Electro Magnetic Field
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Lecture - 9
Energy in the Field

Good morning. The last eight lectures have basically set up the basic equations of electrostatics. And for the next few lectures we are going to apply these equations and get some useful results. Let us again see what we have already done. From Coulomb's law we had that the electric field is equal to 1 over 4 pi epsilon 0 volume integral over any volume, the charge density d V divided by... That is, the electric field is proportional to the charge at any point and inversely proportional to the distance between the charge and the location, where the electric field is measured. It is the square of that distance. From this, we obtain various results. One of the results we obtained was that integral E dot d l with the minus sign to any point r from some reference point which I am not going to specify is a function only of r and we call that function the electrostatic potential phi.

So this electrostatic potential phi does not depend on how we went from the reference point to r. It only depended on where r is and from this we also obtain that the electric field is minus the gradient of phi. Also from this equation we obtained that surface integral over any surface, closed surface, that is what this 0 means, of E dot d S is equal to the total enclosed charge divided by epsilon naught; and this implied in turn divergence of E is equal to rho over epsilon naught. So, E as a function of r, rho as a function of r.

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So, these two equations are the fundamental equations we work with. One says that there is something called electrostatic potential and if you take its gradient, we get the electric field. The second says that if you take the divergence of the electric field, you find the source of the field namely, the charge density divided by epsilon naught. If you combine these two equations, last class we saw, which essentially saying divergence of minus gradient phi is equal to rho over epsilon naught, which is rewritten as del square phi equals minus rho over epsilon naught and this is what we call Poisson's equation. So, if I have to give names - this is Coulomb's law, this is electrostatic potential, this is gradient, this is divergence, this is Gauss's law and this is Poisson's equation. So, this is what we have reached as of the last class.

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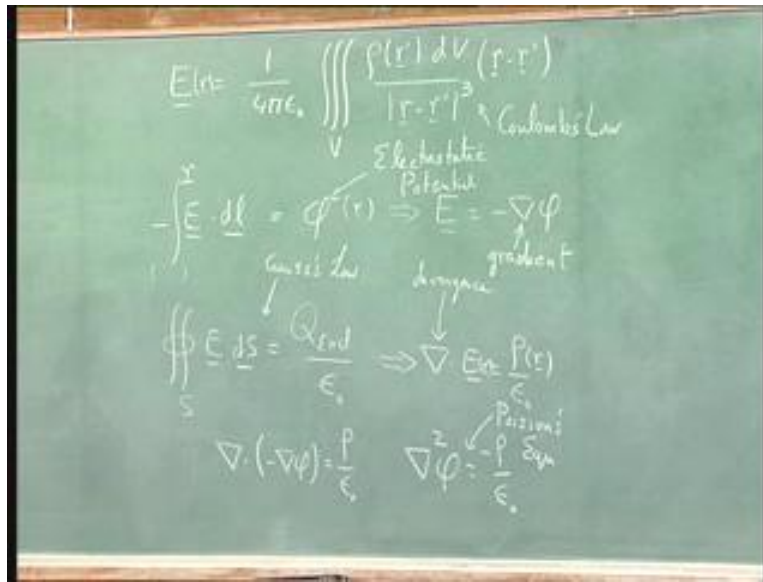
It is not very much. Almost all of it you already know. Now what we would like to do is to apply all these equations and do something useful. So for that, let me first start with the concept of energy in the field. Now we have already have met this concept before. If I have a charge Q and I have another small charge little q, we already know that however I move this charge from a position r naught to a position r, the amount of work I do to take the charge from r naught to r, work done on charge, which is equal to minus integral q E dot d l some r naught to r; it is a function only of r naught and r and therefore this came out to be q phi.

So, we have already met work. Now the question is, supposing instead of this problem, I had a slightly different problem. Supposing I had a charge Q and I wanted to bring another charge, say minus Q, close to it. Now obviously I do not have to do any work. In fact, I have to do negative work to bring the charge there. But where does that work go? If I brought a plus charge here? I did work to bring that charge there. So I did some work but where does that work go? One thing I know is that if I now let this charge free to retrace the step that it took to come here, I do some work. If I release the charge, the forces of electrostatics will accelerate the charge.

So in the end, I expect kinetic energy of charge would be equal to work done. That is, this work that I did to bring the charge up to here. If I let the charge free to move along this path, then the charge will start speed up and by the time it reaches out here, it would have gained enough kinetic energy exactly equal to the amount of work I had to do to push the charge up there. This is what I expect and I expect this from analogy with gravitation.

I know that if I have a mountain and I have a rock and I roll up that rock to some point, I did work on it. Now the amount of work I did on it was m g h. If I let the rock go, then the total kinetic energy of that rock - both the linear momentum and the rotational momentum - all of it put together must be equal to exactly this m g h. That is, it rolls down the hill and converts gravitational potential energy to kinetic energy. This by the same analogy, I expect that if the charge moves along that same path without me slowing it down, it will pick up kinetic energy like the rock. Exactly equal to the amount of work I had to do to push it up. The m g h we call as potential energy and the work done here we call as electrostatic potential energy. So it is the same concept.

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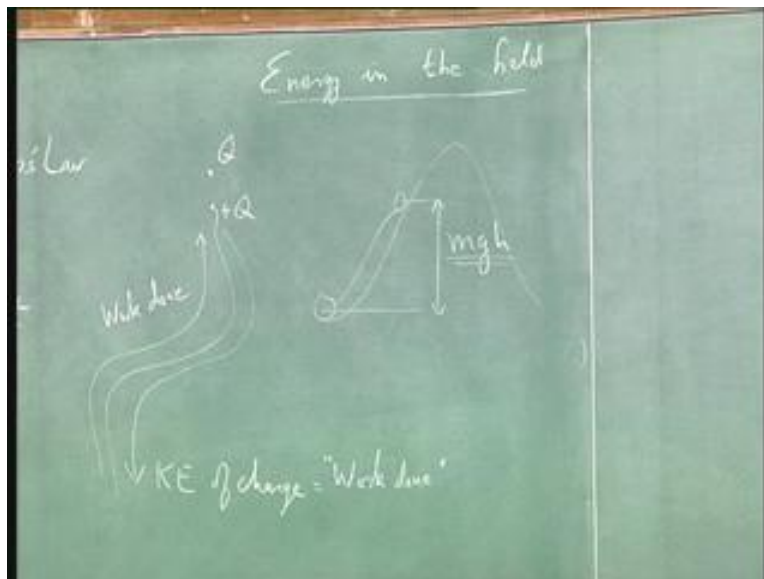
This is gravitational potential energy, this is electrostatic potential energy. Well, that is a nice idea but now my question is supposing I had a bunch of charges, N charges. How

much work do I have to do to put those N charges together? In other words, how much potential energy does...do the end charges represent? Now the answer is like this. Let us say, I have charges Q_i at positions r_i. So, how will I do it? First, I will take, let us say that this is very far away, infinity. I will bring Q_1 to point r_1. There are no charges in the universe. So there is no force; electric field is zero.

So this Q_1 comes through without any work done on it. So, W_1 is equal to zero. I did not have to do any work to bring charge 1 to the point r_1. Let us now bring the next charge Q_2 into this region. Q_2 is brought to this position r_2 from infinity; but as Q_2 is brought in, it must fight against the electric field exerted by Q_1. So due to Q_1, there is an electric field and this electric field is present all along the path taken by Q_2.

So Q_2 does work. How much work? Well, Q_2 does work, W_2 equal to Q_2 times the electric field due to Q_1. So, Q_2 times Q_1 over $4\pi\epsilon_0$ 1 over magnitude r_1 minus r_2. This is Q_2 times, phi. Now I bring the third charge in Q_3 to position r_3. Now Q_3 must do work not only against the electric field of Q_1, it must also do work against the electric field of Q_2. So, W_3 which is the amount of work I had to do to bring Q_3 in would be equal to Q_3 times the potential due to Q_1 which is Q_1 over ϵ_0 1 over magnitude r_1 minus r_3 plus Q_3 times potential due to Q_2 1 over r_2 minus r_3 and so on and so forth, till we bring in N charges.

So, how much work do we have to do over all? We have to do W_1 plus W_2 plus W_3 plus etcetera, up to W_N. So the total work we have to do, we will call it W is equal to a sum i equals 1 to N of W_i where W_i of each of these terms which is sum i equals 1 to N of another sum j equals 1 to i minus 1 Q_i Q_j over $4\pi\epsilon_0$ 1 over magnitude r_i minus r_j. Where did I get this? Take any particular i that corresponds to a particular W_i. That W_i contains Q_i times the summation of terms and what are those terms? They are Q_j over $4\pi\epsilon_0$ r_j minus r_i magnitude, since this magnitude...it does not matter whether its r_j minus r_i or r_i minus r_j.

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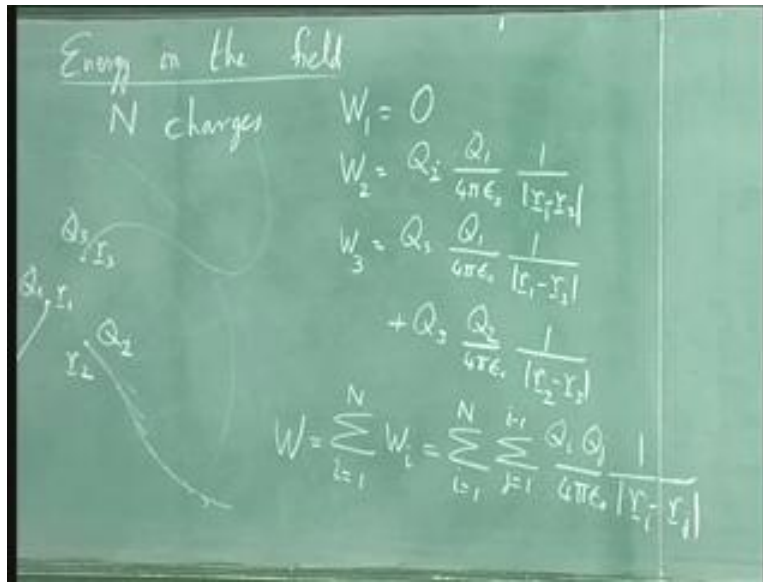
So, this represents the total amount of work I had to do to bring all the charges in and you will notice it is not a very simple expression. We will leave something to do, some work to make sense of what its equal to. It is not a very obvious expression and we do not know how to calculate, what it adds up to; but there is a simple way of understanding what this actually is. Let us see how that simple way is. Your textbook actually gives another way and you should read both approaches and learn from both. Idea is as follows. Let us say I have some charge distribution rho of r and I want to know how much energy is stored up in this charge distribution. This charge distribution could be in the form of N point charges or it could be in the form of a charge cloud.

Now I am going to assume that this charge was built up bit by bit by bit. So at t equals zero, no charge at t equals say epsilon, t equals t naught, I have everywhere epsilon rho of r. That is I have taken one percent of all the charges and put it everywhere. The shape is the correct shape but much less charge; only one percent of the charge at t equals 2 t naught. I have twice epsilon rho of r and so on and so forth. At t equals say N t naught, I have whole rho of r. So slowly, in time, I am bringing charge in and building it up. At any given time, let us say t equals k t naught, the charge density that is present is rho of r

times k epsilon. This epsilon is nothing but 1 over capital N. So you have some fraction of rho of r k over N of rho of r.

So, what is the potential due to this rho? Well, we know the potential. Potential is 1 over 4 pi epsilon naught volume integral k over N rho of r prime over r minus r prime d V. This k over N is a constant. Just pull it out. So, what do you get? K over N is a number, times 1 over 4 pi epsilon naught volume integral. Now the quantity in square brackets is nothing but the final potential due to this final charge. That is the definition of potential. The definition of potential is, if I know all the charge, this is how much it is.

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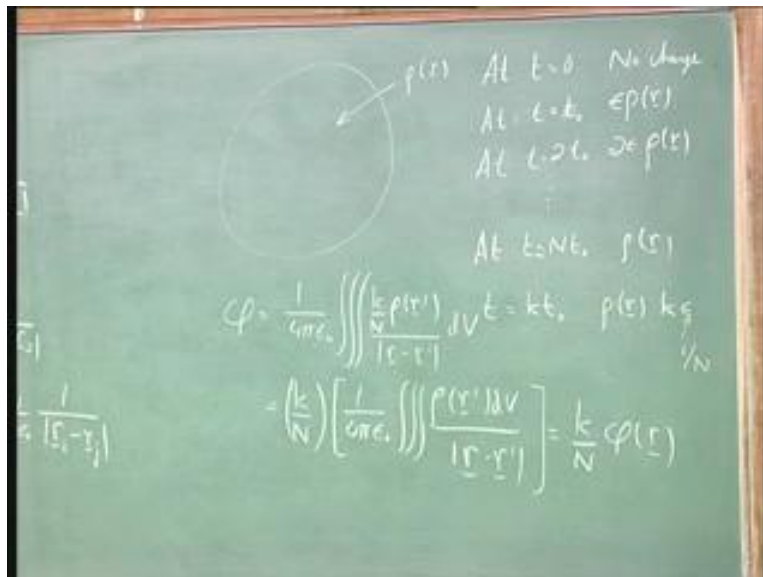
So it is equal to k over N phi of r. So at t equals k, t naught phi is equal to k over N phi of r and I am going to introduce a little bit of charge. How much? 1 over N rho of charge. So, how much work do I do? Well, in bringing in this small amount of charge, I am not changing the amount of charge present here by much. So the potential is essentially constant as I bring in the charge. So I can say the amount of work I do delta W is equal to rho of r over N times k over N phi of r. This is the amount of charge present times d V the volume. So, this is the amount of charge which is being brought against this potential. But now I have to integrate over all space. The total work I do is the sum of all these Ws.

7

So its total W is equal to sum on all delta Ws; but if you look at this expression which are very simple expression because it is a volume, integral rho of r phi of r d V, that part is common, times 1 over N square times sum 1 k equals 1 to N of k. k is the only varying part. Everything else is constant, but we know what sum 1 to N of k is. That is k into k minus 1 over 2, approximately. I am going to say that this piece is N square over 2; the N square cancels out, they are left with r.

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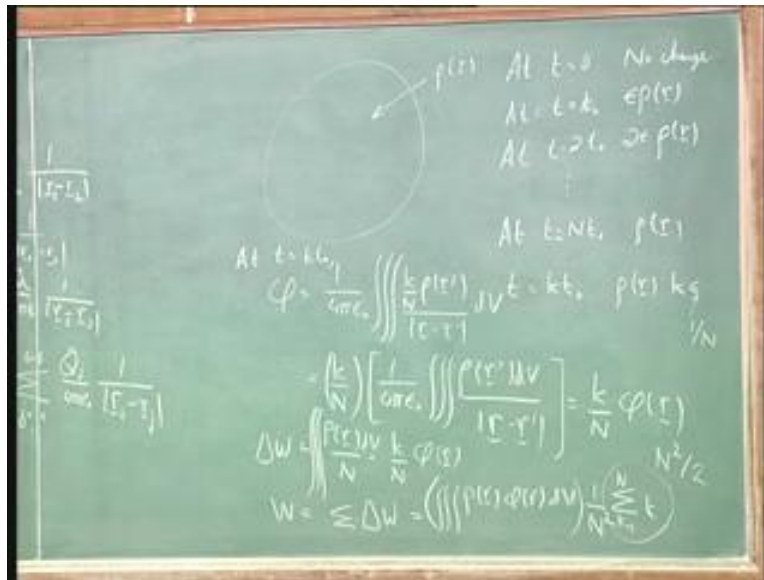
This is a very important result. Let me write it down. You can derive it in many ways. This is not the only way to derive it but what it is saying is, energy is required to build up a charge distribution and the charge distribution is rho of r is equal to 1 over 2 volume integral rho of r times, phi of r d V. Let me just sort of review because I think it probably went too fast.

I have a charge distribution here. It can be anything. In order to calculate how much work I did, I bring in this charge distribution little by little, 1% at a time, 0.1 % at a time. So, when I am bringing say the 50th % of this charge distribution, I am going from 49 % to 50 %. So the potential that is there is 49 % of the final potential. That is what this k over

N is telling me. Potential has the same shape because I have kept the shape constant but the magnitude is less.

Now the amount of work I did, well, I am working against a weaker potential but exactly the same shape. So I do less work. How much less work? k over N of the work. What is the total work I do? Well, it is fairly obvious I think. Initially I do no work at all. Finally I do full work, half way around, down 50 % - I do half work.

So really, the work I do is a triangle. The delta W versus say time goes like a triangle. Initially i did not do any work because to introduce the first percent of charge did not take any work at all. There was no field to begin with. So there was no work done. Finally when I introduce the last percentage of charge, I do a lot of work because I have to introduce it against the full potential.

So, it is a triangle that I have and each time I am introducing a bit of charge, I am doing this much work and the total work I do is the area of this triangle; and you know that the area of a triangle is the base into height divided by 2. That is why this half...because rho into phi integrated over all space is like assuming that from the beginning to the end I was working against this potential.

Supposing to introduce the first charge and to introduce the last charge I have, I had to work against this potential. Then the amount of work I would have to do is the full area of the rectangle; but I know that actual work I did was only one half of that rectangle. Initially I had to do very little work. Finally I had to do lots of work. So that is where the half comes. It is a very important factor of 2.
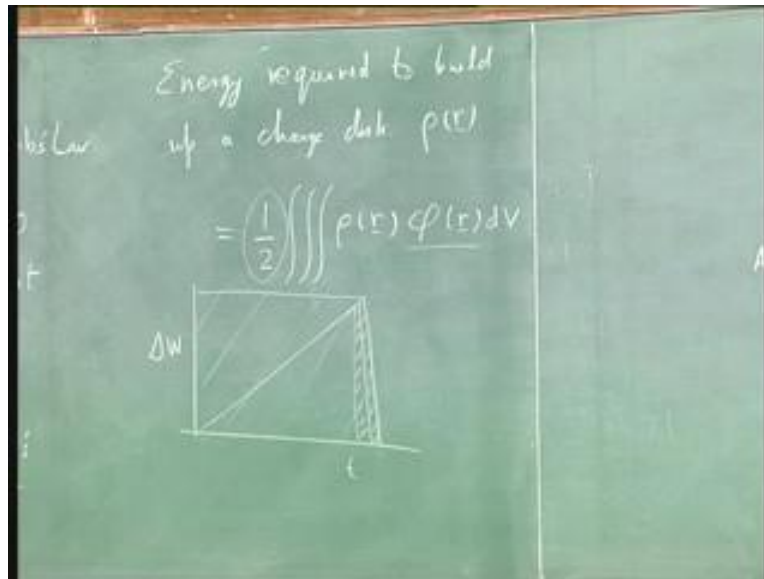
It is...it will keep coming back in all kinds of places. So you should feel comfortable when you see one half integral rho phi d V. So that is the energy that is required to build up any charge distribution. So what that means if I have four charges in a square and they are all separated by a distances, a, it is fairly easy to calculate the potential because you know the distances. This distance is a route 2. So you can calculate potential of any of these charges. So the potential at all the charges is equal and that phi naught is equal to Q over 4 pi epsilon naught times for this potential. The potential due to the second charge is 1 over a. The potential due to the third charge is 1 over a route 2. The potential due to the 4th charge is again 1 over a.

So this is the potential. How much is the stored energy? Well, it is Q times phi naught. This is the stored energy here plus Q times phi naught plus Q times, phi naught 4 times divided by 2. The factor of 2 comes from the fact that when I brought the first charge, I did not do anything. When I brought the second charge, I had to do something which only when I had brought the last charge that I did the full amount of work.
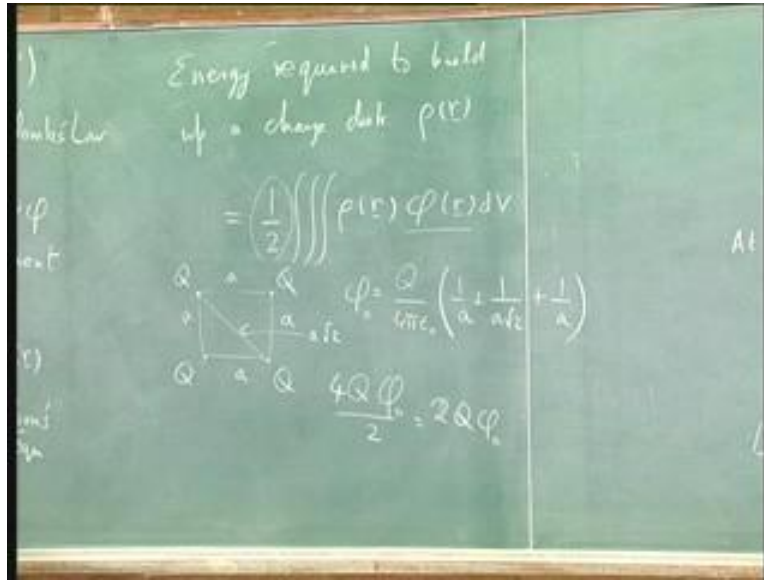
Now, the whole purpose of calculating work in getting a collection of charges together is because it has a lot of importance to electrical engineering and you all know what element that is. That is the element we call the capacitor. You have already seen it in many places. Supposing you have two plates and the plates are separated by a distance d. The plates have an area A. I am sure you have seen the formula. The formula is saying C is equal to epsilon naught A, over d.

Now, what exactly is this expression saying? What this expression is saying is if I take the top plate and add a charge Q to the top plate and if I look at how much work was done to introduce the charge Q, I can work out how much stored energy there is and the stored energy that is present there is in the form of positive charges sitting on the plate due to this charge Q associated with a potential a voltage on that plate.

So you have seen the formula Q equals C times V and you have seen the formula that energy equals one half C V square or is equal to one half Q times V. Now, if you look at this formula, you can see that is essentially the same thing. This is nothing but Q. This is nothing but V and you have your one half. So the formula for capacitor is really saying the same thing as energy in a electrostatic field, and what we are going to do for the next

lecture or so is we are going to look at the capacitor but we are not going to look at it as a circuit element. We are going to look at it as storage energy, storage due to an electric field. I have one preliminary that I have to complete.

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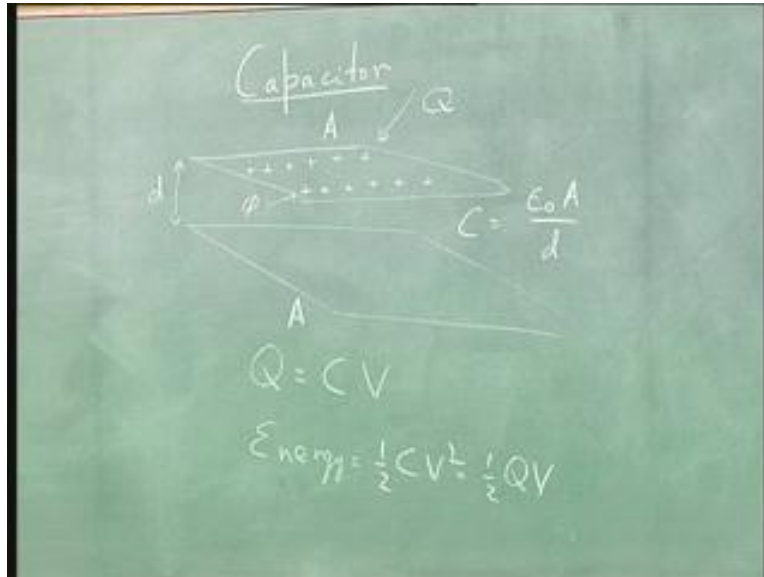
So, let us take get that out of the way. Last time I had told you that if you have a metallic body, a conductor, then inside the conductor electric field is equal to zero identity. I did it by two different arguments. One argument said, I cannot have circulating currents and the other argument said, I cannot have radial currents. The argument against circulating currents was that if I have r 1 to r 2 of E dot d l and I call this r 1, I call that r 2. If I go this way, I would be going with the electric field. I went this way, I would be going against the electric field and so the results would be different. Unless this was identically zero that was for fields that were going round and round what about the fields going away from a point well if you have any such region look at it from above what you have is an electric field that is pointing outwards that electric field is steady which means you can use Gauss's law and say there is a charge inside however that electric field also represents a current.

So this electric field is also equal to the current divided by conductivity sigma. So, on the one hand you had surface integral E dot d S equals Q enclosed over epsilon naught equals constant because we are talking about equilibrium. On the other hand you had that this thing is also equal to surface integral j dot d S with the 1 over sigma which meant that charge was continuously going out of this cylinder. But if charge is continuously going out of the cylinder, it is actually equal to minus 1 over epsilon naught d Q d t. The amount of charge that is inside must be reducing if charge is going out. This says charge is constant, this says charge is reducing. You cannot have both. The only way to make sense of having two separate statements like this is to say that electric field is equal to j is equal to zero. So, that is what it says: electric field is zero inside a body.

Now, if that is so, then there is an obvious conclusion. The conclusion is, if I have any metallic body and I take any two points r 1, r 2 and I draw a line connecting r 1 and r 2, that is entirely inside the body. It does not leave. It stays inside the material. Inside the material, electric field is 0. What does that mean? It means integral r 2 to r 1 E dot d l. Let me put a minus sign there; is equal to phi of r 1 minus phi of r 2.

So this E dot d l integral is nothing but the potential difference between r 1 and r 2. Electric field is identically zero; left hand side is 0. Therefore, phi of r 1 minus phi of r 2 is equal to 0. So it means that every point inside or on the surface of a conductor is at constant potential. Any two points have the same potential. So, this is why we are able to give voltages to conductors and this is why when we come to a capacitor, we are able to talk about voltage of a plate. We are able to talk about voltage of a plate because if I take a point here and I take a point there, a line connecting those two points which lies entirely within the plate, the electric field is 0 and therefore the potentials are the same. Therefore, we are able to say that every point on the plate is at same potential.

So instead of talking about phi of r so there is a, it is a function of position we simply give the voltage we do not say that this is the voltage of the plate at this point whereas that is voltage of the plate at that point at all points on the plate voltage is the same
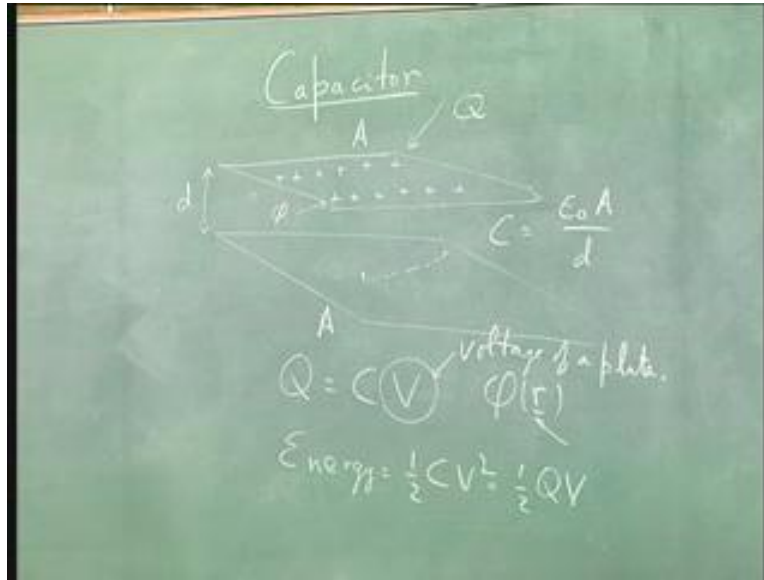
Now, at the end of last class I had derived Poisson's equation. Let me write it down again. It says del square phi is equal to minus rho over epsilon naught. Now, what does this say? For one dimensional problem it says, supposing phi and rho are functions of x only. Let... Then I can write this equation. Now it says d square phi d x square where phi is a function of x is equal to minus rho of x over epsilon naught, alright?

Now this equation happens to be very important. It is important because supposing I have a capacitor system. I have a plate and another plate and let us suppose that this direction is x. I choose to ground one plate and the other plate I put at voltage V naught and I want to know how to solve for rho and how to solve for phi.

So, I have to solve for this differential equation, but I have two pieces of information. Phi of 0 is equal to 0, phi of this distance is given as d phi of d is equal to V naught. It is a differential equation, second order ordinary linear differential equation, very simple equation to solve; but there is something that makes it simpler still. The region between the plates is air, which means there is no charge there. So, I also have for zero less than x less than d d square phi d x square is equal to 0. Now we know all about solving this particular differential equation. We know that the answer is equal to phi is equal to A

14

plus B x. Just integrate this with respect to x and the integrating constants will give you A plus B x.

So, if you now substitute A plus B x and try to solve the boundary conditions, you will have A plus B times 0. So A is equal to 0 and on the other side A plus B times d is equal to V naught. The first equation allows us to get rid of A; A is zero. So there is no A here either. The second equation allows us to get B; so that implies B is equal to V naught divided by d. So I can give the final answer. I can say that the potential phi is equal to V naught over d times x zero less than x less than d at x equals 0; phi is 0 at x equals d phi is V naught. So it satisfies the boundary conditions.

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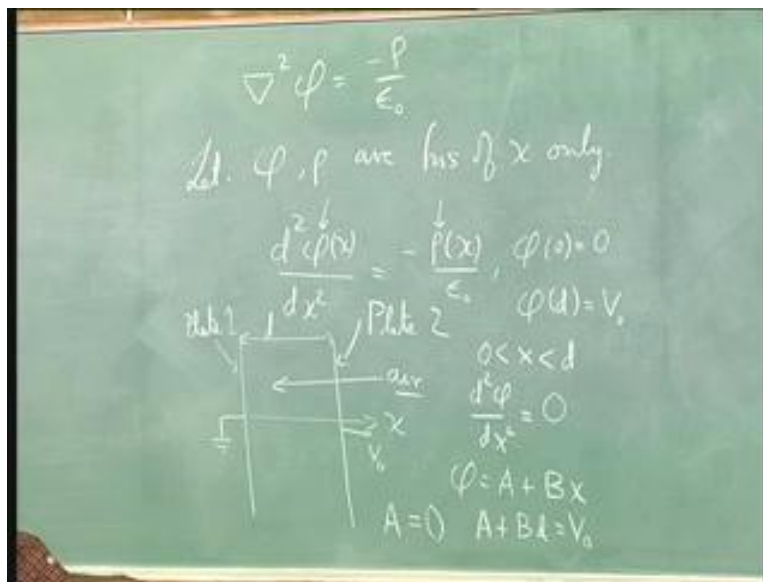
So from this I can find out electric field. The electric field is minus of the gradient phi but here only one direction matters. So it is minus d phi d x. So it becomes equal to minus of V naught over d. So, if I have a second plate at voltage V naught, first plate grounded, the electric field is in this direction. It is negative. It is negative because for a positive voltage here, positive charge will come here, negative charge will come here and so the electric field is towards the first plate.

15

Now, this still does not tell us what we really want to know. It still does not tell us what is rho, alright? What is the charge on the plates? Well, for that, we go back to some more high school physics. We apply Gauss's theorem. Let us look at it. For example, let us look at plate at x equals d. Now we have from here that the electric field is a constant and is pointing towards the first plate. So, if I have such a situation, what I can do is I can say I will take a surface that looks like this. This surface has a flat region with area d A. It has a sloping region with the length, call it h.

So the flat area is d A, sloping area is 2 pi r. This is r times h and d A is going to be equal to pi r square. Now the electric field is coming straight out, means it is tangential to the slope area. So I am only going to see the contribution of the electric field on the flat surface. What is more inside the material? I know that electric field is zero everywhere. So this piece does not give me any contribution.

So if I do surface integral E dot d S, I only get one surface contributing; that is 2 pi r times...sorry, pi r square times this electric field. That is minus V naught over d. However, this normal to the surface is pointing in the negative x direction. So this minus sign becomes a plus sign. So this gives me the area times the electric field must be equal to the charge divided by epsilon naught. Well, what is the charge? Charge is nothing but the amount of charge on the plate times the area of that region. So pi r, square over epsilon naught. So I can cancel out pi r squares and I get an answer which is very familiar, which is that the electric field, what I call the E normal in which, in this case is V naught over d is equal to the surface charge divided by epsilon naught.

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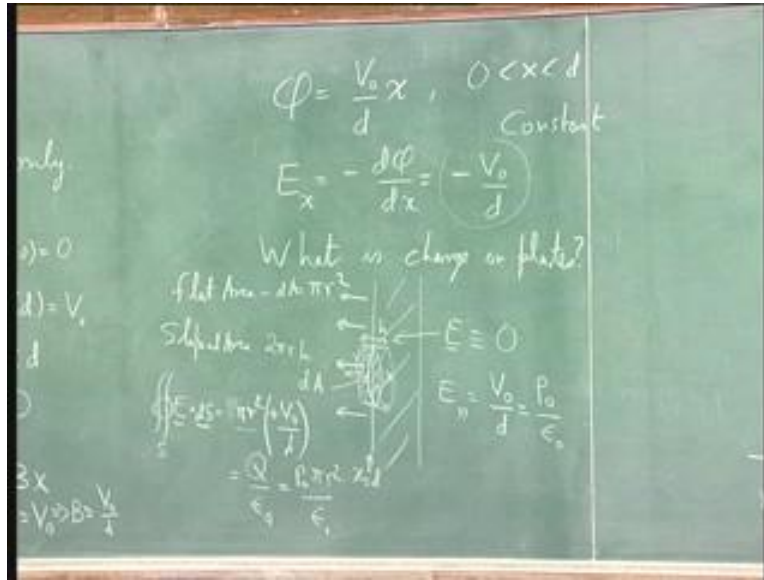
I have skipped a point here. So let me cover that before I lead this problem. When I talked about a metallic body, I said that electric field is identically zero. If the electric field is identically zero, then the question comes - where is the charge? Well, you know the construction. I take an imaginary surface that lies entirely inside but near the surface of the body. So at every point on that imaginary surface electric field is zero, which means surface integral E dot d S is 0, which means Q enclosed equals zero. So, there is no charge enclosed inside a metallic body yet. A metallic body can carry charge. It can carry charge because we can always take an isolated metallic body and put charge on it. So, where is the charge? The charge must be lying on the surface. The charge arranges itself on the surface in such a way that interior of the metallic object has no electric field. So now, going back to this problem, I have a normal electric field. I have no electric field in there in inside the material; I have some surface charge.

I have to relate the normal electric field to the surface charge. So, what did I do? The area times the normal electric field is equal to the charge density times the area divided by epsilon naught. So, this allows me to connect the normal electric field to the charge density. So what do I get? I get rho naught is equal to epsilon naught V naught over d. Now the charge on the plate is this charge density multiplied by the area of the plate. The

17

Q is equal to area times the rho naught which is equal to epsilon naught A over d times V naught and this is where we come to our standard result, which is Q is equal to C V. This number quantity C we call capacitance is the thing that we are used to. This is what we see in circuit theory.

However in electric fields, when you study it in electrostatics, it is a more general thing because it is entirely possible that I create say a sphere. I ground the bottom. I hold one part of it at V naught. I hold another part of it at minus V naught and I asked how much energy is stored. It does not fit into the simple form. This simple form is only good for 2 plates. What happens if I have multiple plates? We will see in a short while. May be in the next lecture or 2, that no matter how many plates you have, you can generalize this concept of capacitance and define a matrix. Which matrix equation? Which basically says the same statement? Let us take one more example to understand this, these concepts.

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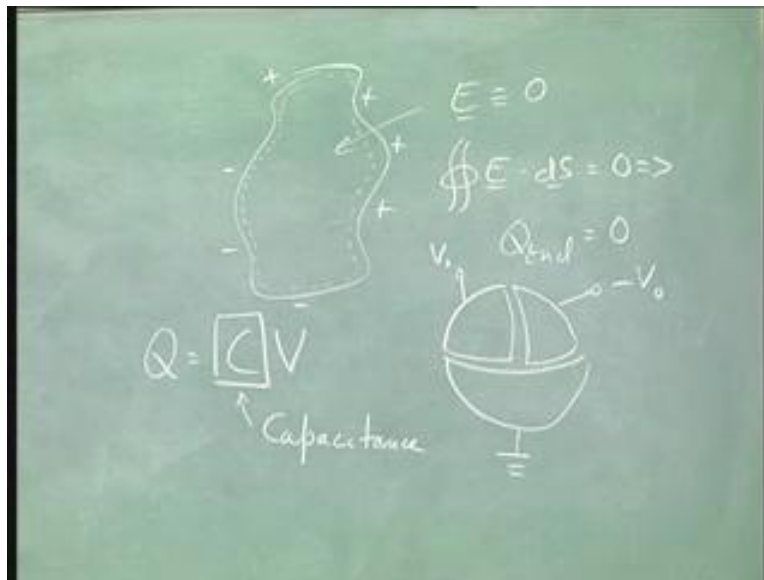
Supposing we have a spherical region filled with charge and you are told that the charge is a function of radius only. It does not depend on theta. It does not depend on si. It is radial and you are told that the electric field inside this material is constant once it leaves the surface. Of course it is not constant, but inside the electric field E r is a constant.

18

So the problem is to find out how much charge there is, what is the kind of charge there is and to find out the capacitance of an equivalent metallic sphere containing the same charge. Problem is straight forward. All we have to do is apply divergence theorem. If I take an imaginary sphere of radius r...let us assume the sphere itself is radius a; then I know that surface integral E dot d S which is equal to E r times 4 pi r square. This is the surface area, this is radial. Electric field is equal to charge enclosed divided by epsilon naught. What is the charge enclosed? It is nothing but integral rho of r d r times 4 pi r prime divided by epsilon naught. Notice here that I have put Q enclosed. So, what it means is the electric field here depends on the charge that is inside the sphere. It does not depend on the charge that is outside.

We found that if you took any region and you put a charge outside the charges, field lines do not contribute to E dot d S. So you can now have that the electric field is related to charge density. Now, this is a, unwealthy problem to look at but luckily we can go to the differential form of this which is divergence of E, which is equal to rho over epsilon naught. Usually you prefer to work in this because of the symmetry, but since I am giving you the electric field and I want to calculate rho, it is easier to work with the differential equation.

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19
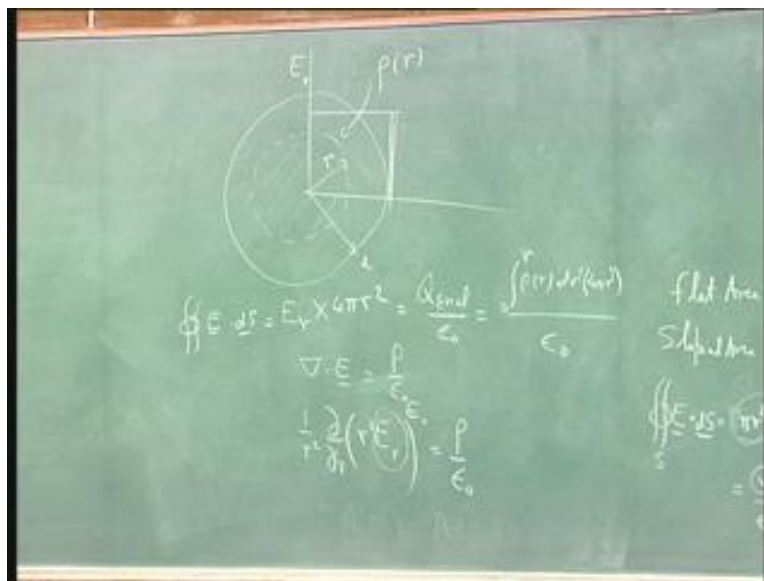
For spherical coordinates, this is what the equation becomes and all you have to do is to differentiate, because E r is nothing but a constant E naught. So substituting in, we can easily obtain rho. I will complete the problem next time, but I want you to try it out yourself because most of the ideas we have talked about in electrostatics can be tested out by solving problems like this. For example, look at what happens if you choose a gaussian cylinder that is larger than a, and try and plot out the electric field and potential as a function of r.