minus j omega t dt. And this straight away, seen to be minus infinity to plus infinity of f of t e to the power of minus j omega minus omega c.

Now, if you add f of t e to the power minus j omega t dt this would have called it f of j omega. All we have now is instead of omega we have got omega minus omega c. So, it is almost same of the defining relationship of the Fourier transform of f of t except that instead of omega we have got omega minus omega c. So, this obviously will be f of j instead of omega we have got omega minus omega c which means that, the Fourier transform of this is f of j omega. This is the same Fourier transform except that the spectrum gets displaced by an amount of omega c.

So, whatever value you take at a particular value omega takes the same when omega minus omega c takes the particular value; the function behaves the same values. So, this is again an important relation which will be useful in the contest of modulations of signals, amplitude modulations signals. Where f of t is really multiplied by instead of e to the power of j omega c something like cosine omega c t or sin omega c t we will take this up we show that presently.

But, you notice the duality between the property of translation in time and the translation frequency. When you are translating the function of time then, the transform domain the transform gets multiplied by exponential term. If you are translating into the frequency domain as we have done here then, in the time domain it gets multiplied by e to the power of j omega c t. So there is the duality between the transform domain and the time domain. Now, let us carry this illustration of this let us take up a case where f of t is multiplied by not e to the power of j omega c t but cos omega c t.

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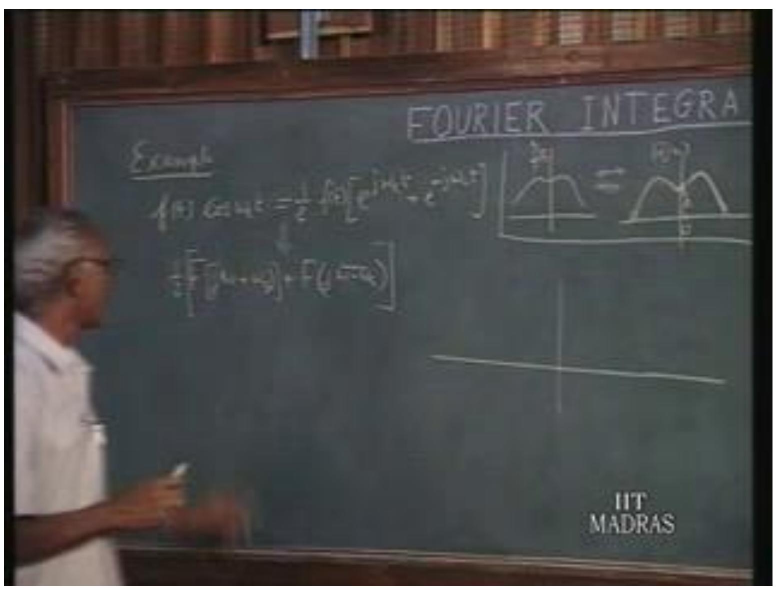

As an application of this property let us take this example, where f of t is multiplied by cosine of omega c t, we would like to find out its Fourier transform. Now, we can express this of course as 1 half of f of t multiplied by e to the power of j omega c t plus e to the power of minus j omega c t because, cosine omega c t is expressed as e to the power of j omega c t plus e to the power of minus j omega c t divided by 2.

so obviously then we have 2 terms: half f t e to the power of j omega c t, half f t e to the power of minus j omega c t.

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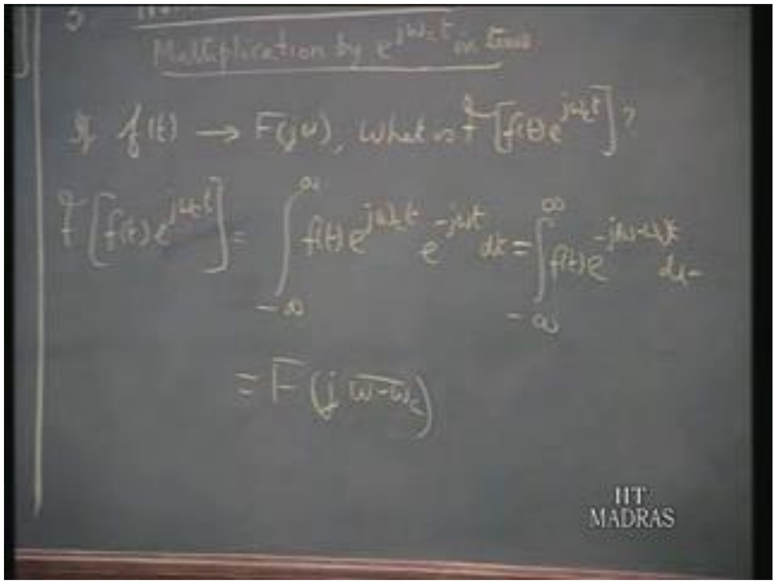

We observed in this property that if we have f of t e to the power of j omega c t then, the Fourier transform is f of j omega minus omega c. If you have minus omega ct this would have been omega plus omega c. So, we use those 2 properties and then find the Fourier transform of this. So, the Fourier form of this would be 1 half of f of j omega plus omega c and f of j omega minus omega c that is what we have got.

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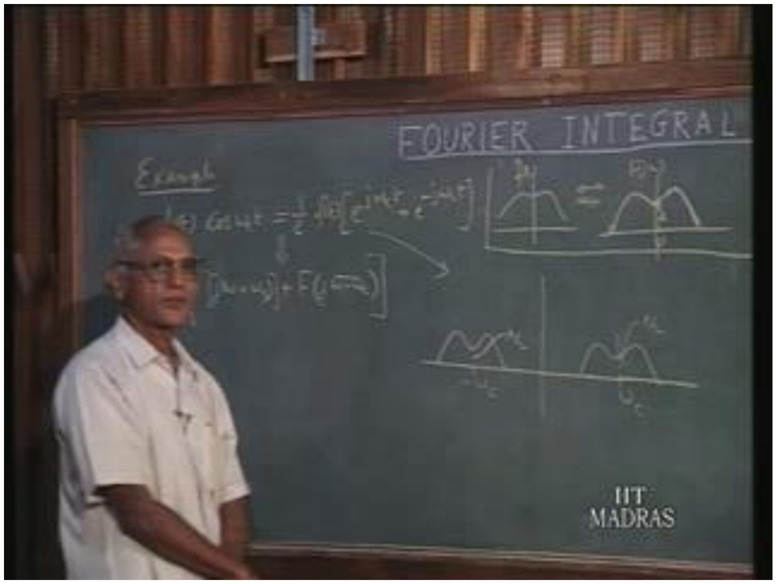

So, what is the implication of this? Let us see, suppose I have f of t and the corresponding Fourier transform Fourier spectrum let us say it is like this it is amplitude spectrum which is symmetrical. Something like this suppose, we call this just for the sake for reference let us call this a, this is 0 these 2 format Fourier transform pair.

Now, when you are talking about f of t cos omega c t. The spectrum for this now consists of 2 sections: 1 is half f of j omega plus omega c, another is half of f of j omega minus omega c. This portion is now centered around omega c and then it will have the same spectrum is reproduced here except that it is centre around omega c instead of the origin. Therefore, it will have something like this if this is a this would be a by 2 because it is half of that.

Similarly, f of j omega plus omega c is the spectrum which is similar to this except that reduced by the scale factor half and it is advanced by an amount omega c. Therefore it will be centered around minus omega c, so it will be like this. So, this is the spectrum of this 1 and what is the how this spectrum is related to original spectrum. We can say that the original spectrum gets split into 2 half equal half and 1 half shift forward and sit around omega c instead of 0. The other half is shifted in the reverse direction and centered around minus omega c instead of 0.

That means, the essential shape of the spectrum remains the same except that has been centered around the origin in the frequency domain it is centered around omega c and minus omega c. And this is common place operation in communications and the amplitude modulations. Where, information is contained of the signal is purposely shifted from the instead of being centered around the d c it is shifted to a more convenient location of the frequency axis .It is shifted to be centered around plus omega c or minus omega c this is called the carrier frequency. This is done in the interest of 2 things: 1 first of all the for the transmission of the information it will be more convenient the information is centered around a higher frequency rather than the d c.

And secondly the same channel communication channel more effectively used by having several items of information in a more convenient location. Instead of that is you can have another information channel here, another information channel here. And therefore, it makes for more effective communication in a given channel because of the reasons it is convenient to modulate a signal f of t by multiplying by trigonometric function of this type. These details of this of course will have in your communication theory but, as far as we are concerned we just want to find out an application of this property of the Fourier transform.

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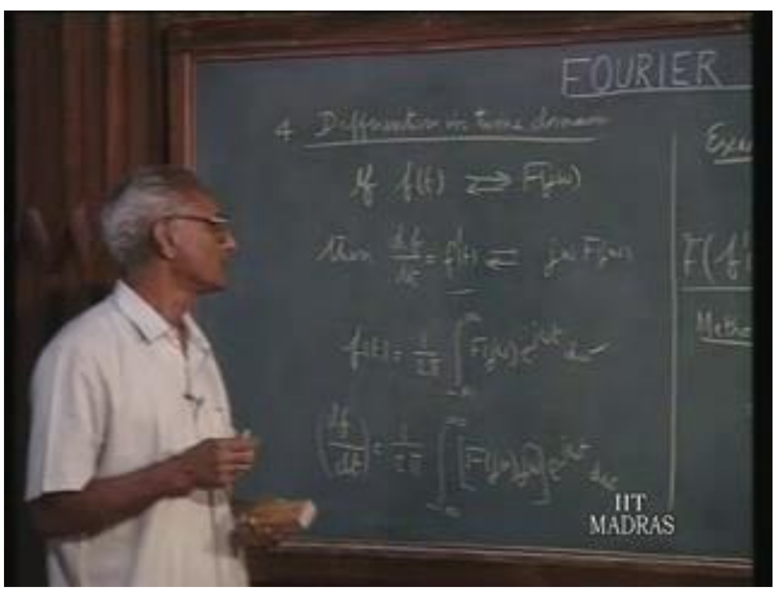

Now, let us take up the next property let us now enquire into what happens when f of t is differentiated in time domain? What is it affect in the transform domain? So, we will say differentiation in time in time domain. What is it affect in the transform domain? If f of t and f of j omega form a transform pair then, this particular property tells us df by dt which may like to call f prime t has its Fourier transform which is obtained from the original Fourier transform by multiplying by j omega.

So, the differentiation time domain corresponds to multiplication transform domain by j omega and this is 1 of the very useful prosperities of a transform of a particular time function. Because, the calculus that is involved in the differentiation is replaced by an algebra algebraic multiplication j omega and this is a trend which we noticed as we go along not only in Fourier transform but laplace transform also. Operation involving in calculus integration and differentiation or converted into multiplication or division by j omega and makes for the great convenience as far as the working is concerned.

Will talk more about later but let us see why this is so, f of t can be obtained from f of j omega to the inverse transformation. That is f of t can be written as 1 over 2 pi from minus infinity to plus infinity of f of j omega e to the power of j omega t d omega. Now, let us differentiate this signal, so d f by dt the derivative of this with respect to time df by dt. And assuming that we can differentiate under the integral sign here, if we carry that out here this will become 1 over 2 pi from minus infinity to plus infinity f of j omega j omega e to the power of j omega t d omega. Because, this is the only function of time inside the integral sign.

So, what we have here is a quantity f j omega j omega multiplied by the e to the power of j omega t d omega. So obviously, whatever we are having here is obtained is 1 over 2 pi times minus infinity to plus infinity plus some quantity function of omega into e to the power of j omega t d omega. This is the defining relationship of a inverse Fourier transform. Therefore, d f by dt must be the inverse Fourier transform of f of j omega j omega, so that is what we have here.

If we have j omega f of omega its inverse Fourier transform is df by dt or in other words d f by dt f j omega j omega f j omega is the transform of d f by dt.

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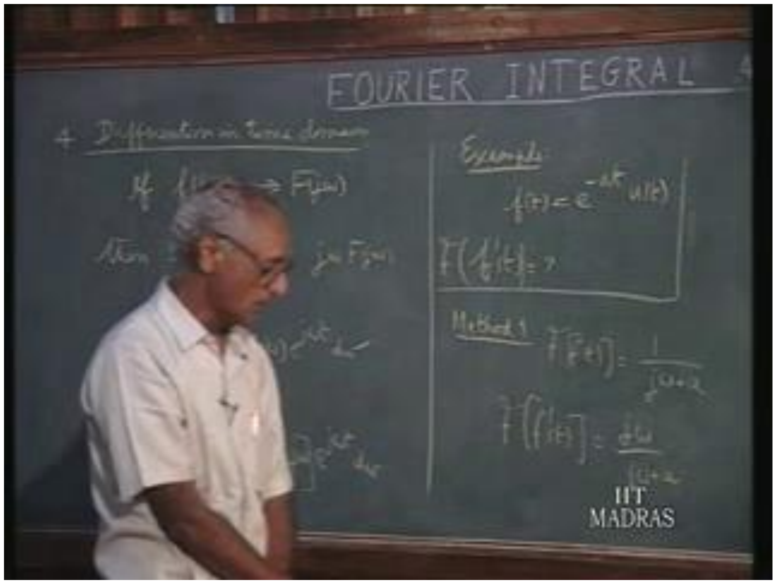

So, this is the very useful result and to illustrate this let us work out an example once again. Let us take a simple signal f t is e to the power of minus a t u t. Let us work this the Fourier transform of f dash t in 2 ways 1 by straight forward method by differentiating this and then trying to find out its Fourier transform. Or alternately finding out the Fourier transform of this and then finding Fourier transforms the differentiated signal.

So what we look to find out is Fourier transform of f prime t is what. So the method 1: Fourier transform of f of t this is the function which we have been talking about all the time is 1 over j omega plus a. so the Fourier transform of the derivative function f prime t is j omega over j omega plus a. That is all it illustrates, it is quite simple and straight forward. Now, we like to check on this result by actually carrying out the differentiation in time domain and finding out its Fourier transform.

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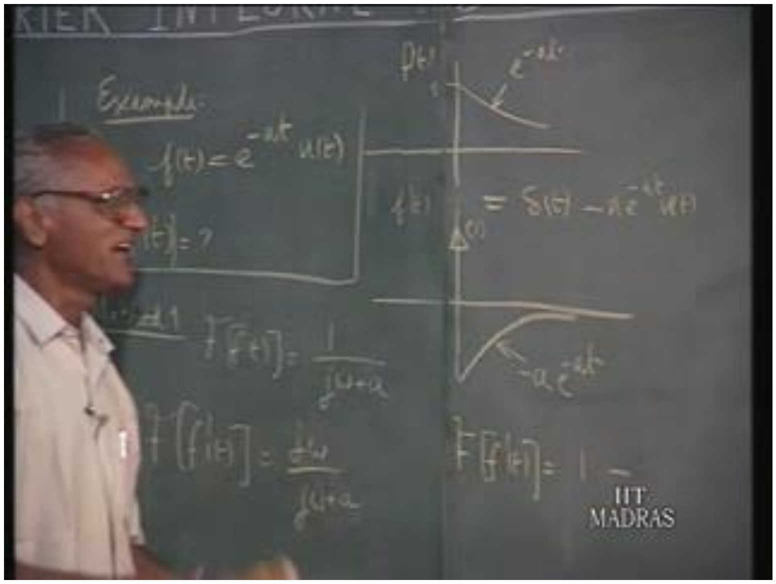

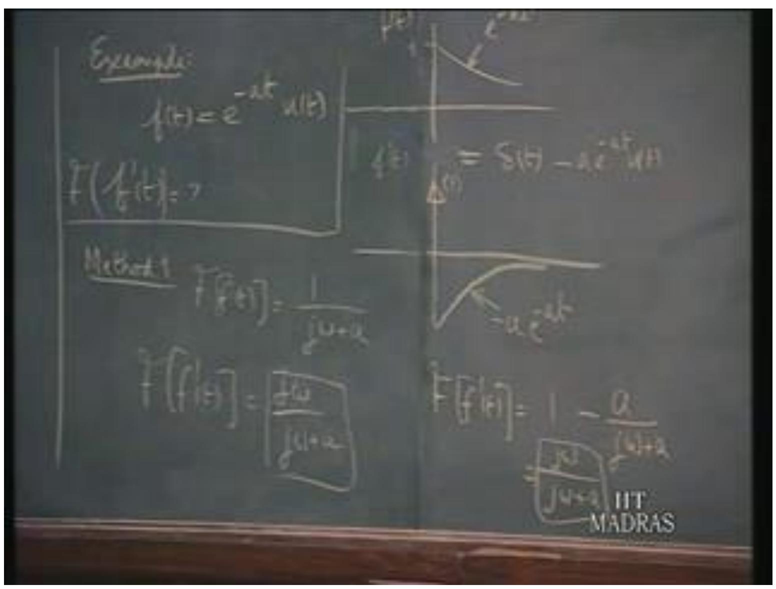

So, let us see what it is. This is the function f of t this is 1 this is e to the power of minus 1. If you plot the f prime of t its derivative e to the power of minus a t its derivative will be minus a e to the power of minus a t. So, you have minus a e to the power of minus a that is now that is not all, why? The Fourier transform can considers the time function extending from minus infinity to plus infinity therefore; we must take the derivative at every point.

Now, you observe this f of t all 0 all along and suddenly it jumps to the value 1 therefore at this point jumps from 0 to 1. Therefore, there is the derivative must be infinitely large and that quantity when integrate at must be raised to 1 therefore obviously what we are having here is the delta of magnitude 1. So, f prime t really is not simply given by this alone but also you must have a delta function.

So, in other words f prime t is equal to delta t plus this quantity which is minus a e to the power of minus a t because, this function starts only from t equal to 0 onwards. So, f of j omega for this the Fourier transform of f prime t would be the Fourier transform of this plus the Fourier transform this. The Fourier transform of delta t is obviously 1 and e to the power of minus a t u t as 1 over j omega plus a except we have 1 more a here and negative sign minus a over j omega plus a.

This is in the j omega over plus a result which is an accord with. So, the important factor fact that example brings up is for 1 thing when we are taking the differentiation you must also consider the derivatives associated in the jumps as shown here. And secondly, to find out the Fourier transform of differentiated signal we do not have to carry out the differentiation time domain. If you know the Fourier transform of this signal all you have to do is multiply by j omega.

so this makes as a pointed out for great convenience in manipulation of differentiated signals. Now, I would just introduce a topic we will not carry this out. What 1 might argue is if the differentiation is equivalent to multiplication by j omega in time domain would it mean that integration will be equivalent to division by j omega.

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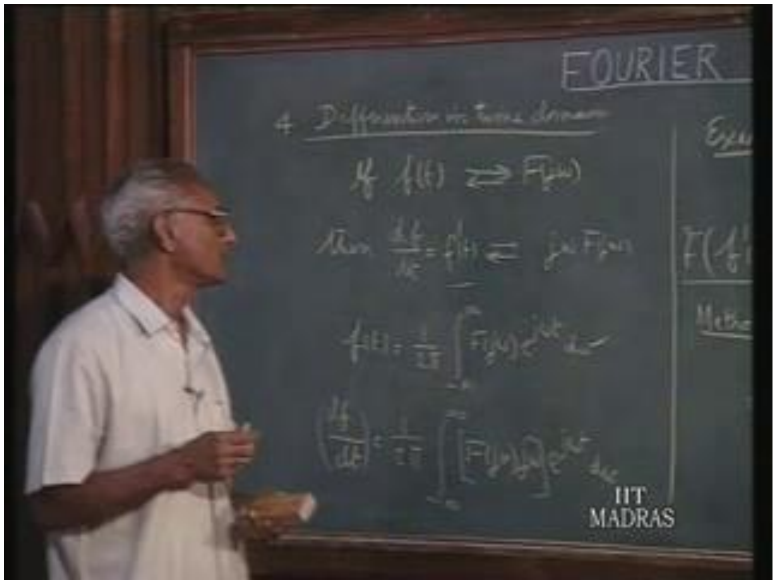

After all this signal is obtained by integrating this signal therefore, if you have some Fourier transform for that. The Fourier transform the original signal must be obtained by dividing j omega. This is true up to a point but this is not quite true. In other words what you like to point out postponed the discussion to later stage.

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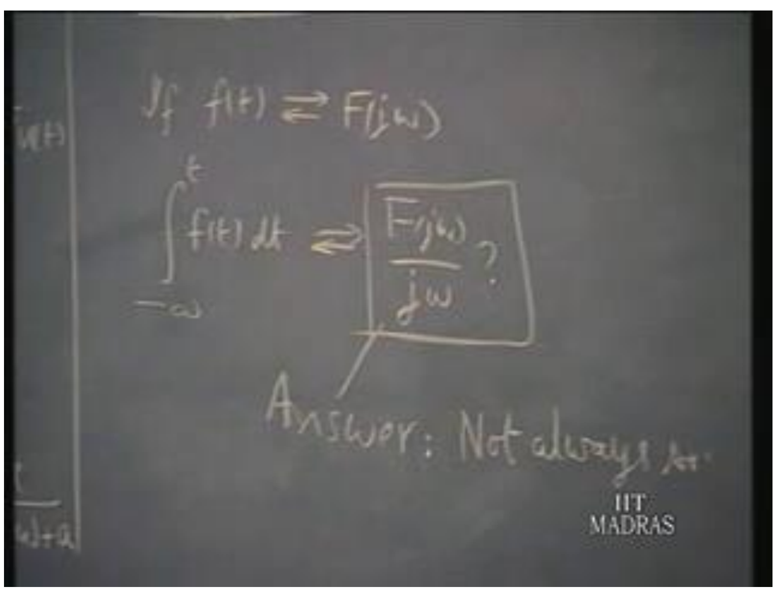

Integration if f of t has the Fourier transform f j omega then integration then also say minus infinity to t f of t dt. What its Fourier transform? Is this f j omega by j omega? This is the question which we like to ask, the answer is not always so. So, you must be careful try to integrate the signal in finding out the Fourier transform. We will discuss this at later point of time after we find out the Fourier transform of the step function.

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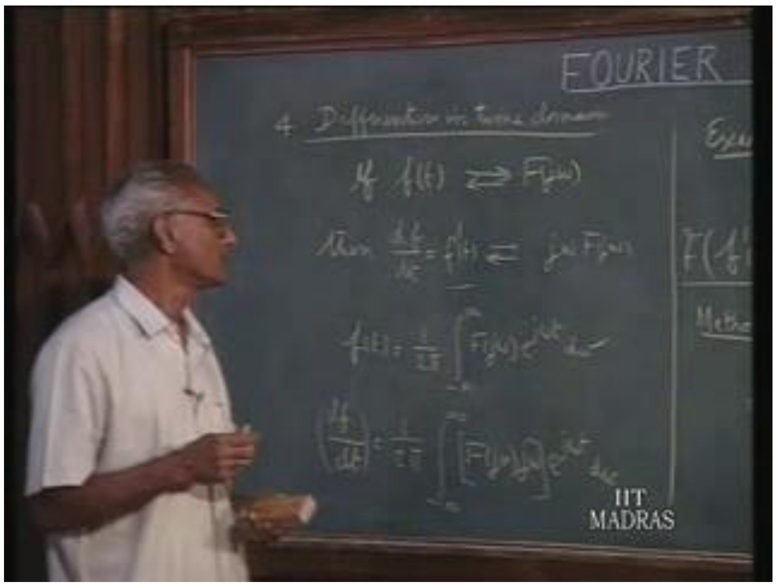

So, we will postpone the discussion of this what I would like to caution you at this stage is you should not jump to the conclusion that because, differentiation involves multiplication in j omega integration means division by j omega in the transform domain that is not so. We have so far seen how certain operations in time domain how they reflected in transform domain for example; we thought we saw how the function of time shifted in time what it affects in the transform domain.

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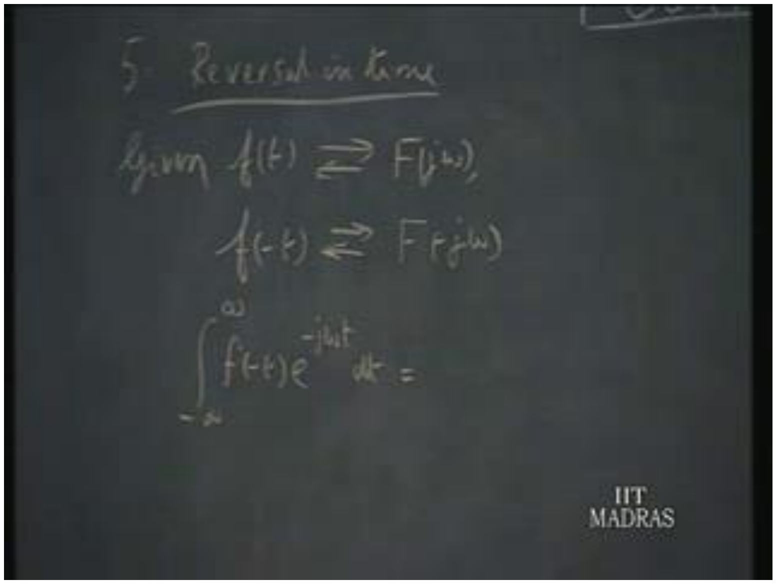

When it differentiated in time what is the effective transformation domain these property we have studied. Let us now continue this trend take up now the case where it is reversed in time. So, if f of t and f of j omega form a transform pair we want to know what the corresponding f of j omega for f of is minus. In the sequence of even takes place in the reverse direction what is the corresponding f of j omega? The answer simply is f of minus j omega.

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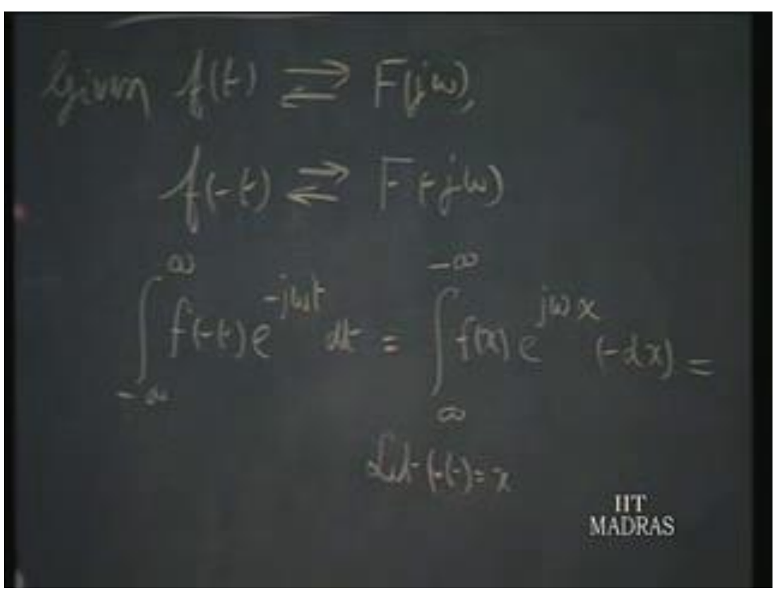

It is straight forward to show this because after all you have now minus infinity to plus infinity of f of minus t e to the power of minus j omega t dt. You can replace, let minus t to be x then I can write this as: f of x e to the power of j omega x minus dx dt I will put minus dx. And when t equal to minus infinity x will be plus infinity and when t equal to plus infinity x equal to minus infinity. And this can further be written as this minus sign and that integral limits can be taken care of by putting this plus sign and reversing the integral limits.

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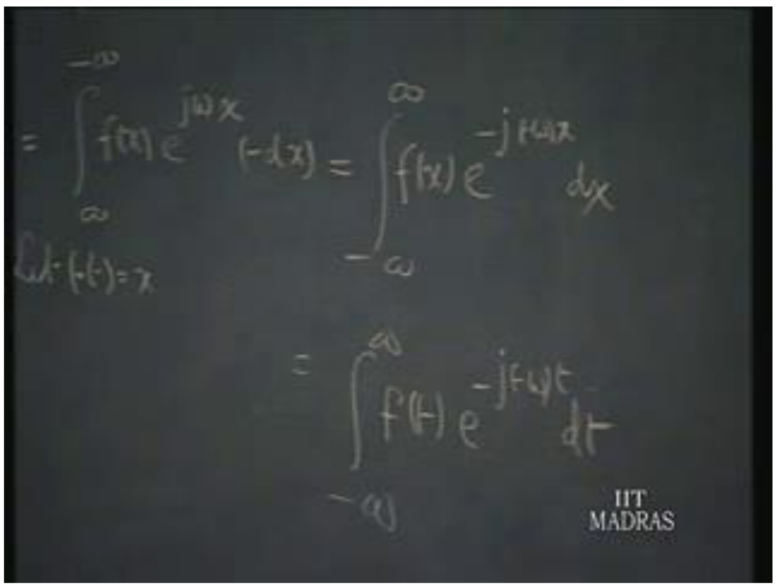

Therefore, I can write this minus infinity to plus infinity of f of x e to the power of minus j minus j omega x. So instead of plus omega x I put minus omega x therefore I introduce the minus sign here this of course written as minus infinity to plus infinity after all x is the dummy index this is f of t e to the power of minus j minus omega t. So, the difference between the relationship of the Fourier transform of f of t and this is instead of plus omega I have got minus omega. Therefore, this is f of minus omega right.

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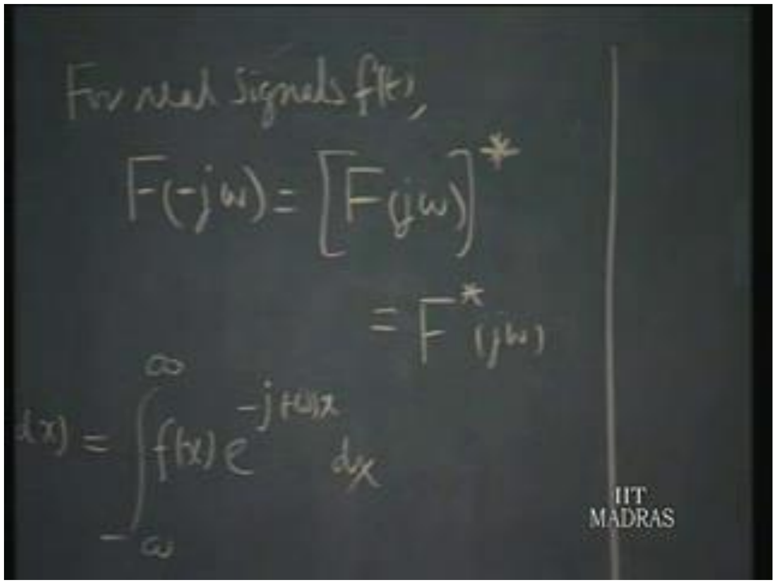

What is the further significance of f of j omega? For real signals f of t that means, when ever substitute real value of time the f of t is also turns out to be real then it turns out that f of minus j omega equals f of j omega conjugate. That is the angle will be reversed the magnitude will remain the same you can write this also f star there that is in other words conjugate f of j omega.

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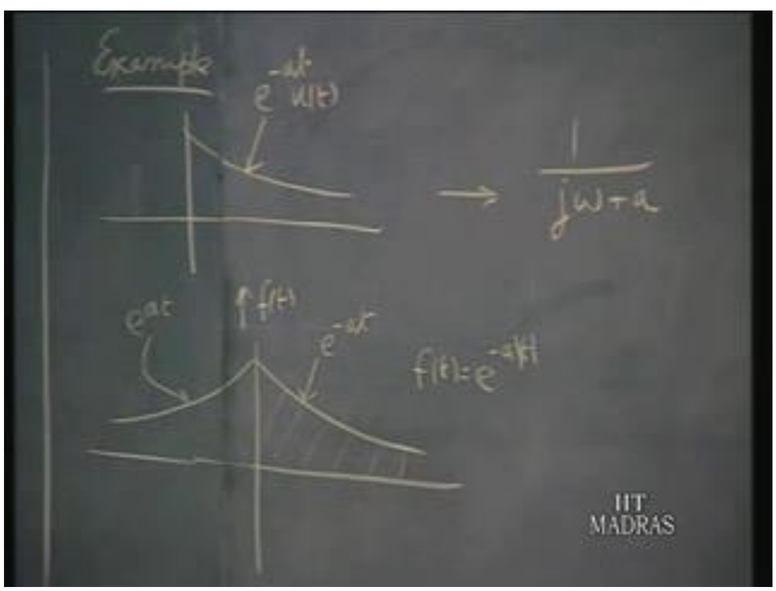

Again according to our practice let us work out an example. This time function e to the power of minus a t u t we know the Fourier transform 1 over j omega plus a. Now, I would like to enquire about the Fourier transform of a time function which is given by; e to the power of minus a t for positive t and e to the power of plus a t for negative t. That means, this portion is e to the power of minus a t. It is a reflection on the negative side this will be e to the power of a because, for negative t the exponent will become negative and again decrease.

So, this f of t that we have now can be described as f of t is e to the power of minus a t magnitude. So, the composite curve can be described analytically in this fashion. Now, what is the Fourier transform of this it is quite straight forward because, we know the Fourier transform of this section. The Fourier transform of this section is 1 over j omega plus a we know that. And what is this time function? How it is related to this?

If this is f of t this is f of minus t the whole sequence of even that takes place in the positive time axis will now be taken in the negative time axis in the same order. Therefore, the Fourier transform of this if it is f of j omega the Fourier transform of this is f of minus j omega.

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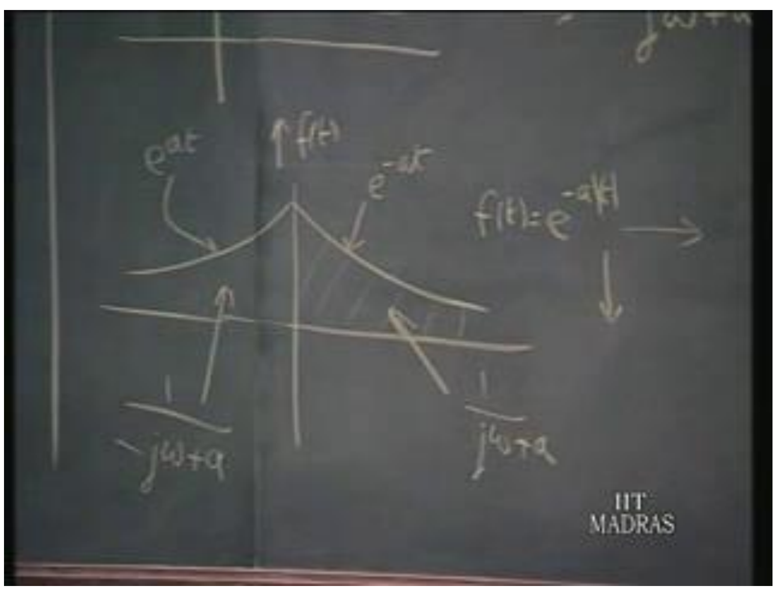

After all, what we have here is u f of t is the summation of these 2 curves. Therefore, we can say its Fourier transform is the Fourier transform of this is 1 over j omega plus a and the Fourier transform of this portion is 1 over minus j omega plus a.

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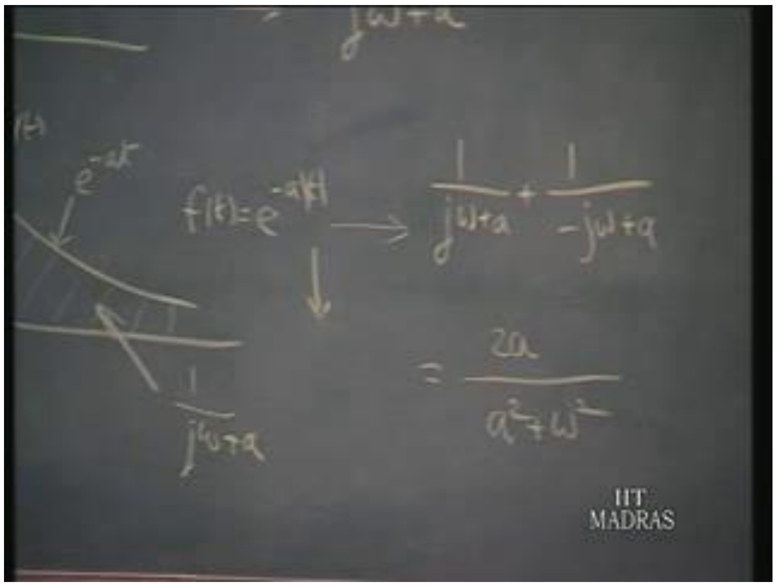

So, this particular f of t which we have given e to the power of minus k magnitude t we have the Fourier transform minus k magnitude t we have the Fourier transform which is 1 over j omega plus a plus 1 over minus j omega plus a. And combine these 2 you will get denominator a squared plus omega squared and in the numerator we have 2 a. Now, the essential motivation for our study of the various properties of Fourier transform is to find shortcut to find the Fourier transform in this manner. You do not have to try to do integrate this in both direction and get its result.

Once we know the property we easily deduce the Fourier transform of this in this manner. So, this is the real motivation for our study of the various properties of the Fourier transform. So, that we can intelligently obtain the Fourier transform for such a functions and also find the inverse Fourier transforms of certain functions, once we know the various properties. Let us now, continue our discussion additional property where, we are thinking of scaling in time domain. So far, we talked about translation time domain, reversal in time domain, differentiation time domain.

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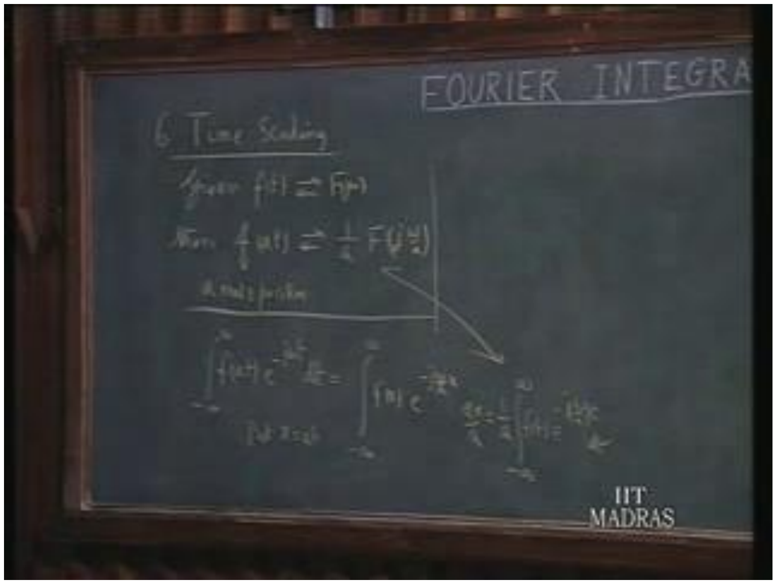

Now, let us consider what happens if you scale the function in time domain. Again, we start with basic f of t f of j omega pair. Then, I would like to know what is the Fourier transform of f of a t where a is real and positive. Then it turns out to be this 1 over a f of j omega over a. Now, how do establish this? Proof is again straight forward we take the Fourier transform of f of a t which is minus infinity to plus infinity of f of a t e to the power of minus j omega t dt.

Now, by this time you must have got an idea how to go about this we must substitute some x for a t. So that we try to bring this into a form which we can compare with the Fourier transform of f of t. Therefore, if you substitute x for a t then this becomes when t becomes minus infinity x also becomes minus infinity. When t becomes plus infinity x also becomes plus infinity f of x and e to the power of minus j omega t equals to x up on a and d t is d x up on a.

So, this is obviously 1 over a from minus infinity to plus infinity of f of x e to the power of minus j omega by x d x. And since, the dummy variable x in the integration I can as well replace it by t. This will be f of t e to the power of minus j omega by a times t dt and this clearly is 1 over a times f of j. Instead of omega you have got omega by a therefore in the defining relationship of f of j omega t you would had f of minus j omega t.

But, instead of omega you have got omega by a so instead of being f of j omega it becomes f of j omega by a and this a is stitching outside.

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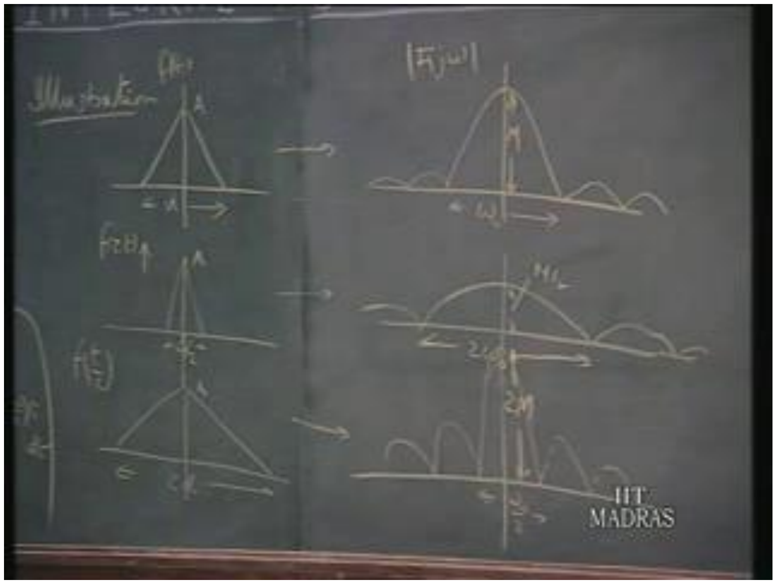

So, this again has got some important aspects in relation to scaling of time functions. Let me illustrate this; illustration of this suppose we had a time which is given like this a and stitching from over an interval d. And let we imagine that its Fourier transform would be like this magnitude spectrum I am plotting and this is equal to let us say some omega I will say some omega naught whatever it is. And then, this is a this is the Fourier spectrum for this magnitude spectrum. Now, let us see what would happens if I compress the in the time scale that means, the evens now come closer together.

So, I have the same a but the evens occurs faster so this is d up on 2. So, the same information sequence of information evens now compressed in times they are carrying over a small interval of time t up on 2. So this is f of t this will be f of 2 t because, at a smaller value of t you get the same original values.

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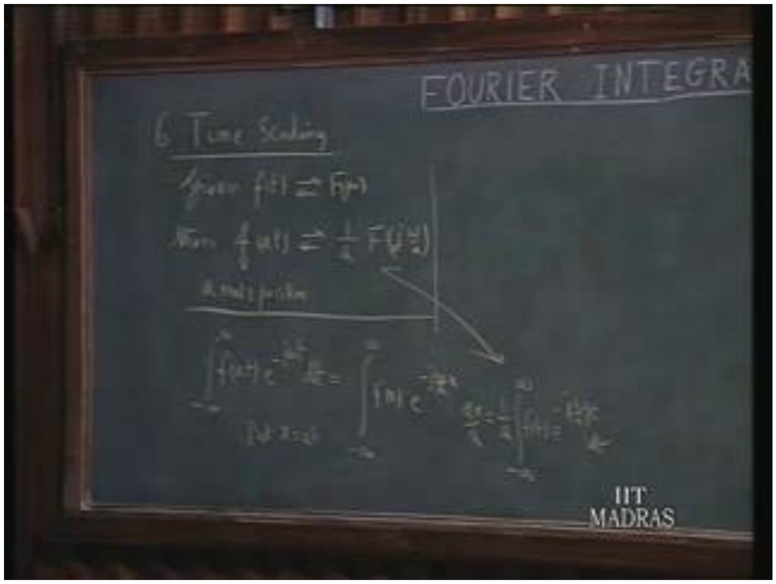

What is the consequence of the spectrum? From what we have seen here first of all f of a t 1 over a the amplitude must go down by factor half and then omega by a. So, it means j omega by 2 it becomes that means it get spread out. So that means the amplitude like this so, this will become m up on 2 and this will become 2 omega. On the other hand if I allow the things to take place more. So, let us say this is 2 d and this is a this will be f of t up on 2. That means, it takes the longer time for a particular value to reach compared to this, the t must be doubled the same values what it had in the case of f of t.

So, in this case a is equal to half in the original formula that we have here a is equal to half now therefore, it means as far this spectrum is concerned it will be 2 m. This will become larger but 2 f of 2 j omega therefore, in the frequency spectrum things becomes crowded now. So it means, it will be very large compare to this and this become omega up on 2 omega not up on 2 and this will be 2 m.

So, what it really means is the significance of this is, that if you allow certain events to come faster become faster. Then it requires as the larger frequency components; larger frequency component compare with this you have got certain frequency component. And then this spectrum is spread out that means you have more high frequency components.

So, if you have a time signal in which events occur faster then it requires higher frequency components. On the other hand the same sequence of even take place more lesser fashion that means, they get more of lower frequency terms coming to the picture and their amplitude also gone. That is what it is things which occur faster require high involve high frequency component and thing which occur slow fashion which has a greater concentration of low frequency components.

So, in other words the slower the rate of change then you need to have a smaller band information transmittance is concerned. But, the other hand you like thing to happen fast if you have got faster changes then you require a larger band between higher frequency components both. So, this is the important property in the information transmission which we have we can say information centered a faster rate needs to have as higher frequency component. And therefore, the channel must also have the corresponding bandwidth to handle such signals.

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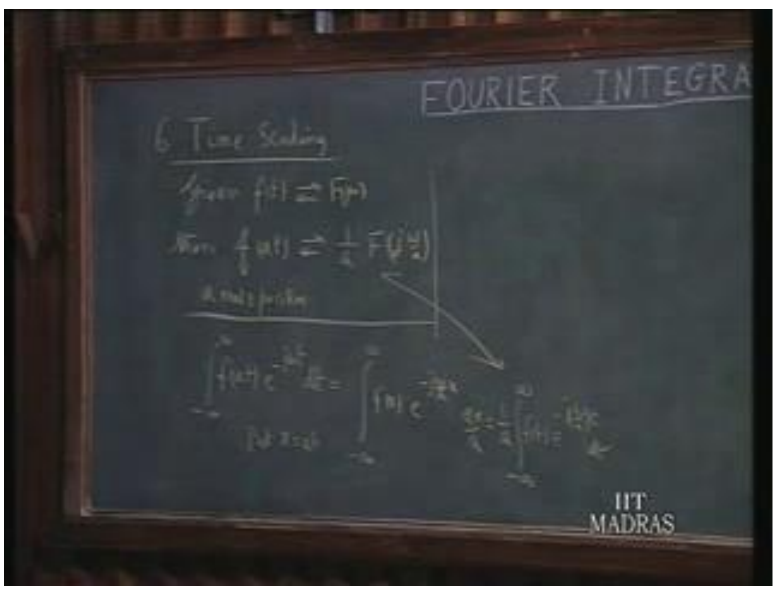

If we have seen now that f of t it goes f of j omega f of a t is 1 over j omega by a. And we assume a is positive real and positive so that, we are talking about a b is greater than 1 and less than 1 and we talked about this. It will be of interest for us to know what happens when a is negative.

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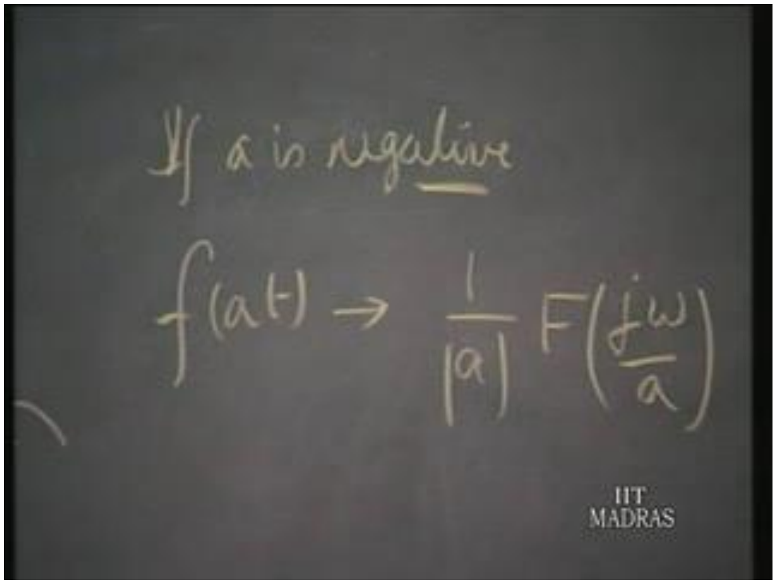

Then it can be shown that if a is negative f of a t as far as its Fourier transform 1 over a magnitude f of j omega over a. So, all it means is the magnitude of a comes into the picture rather than the absolute value actual value of a. So, this is a result which can be derived on the same basis. So, all it means is the scale factor is now depends of the magnitude of a even if a is negative.

So far, we have studied several properties of the Fourier transform you recall it talked about the linearity of Fourier transforms. Then various operations in the time domain like: translation in time domain, then multiplication by exponential factor e to the power of j omega c t. Then we talked about differentiation in time domain, then we talked about time scaling and we talked about reversal in time domain. And saw, what all these operations in time domain how they are reflected in the transform domain.

The study of all these properties will help us to find out the Fourier transform of certain functions in much more in simpler way than if you have desire to direct integration. And also when sometimes we want to find the inverse Fourier transform that means, f of j omega is given and we want to find the corresponding f of t. If we keep these properties in the back of our mind sometimes, we get the solutions in more straight forward way.