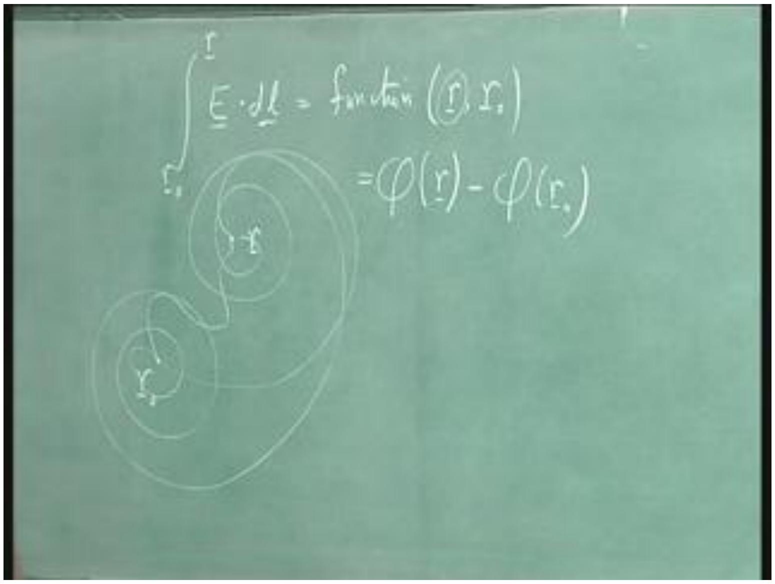
**Electro Magnetic Field**
**Dr. Harishankar Ramachandran**
**Department of Electrical Engineering**
**Indian Institute of Technology – Madras**

**Lecture - 8**
# Poisson's Equation

Good morning. In the last two lectures, we had introduced two topics: one was the electrostatic potential and the other was the divergence theorem. So, this lecture what we will do is use these ideas in some problems and finally lead up to Poisson's equation. So, let us do some review. We had started with a very important idea in electrostatics that we derived. We showed that if you took the electrostatic field and worked out this integral E dot d l going from any point r naught to a point r in some path. So, if you did this integral from r naught to r, the value of this integral depended only on the function, only of the end points. It depended on the value of r, it depended on the value of r naught. But it did not depend on how you went from r naught to r.

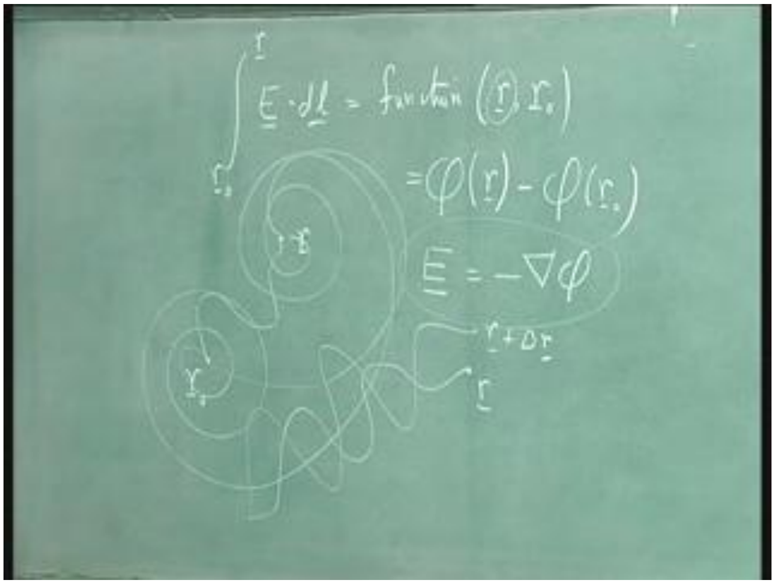
You could have gone this way, you could have gone this way; you would have got the same answer. Since the answer did not depend on how we went, the answer must clearly be a function of the positions r and r naught and so we identified a scalar function because this was a scalar vector dot, a vector. So, it is a scalar. The integral of a scalar is a scalar. So, it is a scalar function and we give it a name, electrostatic potential phi; and it has a value at every point in space. And when I do an integral from r naught to r, the answer I get is the difference between the value of this potential at r and the value of this potential at r naught. So, we have derived this and as I told you, it is something very important about electrostatics, if you are able to do this.

(Refer Slide Time: 03:37)



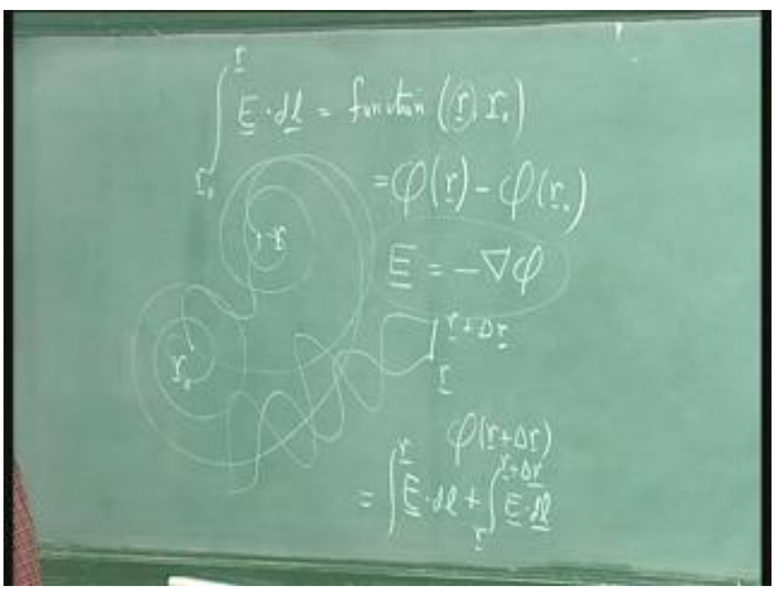
Now, another equally important thing that we found out was that you could relate electric field and potential namely, you had a relation that the electric field was got by taking this scalar field phi and applying the gradient operator to phi E was equal to minus gradient of phi. The way we showed this was to say the following. I have a point r. The electrostatic potential at this point is got by going from some reference location through some path to r. Now, if you look at some point near r, a point I will call r plus delta r, the potential at that point comes by doing this integral from the same reference point on another path up to this point, r plus delta r.

(Refer Slide Time: 05:10)




Now, it could be any path at all; but as we discussed last time, I will choose a specific path to get to this point r plus delta r. I will follow the same path up to r and then go in a straight line from r to r plus delta r. So, I have potential at r plus delta r; that is, this value is equal to integral up to r of E dot d l. That is, this first path plus integral r to r plus delta r of E dot d l.

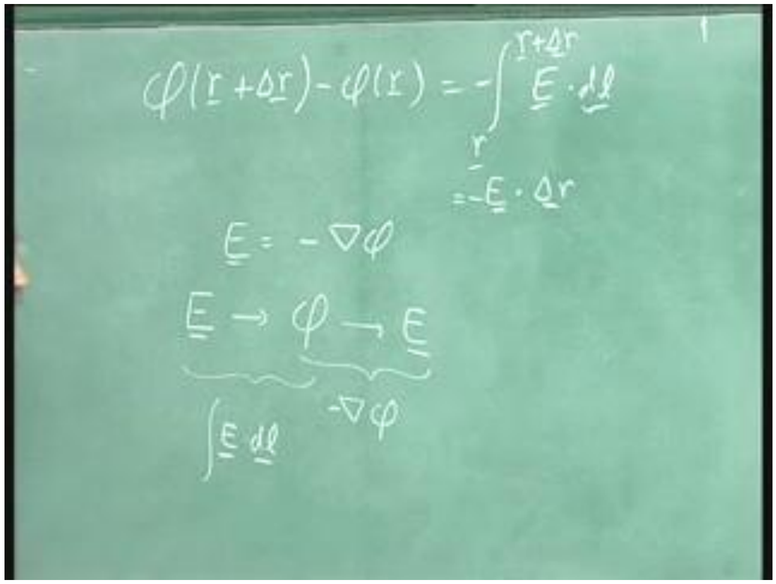
(Refer Slide Time: 06:08)



But, this is nothing but phi of r. So, I get an equation. It says phi of r plus delta r minus phi of r is equal to integral r to r plus delta r E dot d l. As we discussed last time, this is a very short distance and it is from straight line. You can say this is equal to E dot delta r and when you study the properties of phi, it is easily possible to show that this implies the electric field is equal to well...it is the minus sign that I have missed. It is minus of... so, it is minus E dot d r delta r E is equal to minus gradient of phi. So, there are two very interesting relation between electric field and potential. First, I have that electric field; and once again there should be a minus sign here. Please note the electric field integrated from one point to another gives me the potential and if I take the potential in a placer gradient to it, I will get back the electric field.

So in other words, from the electric field I can go to potential; this is by integral E dot d l. And from the potential I can go back to the electric field; this is by gradient of phi, alright? That was one concept that we have I think looked at quite thoroughly. Now the second concept we looked at last lecture was the idea of electric flux and the idea here is that when you have a charge, group of charges...in fact, you have an electric field that develops and at any point the electric field that points have the direction in the magnitude is equal to the force exerted on a charge divided by the charge itself, because F is equal to q E.

(Refer Slide Time: 08:20)



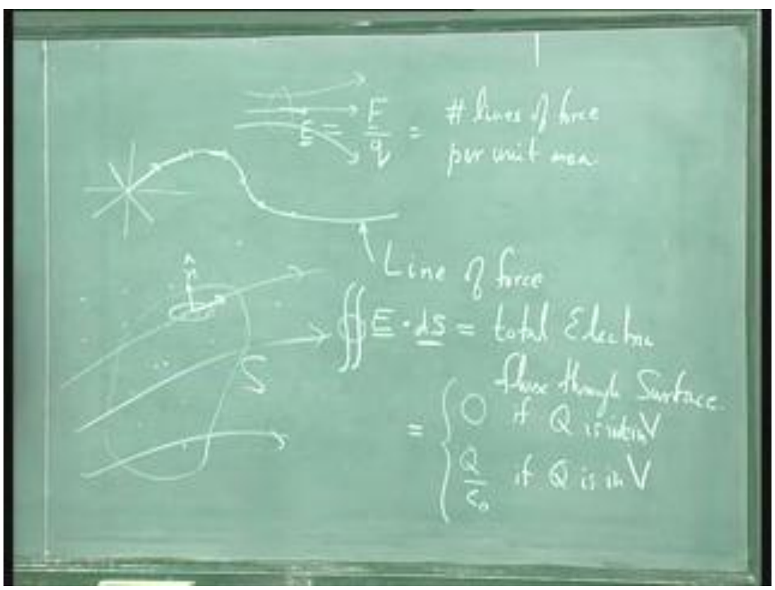

But there is another way of looking at electric field and the other way of looking at electric field is, let us assume that every charge, some electric field lines leave. You have many charges and from every charge some electric field lines leave. How many electric field lines? As many as the magnitude of the charge, if this charge is one unit, may be eight electric lines leave. In this, charge is two units, sixteen electric lines leave that. These electric lines, because the electric field has general shape, the electric field changes direction. The line that joins all these electric fields is called a line of force. I have not proven it, but you can show that lines of force are a very good concept that in fact, it is not just for drawing. And another way of talking about the strength of the electric field is to count the number of lines of force that cut through unit surface area at a point.

So, electric field is F over Q, but it is also equal to number of lines of force per unit area. Now we use this idea. Last lecture we said that supposing you have a volume. I am going to call it S and supposing I have lines of force and this line of force cuts through the surface at some point. So the electric field is in the direction of line of force whereas at that point, the line of force has a normal pointing some other direction. If you want to count how many lines of force go through the area d s, we worked it out last time. It is equal to E dot d s.

So, this is a little bit of flux that cuts through this area d s. So, we can integrate this number over the entire surface. So, we can construct a surface integral and this circle means over the entire surface of E dot d s. This is equal to total flux, electric flux, through surface and we worked out last time that this E dot d s for a single charge, if the electric field is due to a single charge, is equal to 0 if Q is in V and is equal to...sorry, is not in V; is equal to Q over epsilon naught if Q is in V.

So, it is a binary kind of relationship. If the charge is outside, there is no net flux if the charge is inside there is net flux if you put two charges then this applies to each of the charges separately because we have superposition so we can actually talk about n charges sum of the charges inside the volume sum of the charges outside then when we do E dot d s this will apply in turn to the electric field due to each of these charges but only those charges which are inside the volume will give me flux the charges that lied outside the volume give me 0.

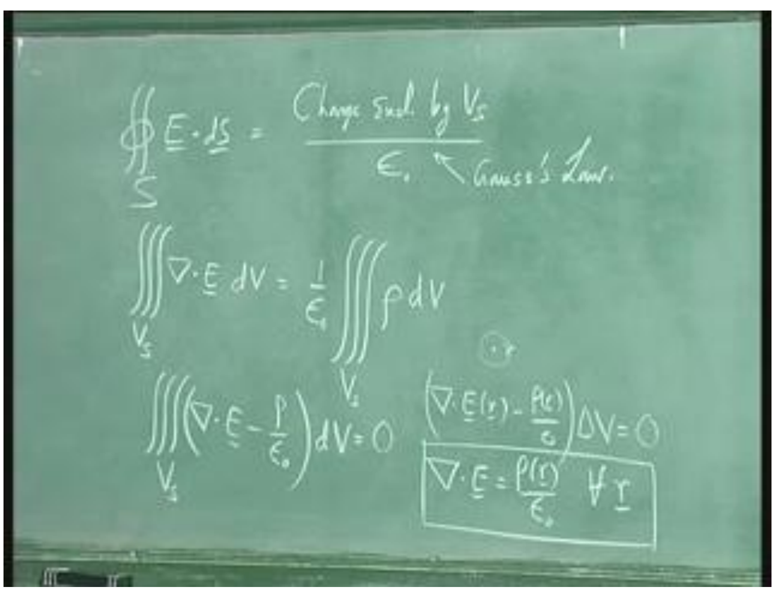
(Refer Slide Time: 13:30)




So actually, we get a result which says surface integral E dot d s due to many charges is equal to charge enclosed by volume. So, this is the surface and this is the volume enclosed by the surface divided by epsilon naught and this is called Gauss's law. Now, there is only one more step which is that I can take - do some mathematics on this surface. We showed that you can write this surface integral as a volume integral over the same volume divergence of E d V. We showed last time that these are saying the same thing and what is charge enclosed by V s? It is nothing but the volume integral over the same volume V s of charge density rho d V.

I can combine the two volumes and so I get a single volume integral V s divergence of E minus rho over epsilon naught d V which is equal to 0. Now, this by itself does not tell us anything. I mean it is a volume. Integral of a function is zero. That can happen to many functions, but it is saying much more than that. It is saying that the volume integral of this quantity is zero for any volume because we did not specify what this volume was. It is true for each and every volume which means I can take a microscopic volume about any point r. I can take a volume just around that point which means that this volume integral will become divergence E at that point. r minus rho of r over epsilon naught times delta V is equal to 0. Delta V is not 0. So it must be that this bracket is zero, which means divergence E is equal to rho over epsilon for all. This is known as the divergence theorem.

(Refer Slide Time: 16:42)



It is equivalent to Gauss's law. There is no...nothing that one has that the other does not have, because we use Gauss's law to derive this. And if you knew this, this must obviously hold because if this holds at every point, the volume integral holds at every point for any volume as well, so, if these two ideas that we are going to use this lecture and eventually derive Poisson's equation.

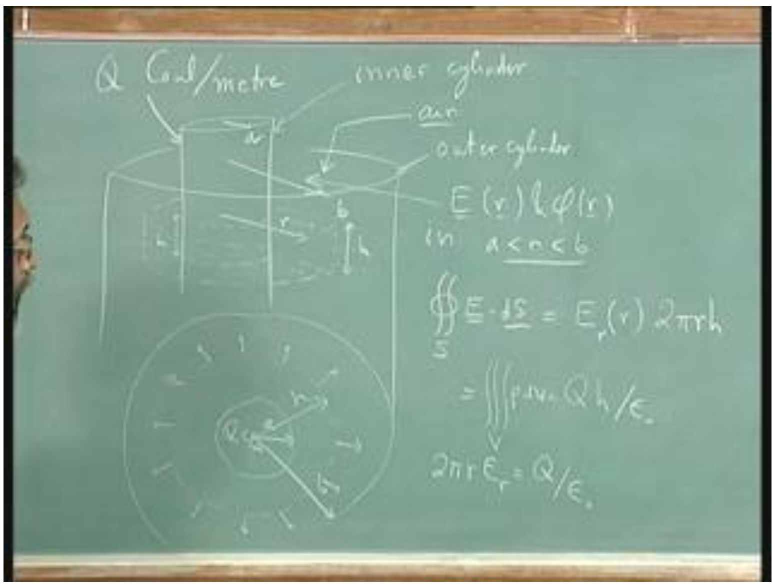
The first example I am going to take is going to be a cable. I have two cylinders. The inner cylinder has say, a radius a; outer cylinder has radius b. The region between the cylinders is empty. It is air. So in this region we have air. Cylinder themselves are made of metal and what I have done is I have put some charge on the inner cylinder. The amount of charge I have put is Q Coulombs per metre. And the question is I want to obtain the electric field and potential in a less than r less than b that is in the region between the two cylinders.

So, if you look from above, then the same looks as follows: I have got two circles; the inner circle has a radius a, outer circle has a radius b. On the inner circle is some charge. The charge is Q; it is Q Coulombs per metre. Now, let us look at a third cylinder which I am drawing with a dotted line at some radius r. Now if I look at this cylinder, due to the symmetry of the problem I know that the electric field is going to point away from Q. So on this cylinder, this electric field is going to point radially away.

So, if I want to compute a relationship between electric field and charge, what I do is I construct the following box. The box has a height h. It has a radius r and it encloses a height h of the inner cylinder. Now the electric field is radially outwards. So it means it is tangential on the top surface and is tangential on the bottom surface. So it means that if I want to do surface integral E dot d s, there is no contribution from the top surface and the bottom surface because the electric field is neither entering nor leaving, and this integral is about electric flux that is passing through the surface.

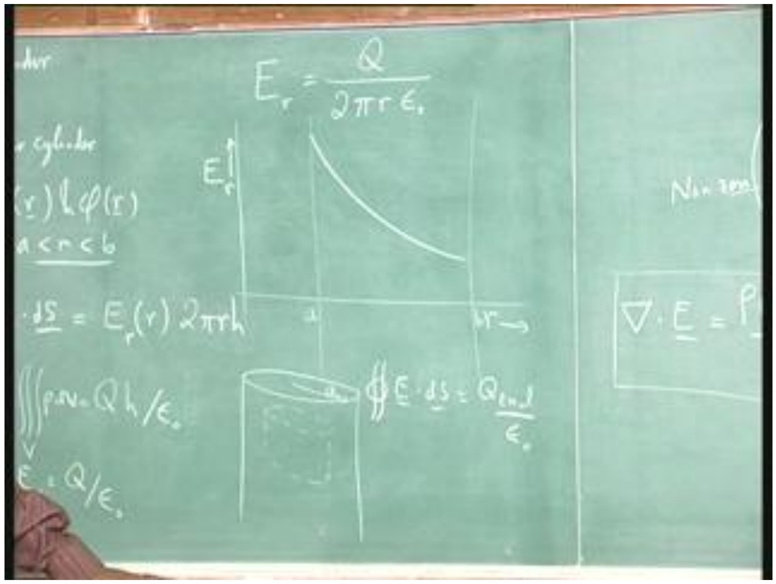
So, there is zero contribution from the top and bottom. There is only contribution from the side. So it becomes E sub r radial component of E which is a function of r multiplied by the area. Well, the area is 2 pi r h. Now, this surface integral E dot d s is equal to the enclosed charge. How much charge is enclosed? It is equal to the volume integral rho d V or in this case since I am given Q, it is equal to Q times h. That is, it is Q Coulombs per metre. This is the height h. So it is Q h Coulombs per inside the box. So I now have a relation between electric field and charge; that is, 2 pi r E r is equal to...sorry, this is Q h over epsilon naught.

(Refer Slide Time: 22:49)




So solving this, I get the electric field. The electric field which is in the radial direction is equal to the charge per metre times 2 pi r epsilon naught. So if I plot this, this is the radius. So the radial distance...my inner cylinder reaches here. So this is a...my outer cylinder is here; so that is b. What I have is that the electric field looks like 1 over r. So this is E r. Now what is E r? Inside the cylinder, if this is a solid, if the inner cylinder is a solid metallic piece, what is E r inside? Well, to answer that, we take the same approach. We try to put the cylinder, this dotted cylinder entirely inside the inner metallic block. So how would that work? I have radius, a cylinder and within that cylinder I put this imaginary surface. So again I would get surface integral E dot d s is equal to Q enclosed divided by epsilon naught. Now the question is what is this surface integral?

(Refer Slide Time: 24:53)



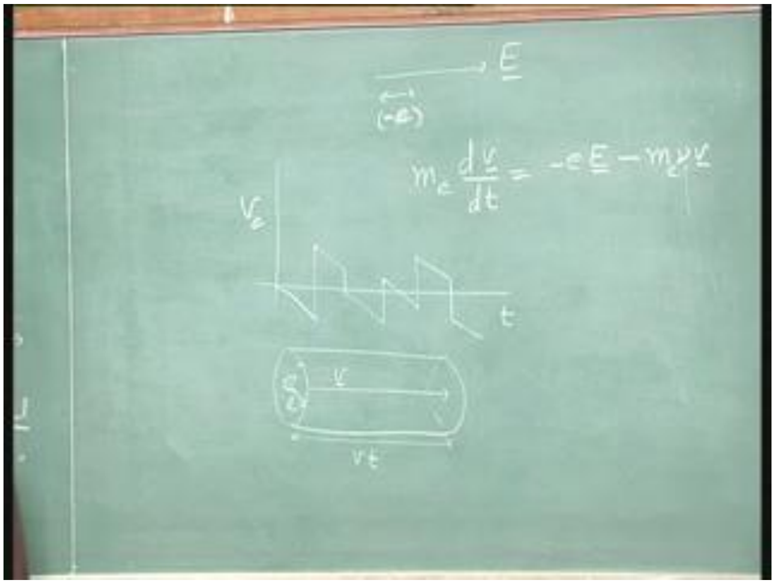
I am sure you learned all this in your school but it never hurts to re-learn, certain. So, let me go through the argument. Now in a metal or a semiconductor or any conducting medium consist of charges that move and in the kind of materials we are thinking of, you know charges which are usually electrons and in the presence of an electric field, let us say in this direction, what happens is that electron feels a force. The force which it feels is in exact opposite direction because the electronic charge is actually minus. Now this electron therefore starts moving. It starts speeding up and if you plot its velocity in time, velocity of the electron, well, it will start decreasing. Let us say it started at zero and it starts decreasing; but in no time at all this electron is going to bang into another electron or into an atom at which point the electron forgets all about its earlier velocity and goes to a new velocity. It may be positive, it may be more negative, it may be zero. Then it starts accelerating again, has another, has a bang and so on and so forth.

Now, in the process of doing this, the electron develops an average velocity. Now the average velocity of the electron is obtained by writing the force equation. The force equation for an electron would be mass of the electron d v d t; that is, d d t of the velocity. So this is the rate of change of momentum and Newton told us, this is equal to the applied force; but what are the applied forces? Well, there is minus e times the electric field. That is an applied force. We do not have magnetic field. In this problem, we will ignore the gravitational field, but there is one more force that is present. It is the force of collisions. Now you can see that this is a kind of instantaneous force that acts at very short period of time, every now and then. But that is very hard to model. So, what we do is we define it as a friction minus m n u v. This is the mass of the electron times the collision frequency or the friction factor times the velocity. The idea is that if an electron is here and it is moving with a certain velocity v, it will interact with other electrons and atoms. It will interact with them provided they are within the range.

So, let us say that the range over which it can interact is some l. So you can draw a cylinder. A cylinder has a radius l. It has a length equal to v t; v is the velocity, t is the time. So, that is the distance that the electron moves, if it moves at constant velocity. Now, if it meets up with another electron in here or another atom in here, it is going to collide. So the faster it moves, the more space it covers, more likely it is to collide.

So that is why the v is there. The more electrons there are per cubic metre, the more tightly packed the electrons are, the more likely they are to collide; that is, this collision factor. And when it does collide, there is a change in velocity and the amount of momentum that changes is proportional to the mass of the electron. So, that is where this term comes from m n u v.

(Refer Slide Time: 29:50)



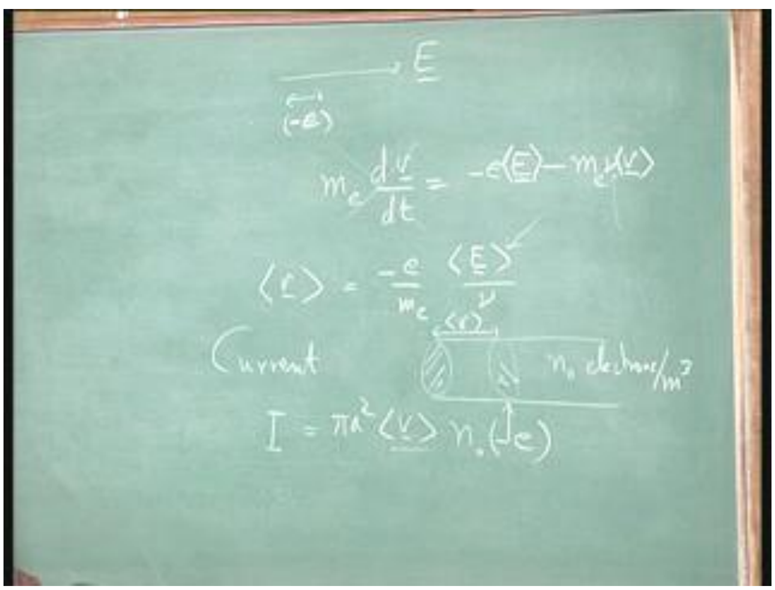

Now, if you take this and you believe it, and I can tell you that it is an approximation, there are plenty of situations where it is not true; but it is true inside a metal. It is more or less true inside metals and semiconductors. Now, if you take this and you say I will wait for steady state, then this piece goes away. Again, this is fake. I am cheating because the diagram I drew showed you the electron velocity jumping up and down.

So actually, what I have to do is I have to average this equation over a fairly long time. Then what will happen is the electron velocity goes up, comes down, goes up, comes down. On average it does not change; and so the d v d t term goes away. So we are talking about average velocity, average velocities and average electric fields. The weight will denote average; it is to put angle brackets around them. So, what do I get? I get that the average velocity of the electron is equal to minus e over m e times the average electric field over n u.

So I now know that the electron drifts and it drifts at a speed which is proportional to the applied electric field. It is also proportional to the charge over mass and it is inversely proportional to how many electrons that are present. The larger the number of electrons, the more they slow it down. So you cannot have much drift velocity. Now, when you ask what is current, current is basically moving electrons. So if I have a wire with a radius a and if the wire has n naught electrons per metre cubed, then the current that flows will have to be pi a square. This v bar, this is the volume swept out in one second times n naught. Why do I say that? I say that because supposing I am watching here and I count electrons for one second. Well, how many electrons will I count? I will have to multiply the space, the cylinder that swept out times the number, density of electrons.

So this will be v bar and this will be the area pi a square; and of course I must multiply by the charge of the electron. So what it means is that ultimately, the current is proportional to the velocity; is proportional to the electric field. If I have an electric field present, there is going to be average drift of electrons and that is going to result in a current. Where does this take us?

(Refer Slide Time: 33:57)

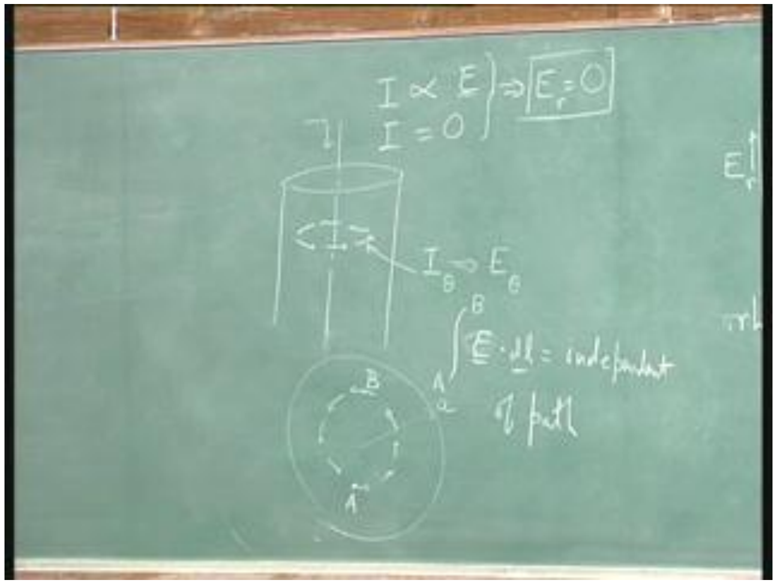



Let us go back to the wire or the metallic object. I am supposed to have radial electric field because it is symmetric. So, if I have my imaginary cylinder and I said I have radial electric field everywhere. What it means is there is current flowing out continuously. So for the next, as long as we wait and watch, may be point 1 coulombs are going to flow out per second from the inside of the wire to the outside, but there is nowhere for the charge to come from. Charge cannot just keep going out without being replaced.

So, there is something wrong with this picture and the only thing that can be wrong...because I do know that the current is proportional to electric field, applied electric field and I do know that I cannot have current. So, if I take these two facts, it implies the electric field itself is zero inside a material which is conducting. If you allow the system to go to steady state, you have to have that the charges settle down in static positions, which means no current, which means no electric field. It is a very standard result. I mean, you learned it in school, you learned it in...I am sure, your Physics 2 and you are learning it again; and is not quite true.

For example, why cannot I have a current that looks like this? Why cannot I have a circulating current? I am drawing it with the wrong aspect ratio. Let me draw it more properly so it is coming out, going back in, going around its cylindrically symmetric... No, charge is accumulating anywhere. I do not have to produce charge out of nothing. Why cannot it happen and why cannot the current just keep moving? Well, if there is a current like this, this is a current that is in this direction and therefore it implies I have electric field in the theta direction. Let me look from above. So, what will I have? I have my material radius a and I have current that is flowing around like this and because of that current, I have electric field that is going around like this. And let me choose one point here which I will call A and one point here I will call B. Now you may remember from two lectures ago, we proved something; we proved that integral A to B phi...sorry E dot d l was equal to independent of path. No matter how I went from A to B, I got the same answer.

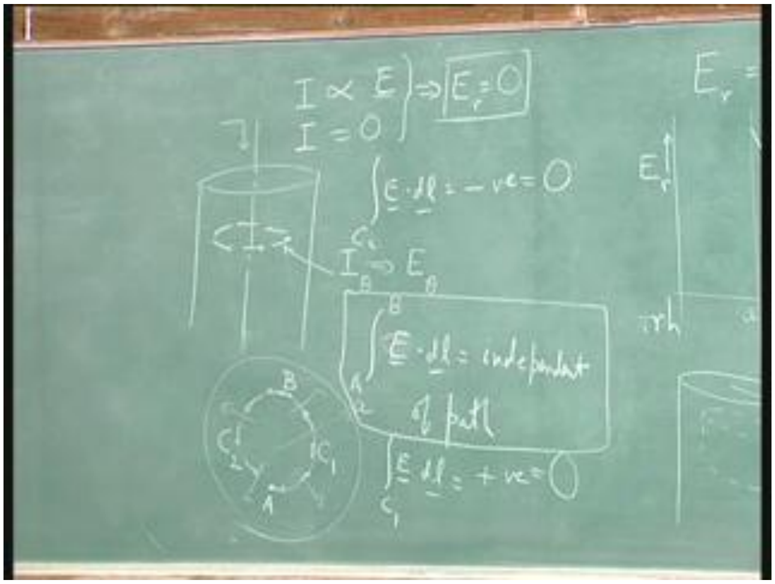
(Refer Slide Time: 38:00)




So I can choose two different ways. I can choose to go this way and I can choose to go this way. So let me label this path; this path C 1 and this path C 2. So, if I do integral along C 1 E dot d l, well, I do not know the answer, but I do know that E is along d l. So, it must be positive if I go along C 2. I do integral along C 2 of E dot d l. Well, the electric field is downwards, d l is upwards. So, E dot d l is exactly minus, so it is negative, but is the same E dot d l, the same E dot d l. One way is positive, one way is negative. There is only one number which is equal to minus of itself and that is 0.

See, you cannot have circulating currents and therefore circulating electric fields inside a material, inside electrostatics. The moment you have Faraday's law, of course you can have. There is a very important topic we will come back to. But in electrostatics, because of this important result, we cannot have any circulating currents and we cannot have radial currents because where does the charge come from? Charge cannot keep on coming. So on average, the radial currents have to go to 0. So the result is that inside a material, E r goes to 0.

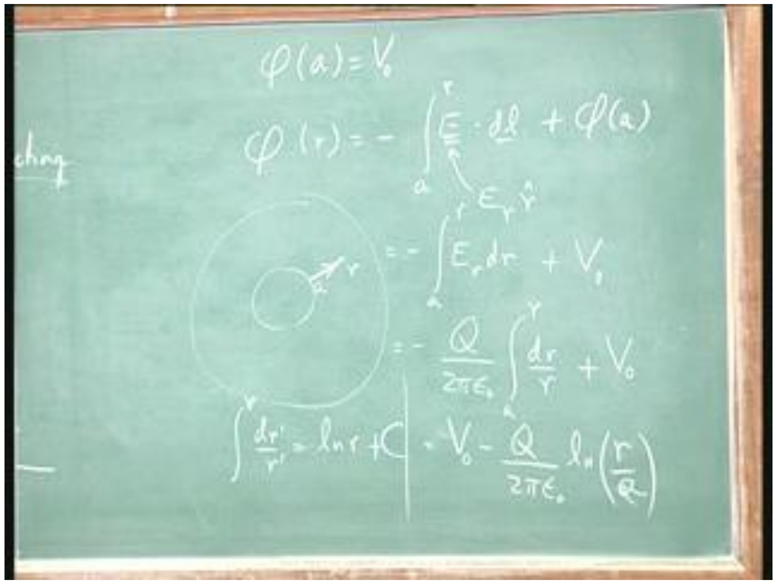
(Refer Slide Time: 40:19)



So, I have plot. This is b. This was a. This was electric field. E r goes to 0 at a and remains 0. The electric field is 0 inside, goes to a value. This value we have worked out already. It has got to do with Q over 2 pi r epsilon naught and then it goes all the way out to b, where it drops to 0 again. This is the electric field and if now in addition I tell you that one of these two - a or b - has a known potential, then in terms of that known potential, I can get you the potential as a function of r. Let us see how that works out. Let us say we know phi at a is equal to some v naught. Then phi at any r is equal to minus integral a to r E dot d l...sorry, phi plus phi; this is only the difference in potential that gives integral a to r E dot d l, but E is radial.

So I can take, if I want the potential at any r, I can take a radial line going from a to r and if I take this radial line, then E dot d l becomes nothing but integral a to r E r d r plus phi of a which is plus V naught. I already know what E r is its equal to minus Q over 2 pi epsilon naught integral a to r d r over r plus V naught. This is a very standard integral. The integral d r over r or I say d r prime over r prime up to limit r is equal to l n r plus constant substitute, that there it is equal to V naught minus Q over 2 pi epsilon naught l n r over a where I have used the property of log, log of r minus log of a is equal to log of r over a.

(Refer Slide Time: 45:00)



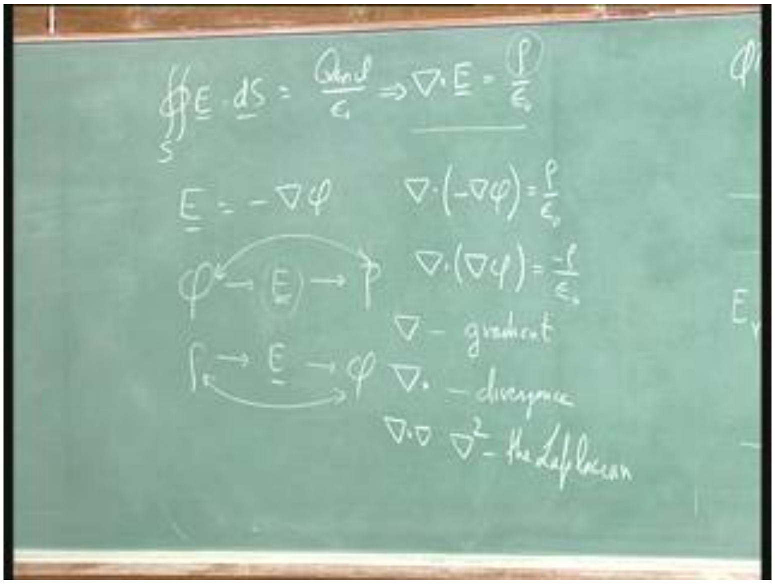

So you can plot this function on the same plot and what you will get is slightly a different kind of function. You plot the log function itself. Let me plot it on the same graph as the previous graph. This is a. If you plot log function itself, log r over a is zero at r equals a and it grows very gradually. This function multiplied by something subtracts from the starting voltage. So, if you take that this is phi, this is the value V naught. You will get a slowly decaying value which is the potential as a function of r. These are important results. They are important because the potential and field between two coaxial cylinders is a very common problem in electrical engineering. It is so common that there are very standard tabulated answers to these geometries; but here I am basically presenting this as a way of understanding how to solve problems using Gauss's law.

So let me review what we have done up to now. We have surface integral E dot d S equals Q enclosed over epsilon naught. This is Gauss's law and we applied it to coaxial cylinders as well as to within a metal and we came up with solutions for electric field. Now, I would like to go back to the previous formula I derived from this. I had given you another formula which was divergence of electric field is equal to rho over epsilon naught is the same equation. It is just that this is true for arbitrary surfaces. It must therefore hold that this equation is also true.

Now let us remember that the electric field itself is got from the potential. This was derived two classes ago. So, you may have forgotten; but...so I will say, just you go back and re-read. So we have two equations: we have an equation that relates electric field to charge and we have a relation that relates electric field to potential. It is very interesting. You start with a scalar field, you create a vector field, take the divergence of the vector field and come back to a scalar field; and what is most interesting of all is that we know, if you know charge density that is like knowing charges. You can calculate the electric field.

So from potential you can go to electric field; from electric field you can go to charge; from charge you can go to electric field. And we know that if we know electric field, you can go to potential. So, this is...both ways it is possible to do this calculation. But why go through this electric field at all? This electric field is a complicated thing and it has three different coordinates. We would like to actually go directly from charge to potential. Then we can calculate the electric field any time we want. How do you do that? Well, I will take the electric field here and substitute this formula. So, let me do that divergence of minus gradient of potential is equal to rho over epsilon naught. I can take that minus sign to the right hand side. So I get divergence of gradient of phi equals minus rho over epsilon naught. Now this particular form divergence of gradient is so common that it has also become a tabulated operator. So far we know gradient and we know divergence. This is called divergence of gradient. It is given a special name del square and is called the Laplace equation.

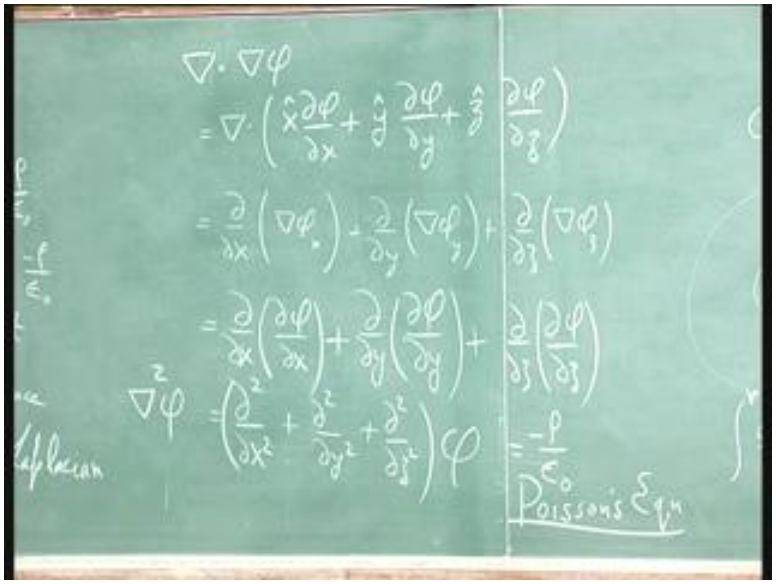
(Refer Slide Time: 51:13)




Now the Laplacian is very easy to write down all we have to do is apply its definition divergence of gradient of phi is equal to divergence of unit vector along x del phi del x plus unit vector along y del phi del y plus unit vector along z del phi del z. This is the vector gradient phi. If I take its divergence, what does it mean? It means del del x of gradient phi along x plus del del y of gradient phi along y plus del del z of gradient phi along z gradient phi along x is this piece. This is the piece that is along x.

So it is equal to del del x of del phi del x plus del del y of gradient phi along y is multiplying y hat del phi del y plus del del z, the piece multiplying z hat del phi del z. So you can see it takes a very simple form - second partial derivative with respect to x plus second partial derivative with respect to y, second partial derivative with respect to z, acting on phi and because of this square operator. That is why it is denoted as del square phi. It is a very very common operator. In fact, it is probably the most important operator in wave theory. All our physics uses the Laplacian and this del square phi is equal to minus rho over epsilon naught. It is a very famous equation and it is called Poisson's equation.

(Refer Slide Time: 54:03)




Now, why is it so important? I mean ultimately, if we know charge, if we know rho, we already found that from rho we can calculate electric field. From electric field, we can calculate potential and we know everything. Why is this shortcut so important? The reason why the shortcut is important is that we do not always know charge. If indeed we knew where charge was, Coulomb's law is all we need; but if you think of electrical engineering, the most important building blocks of electrical circuits, there are three of them: resistors, capacitors and the inductors. And one of the three, the capacitor, consists of two plates which are kept at a certain voltage. One plate is kept at say, ground potential; the other plate is kept at 10 volts and within, between those two plates, an electric field develops which is described by Coulomb's law, which is described by Maxwell's equation.

So that is the problem where we do not know rho, we do not know the value of rho. What we know is the potential at the plates. So let me make...emphasize this again. We have problems where we actually have that one plate is grounded, the other plate is connected to a battery; and we would like to find out what is happening inside and what is happening on the plate. This requires solving a different problem than Coulomb's law because we do not know where the charge is, we do not know how much charge there is, we do not know where the charge is.

What we know is that the potential here is V naught, the potential here is ground and it is for solving such problems that Poisson's equation becomes more useful than Coulomb's law. With Coulomb's law, you will be stuck. You will not be able to solve the limit. But with Poisson's equation, you will be able to take this problem and many more complex problems and solve them easily and this is the major reason why most of electrostatics and this course deals with developing potentials and working out how these potentials interact with each other and solving them. The solution is usually in terms of what are called boundary value problems because I am specifying the potential at the boundaries and I am asking tell me what is happening inside.