Electro Magnetic Field

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Lecture - 7
Gauss's Law

Good morning. Today, I want to discuss two or three very important topics. Let me put down what I want to teach so that we will see what we have to cover and what we end up covering. I want to introduce the concept of electric flux. Following that I want to introduce the concept of divergence of a vector field and along with divergence, I will introduce the divergence theorem. From this we will obtain the most important result namely Maxwell's first equation. The equation we are trying to work towards is divergence of flux is equal to charge density. It is a very important equation and the root to that equation is through a fair amount of mathematics.

So, what I am going to try and do is minimize the amount of mathematics by drawing lots and lots of pictures. So let us make a start. We already know a few things. We know that for point charge, the electric field is equal to the charge divided by 4 pi epsilon naught times r minus r prime divided by mod r minus r prime cube. This is Coulomb's law and it is the starting point of everything because that is the only experimental observation that we start with.

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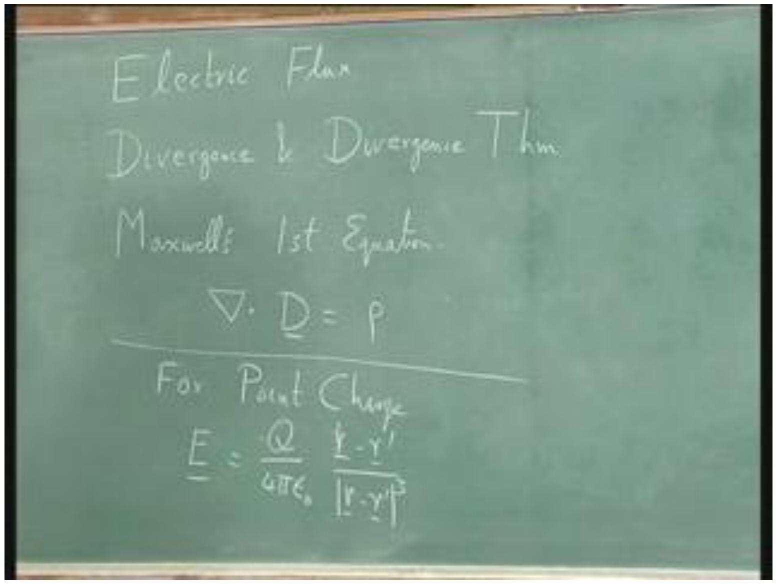

We want to take this observation and go from this observation to something more useful. So, let us make a start. Let us assume that we have a co-ordinate system and we have a charge Q at the origin. We know that the electric field therefore point outwards on straight lines in all directions. So, the electric field is outwards in all directions.

Now, supposing I draw a sphere around this origin. Let the sphere have a radius r. So, it is a sphere of uniform radius r and I want to calculate how much electric field is leaving the sphere. So, what I mean by that is, I want to, I want to integrate over the area of the sphere. I want to calculate the electric field d A. Now there is a problem in defining this integral. The surface of a sphere is not a flat plane. You cannot write d A as d x d y. That is not what it is. Secondly, the electric field is a vector; it has a direction.

So what exactly do we mean when we say area integral of the vector field E over the area of the surface? We have to do something because the answer of this integral will depend on how we define these different pieces. So, what I mean when I say this is, I mean, take a piece of this sphere. The electric field is going right through the sphere outwards because we know the electric field is radially outwards. There is an area; so multiply the area by the magnitude of the electric field. That is what I mean here. So, what I am really saying right now is integral of the area, over the area, magnitude of the electric field times d A.

Well, I know something about the electric field. I know that the electric field is given by Q over 4 pi epsilon naught r square; but how do I write down area? Well, in a spherical coordinate system, the coordinates on the surface are defined by this angle theta and if I drop a perpendicular down onto, so this little area has an image on the x y plane. This is x, this is y, this is z. Then the angle with respect to the x axis is called phi.

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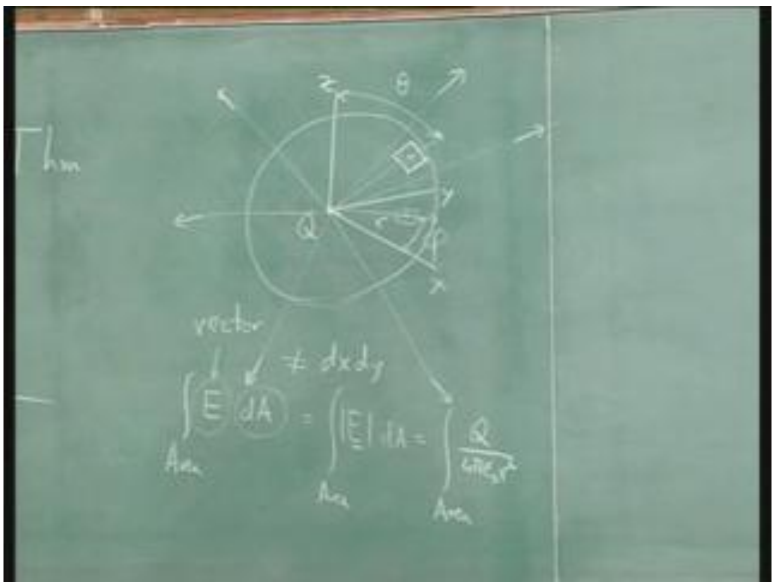

So the coordinate system in spherical coordinates is r theta phi. Sometimes it is called psi. So, if I use such a coordinate system, then the area here is this distance multiplied by this distance. Now, this distance corresponds to drawing a circle at constant phi but varying theta. In that case, this circle actually goes all the way down to the South Pole and comes back. This is what is called a great circle. When we do geography, these circles are very common. They are the circles of longitude.

Now such a circle, if you want to know what this distance is, you draw lines from the origin connecting the two ends of this region. There is an angle between them. That angle, I will call delta theta. The main angle is theta itself. So, this length is delta theta times distance. So it is r delta theta. You would have learned this from your geometry. If you take any circular arc, you know that if this is theta, the distance on the arc is r theta... So the same idea is, if this is delta theta, this is r delta theta. If you go to the other side, that side has a projection and in this projection, theta is constant. That is, this line is sitting at constant theta but varying phi.

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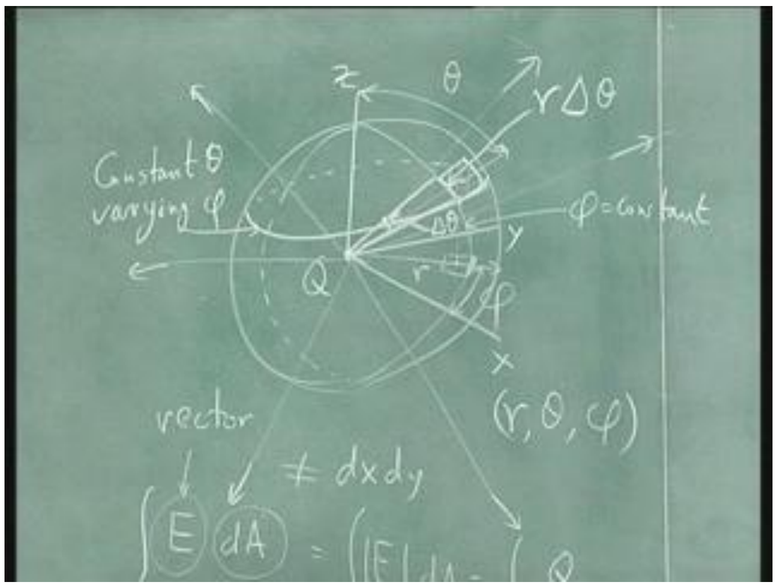

And if you want to know the length, here you have to take the centre of this circle and draw lines. The angle between these two lines - that angle is delta phi. And so this length is equal to this radial length times delta phi. But what is that radial length? Well, this angle is theta, this diagonal is r. So, this length is r sine theta. So, it is r sine theta delta phi. So, now I can write down what d A is. It is multiplied by r d theta times r sine theta d phi. So, this is d A and this is E. So I am multiplying E by d A.

Now, there is a very strange thing that happens. You have learned it in school, you have learned it in college; but it is still strange, namely: there is r here, there is r here which means r square, and there is r square in the denominator which means r cancels out. So, when you do this integral, integral over the area E d A, it becomes equal to Q over 4 pi epsilon naught integral over d theta sine theta d phi. Now, what are the limits of that integral?

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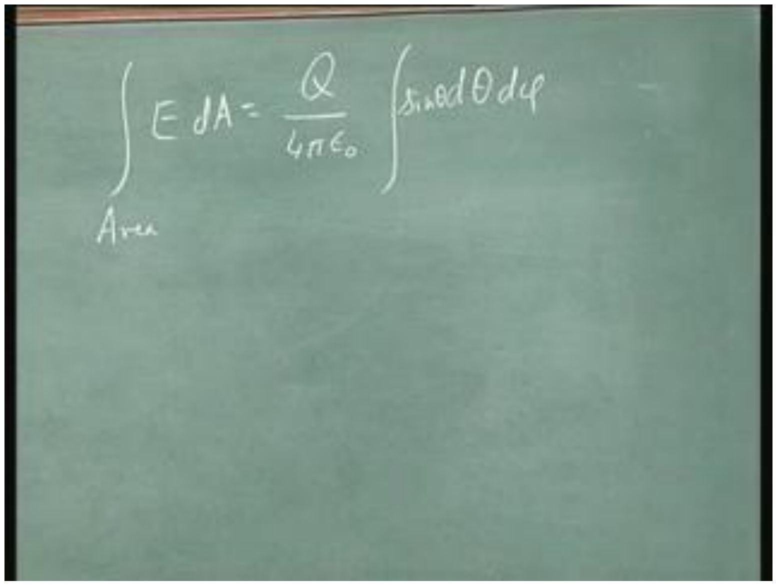

If you look at the diagram, this is x, this is y, this is z. This is some general point. The angle made with the z axis is theta. The angle made with the x axis of the projection is phi. So clearly, theta goes from 0 when it is aligned to the z axis all the way to the South Pole. So, 0 to pi. I work in radians. So pi radians is the same thing as 180 degrees. What about phi? When phi is along the x-axis, phi is 0 and the maximum it goes, it goes past the y-axis, past the minus x-axis, past the minus y-axis and finally it comes right back to the x-axis. So, that is 360 degrees or 2 pi radians.

In scientific work, we always work with radians because that is the natural unit for angle. So the limits of this integral are known. Theta goes from 0 to pi, pi goes from 0 to 2 pi. Now, this is a very easy integral to do. The integral on phi is trivial because sine theta does not depend on phi. So, that becomes this 2 pi. This can be written as 2 pi Q over 4 pi epsilon naught integral 0 to pi sine theta d theta. Well, integral 0 to pi sine theta d theta is 2. I will leave it to you to work it out.

So, this answer finally becomes integral over the area E d A is equal to 4 pi Q over 4 pi epsilon naught which is Q over epsilon naught for all r. Regardless of how big a sphere you work with, regardless of the radius, the answer is the same. When you integrate over the area E d A, the answer does not depend on r, does not depend on how big a sphere you use; it only depends on how much charge was inside.

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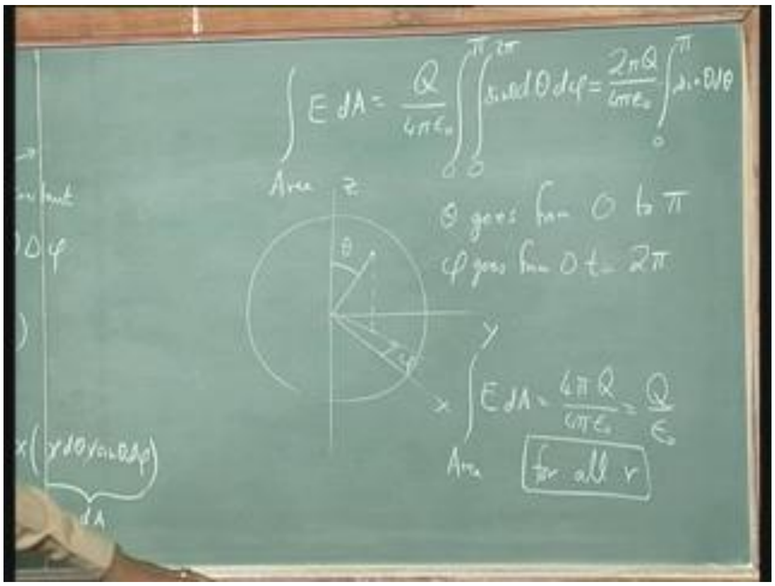

Now, this idea is related to another idea, an idea that we have already talked about, which is, if I put a charge Q, I can draw direction of E in all directions. For example, a direction E here, a direction E here, direction E here, direction E here. Now I can draw arrows and draw a line through the arrows. So, if I take lots of arrows and just keep drawing them and then draw a line right through the arrows, this is called a field line. The place where you would have seen field lines most often would have been in magnetic fields where if you put a bar magnet with a north pole and a south pole, you draw, you draw magnetic field lines and you draw them going from the north pole to the south pole. The same idea holds for electric fields. In fact, the dipole is nothing but a magnet.

So, you can draw field lines and the field lines leave in all directions. For a single charge, the strength of the electric field does not depend on the direction. It only depends on distance. So, you get straight lines in every possible direction that are just leaving the charge. And if you say, that the amount, the number of field lines that will leave a charge is proportional to the angle of the region... so if I, if I define a certain angle, a cone, and I say I will put one field line per degree square, then I will have the whole lot of field lines, approximately 360 degrees, in a certain sense, squared. It is not quite 360 degrees squared; but you will have a large number of field lines going in all directions and if you ask how many field lines will cut through a sphere of radius r, the number of field lines is always the same because no matter how big a sphere you drew, all these field lines are going to cut it and it is going to be the same number; same large, but finite number.

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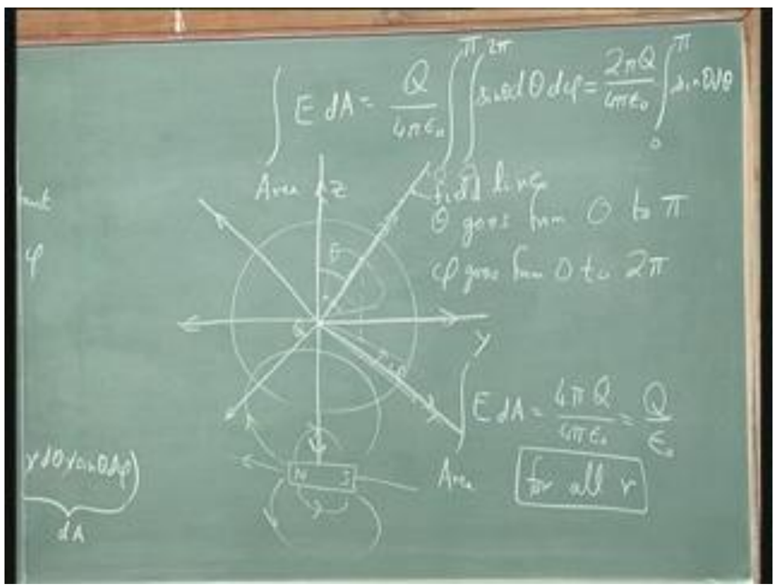

In a certain sense, this integral is measuring the same thing because the electric field is closely related to these field lines we are drawing. The electric field is actually a measure of number of field lines per metre square. If you drew it in this way, if you made field lines very dense and then you drew them and you took any area and you counted, we found out how many field lines are coming out and you counted them all up, the electric field would be proportional to the number of field lines per metre square and then you multiply it by the surface area which was metre square. So, the total number of field lines came out and in a certain sense this statement, this is the same statement as drawing field lines. Keep that in mind because it is a very important idea.

So, that was for a charge sitting at the origin and a sphere centred on the origin; but supposing I put my charge at the origin but I did not centre my sphere at the origin. I put my charge off-centre. This is the centre of my sphere. It has a radius r but the charge is sitting somewhere else, sitting inside the sphere, but is not at the centre. Now the charge is still giving field lines that are straight but something different is happening here. For example, this part of the sphere will be close to the charge. Therefore, it will see a strong electric field. This part of the sphere on the other hand, we will see a weak electric field because it is far away. The question is, if we now did integral E d A, what would it be equal to?

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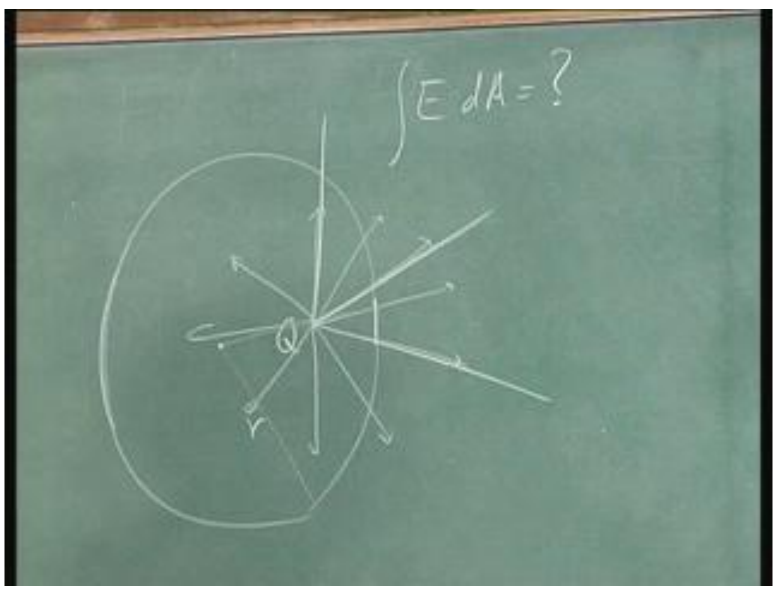

In a certain sense, this is the central problem of this lecture. That is, what is E d A, if I shift the charge off-centre? And the answer comes fairly easily. Answer is as follows. Supposing I take a little piece of area on the sphere. I can make a cone connecting to the charge. Now roughly, all parts of this area are a distance...some distance, call it rho. So, the electric field in this direction E is equal to Q over 4 pi epsilon naught 1 over rho square. That is the magnitude, alright?

Now the problem is, this electric field is pointing this way but if you ask what is the surface pointing, surface is tilted. If I look at this electric field, look along the electric field, let us say this is the direction of the electric field. This surface is looking like this; surface is not perpendicular to the electric field; surface is tilted. Parts of the surface are further in the direction of the electric field, parts are further behind; and the question is what do we mean by E d A when that happens?

In the symmetric case, it was very simple; everything was pointing more or less the way we thought it should. So we did not have to worry about vectors and scalars. But now we better worry about it. What we mean by E d A is if I take this same surface and I extend this cone and make the cone so that it is a symmetric cone with respect to the charge, then there is another area A. This is the symmetric area. When I say E d A, I mean, take the magnitude to the electric field, take this symmetric surface area, multiply them together.

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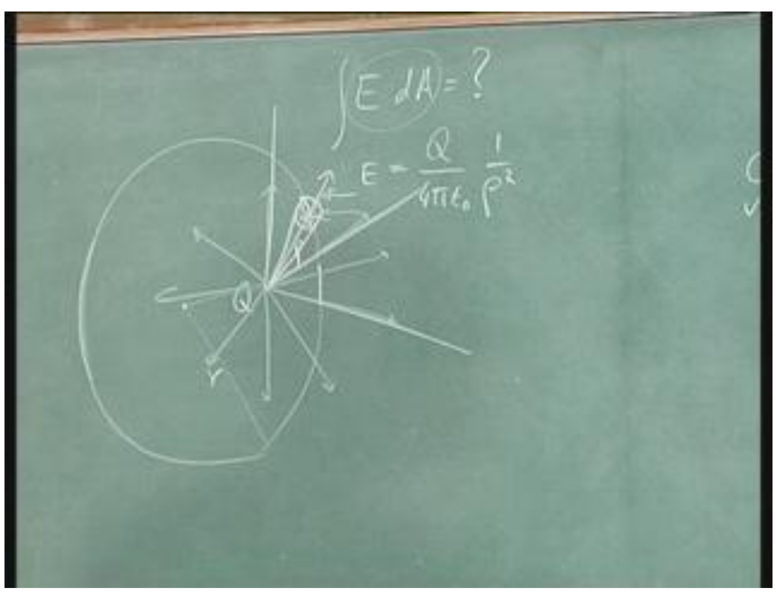

How do we work out such a concept? It is not so easy. Let me draw it again so that you can see what I am trying to get at. I have a charge Q. I have a surface area that looks like this and I am saying I really want to work out my result on a surface area that looks properly symmetric for the cone. That is, I have cut the cone not in the proper symmetric way. I have cut it in some other odd way. How do I decide what fraction of this area is given by this area?

So this is d A. This is some d A. I will call it d A normal. So, how do I figure out how much d A normal is if I know what d A is? Well, the answer is quite simple because we have already done it in the previous problem. What we will do is instead of working on d A n, I will take a unit radius sphere and then I will ask how much area does this cone cut through this unit sphere. It cuts through some area. I am going to call it d omega and then this d A n is equal to d omega times r square where r is the total distance. Why? Because as you go further and further and further, both dimensions of this area keep increasing linearly with r.

So the area must increase this r square. So, it is the same angular size. If I was sitting here and I watched this region and this region and this region, all look the same size but the actual size is increased by r square. The analogy I have for this idea is, supposing I am sitting. I am here and I have put a camera there and there is an anthill, very close by. Far away there is a mountain. If this anthill, if my position, the anthill position and the mountain are correct, the base would agree, the peak would agree; so much so, standing here, the anthill and the mountain look the same size. This is what is known as d omega. This is the apparent size of an object. The d A is the apparent size multiplied by r square. Now, what is this other fellow, the tilted fellow?

Well, supposing my mountain was not like this. Supposing my mountain was like this. Even then it would seem the same size because as far as I am concerned, from this angle up to that angle, the object is there. The fact that the object is nearer here and farther there does not affect the apparent size of an object. Only if I know the depth do I know anything about the size of the object. So, that is the difference. d omega is a kind of standard size. The normal area is d omega multiplied by square of the distance; and the d A is the tilted size.

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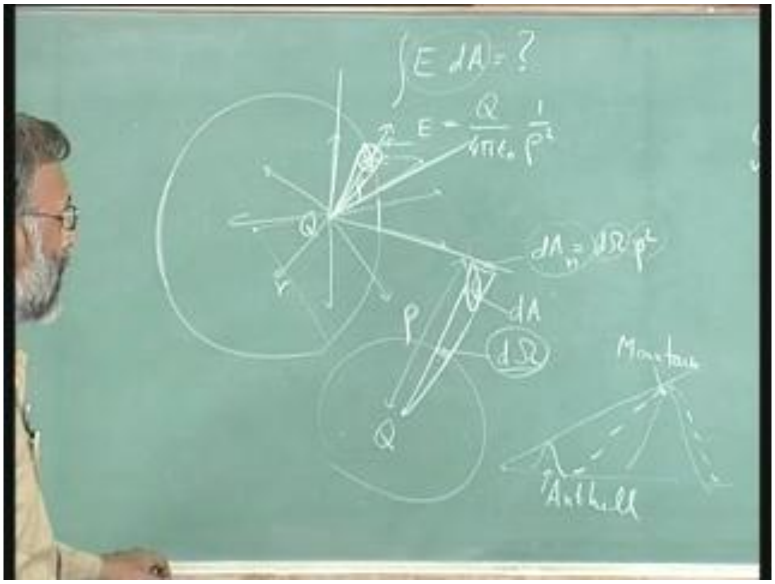

So, what I do here is I say that well, let me take, correct this. I used r here and rho there. I should use a consistent unit. So, I will use rho here again and rho here again. So, when I work out my contribution to the flux, I say E d A is equal to Q over 4 pi epsilon naught 1 over rho square multiplied by d omega rho square. Again, the rho squares cancel. Very strange.

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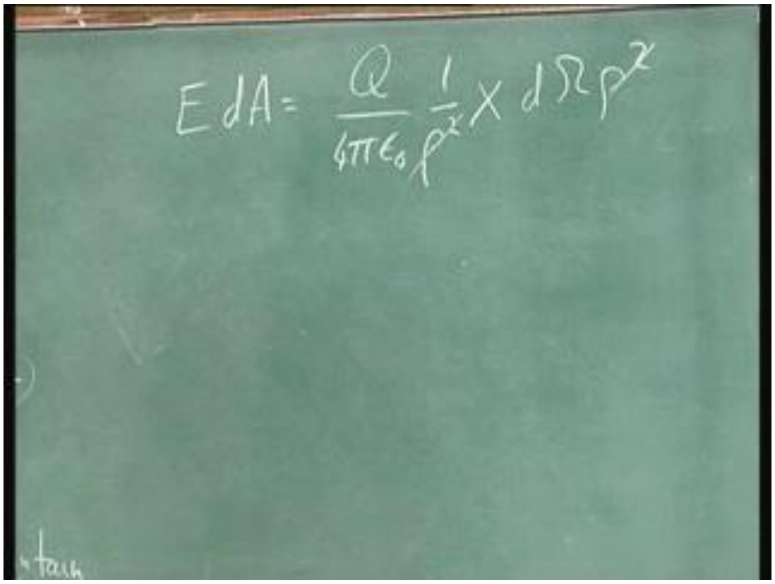

Now when I integrate, you will notice what I am integrating over. Let me redraw the picture. I have a unit radius sphere. I have another bigger sphere and I have a charge somewhere, Q. So I am looking at some cone from the centre which is at a distance from this Q. This distance is rho and I am saying that when I draw this cone, there is a solid angle here and this is what I call d omega. Now, this d omega I can integrate over all possible angles. So, when I do my integration, here is what I get. I get integral E d A is equal to integral and my d omega I do using the same kind of integration ; 0 to pi d theta 0 to 2 pi sine theta d phi of Q over 4 pi epsilon naught; rho squares cancelled and d omega is nothing but this, the same answer.

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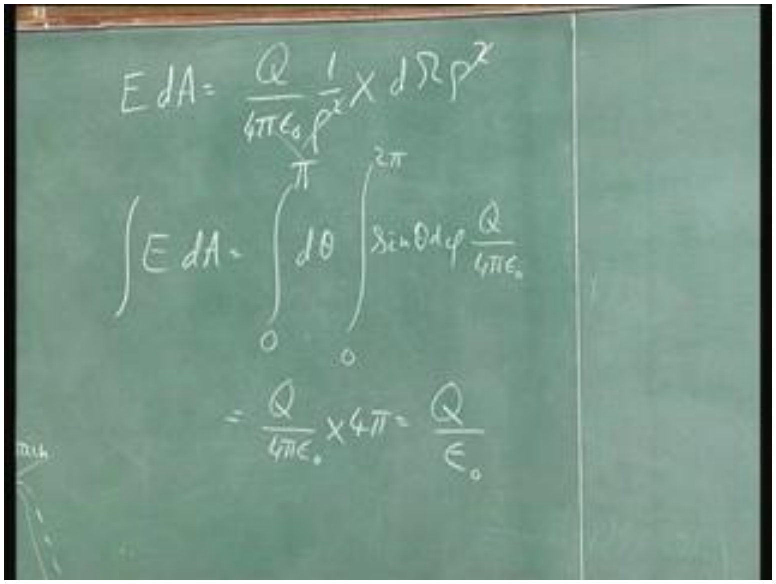

Let me review because I think it is a confusing idea. In this picture, I worked out what happened if you calculated integral E d A for a charge at the origin, for a sphere centred around the charge. It is an easy calculation and when we do the calculation, we find it is Q over 4 pi epsilon naught integral sine theta d theta d phi, which this integral gives you another 4 pi which cancels this 4 pi. So you get Q over epsilon naught.

When I go away from the centre of symmetry, my sphere is centred here, the charge is there. What is happening is that, near points have strong electric field but they have small area; far electric points, far points from the sphere have weak electric fields but large areas.

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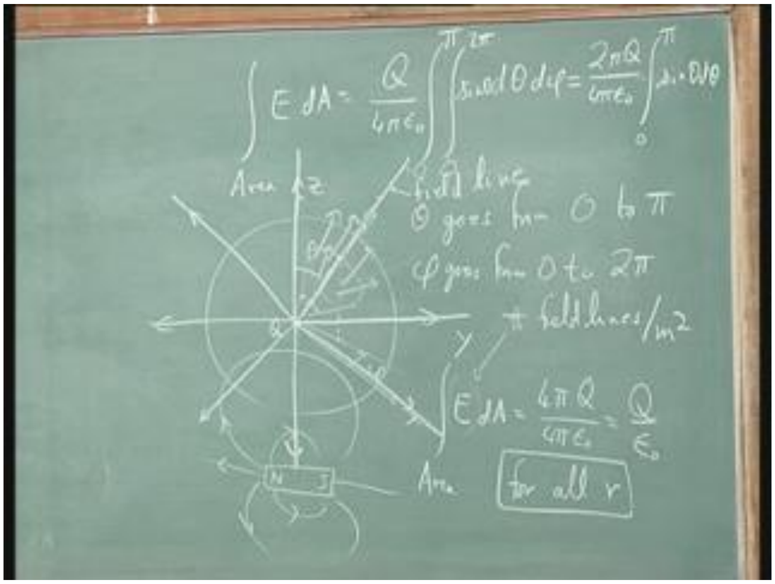

How small and how large? The electric field goes as 1 over rho square, area goes as rho square. So this is independent of rho. Similarly, this is independent of rho and so when you integrate the whole answer, you get an answer that depends only on Q. It does not depend on rho and most peculiarly does not even depend on where the charge is put. Wherever in the sphere you put it, you get the same answer and nothing in here actually cared about the fact that this is a sphere. I was working with what are called solid angles. So I could have generalized this result and made it a general shape. I would have got the same answer.

Now let us do one more calculation. This is the last one and it gives us the result. So we have done sphere charge in the centre. We got integral d A equals Q over over epsilon naught. You also had sphere charge of centre, same answer. So, both of these give you the same answer. Now the third problem to try is sphere, but charge outside. What does E d A look like now? Well, the same thinking we did can be used here. Supposing I take a tiny cone and puncture this sphere with it. The cone has its point at the charge and it has an apparent angular size of d omega which would be equal to rho d theta rho sine theta d phi, alright?

Now, this sphere cone punctures this sphere at two points. There is a point where it enters and a point where it leaves; where it enters, the electric field is going into the sphere and it is, if this distance is rho 1, the amount of flux entering the sphere would be Q over 4 pi epsilon naught 1 over rho 1 square multiplied by size of this, which is d omega times rho 1 square entering. Now, let us look there at the dotted line. Flux is leaving. How much is leaving? Well, it is Q over 4 pi epsilon naught 1 over rho 2 square into d omega rho 2 square, right? This distance is rho 2. So this statement is saying exactly as much charge is entering. The rho 1s cancel. The rho 2 cancel; exactly as much flux is entering as flux is leaving. Total amount of flux contributed by this charge is zero because whatever is entering left, whatever is leaving enter.

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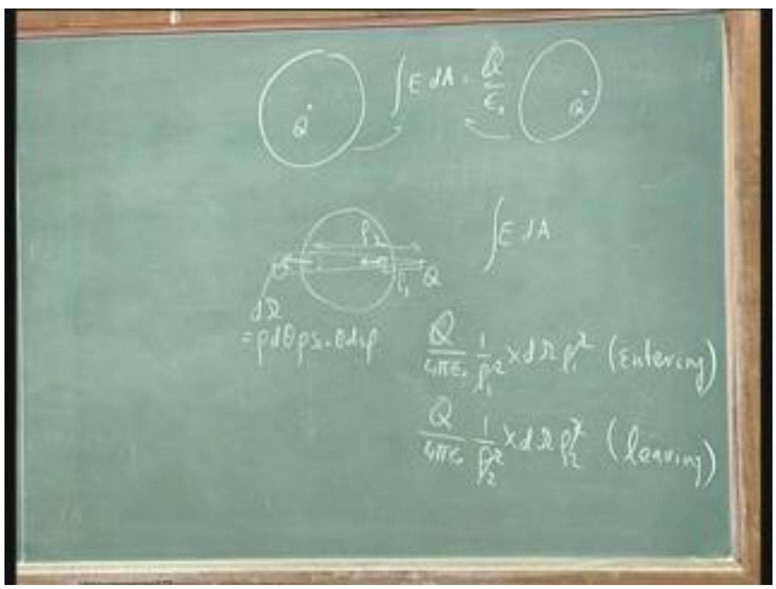

So if you do this integral now, you get is equal to 0. So these are the three results: if you put the charge in the centre, we worked out trivially integral E dot d A or E d A is Q over epsilon naught; if you put the charge on the side anywhere, it was still Q over epsilon naught; but if you put the charge outside, it is zero. So we can we can make this a kind of a general statement. For one thing, it does not have to be a sphere. It can be any kind of surface. A sphere is only convenient for our thinking.

So what is our conclusion? Our conclusion is that integral over the area...and I am going to put a circle. A circle means integral over the entire area of the surface, of any surface S E d A is equal to one of two things: 0 if Q naught in volume Q over epsilon naught; if Q in volume, should be a fairly obvious statement. What we are saying is if you have any surface and you put a charge, lines are leaving the charge in all directions. If the charge is inside the volume, all the lines are leaving.

So when you integrate over the surface, you get an answer. The answer is Q over epsilon naught. If on the other hand the charge is outside, for every line that enters, it also leaves. So, there is no net charge, no net contribution. So, E d A becomes zero. Since this is a vector relation, we have to be a little careful in how we write E d A. I have been sloppy, intentionally sloppy, because I did not want to get into that; but now we better get into it. What we mean is that if the charge Q is here, the electric field is in the direction away from the charge E. Now, the surface may be some other way, may be like this. That is my d A. As I said, the way to make this sense, you draw a cone, draw a unit surface around the charge, find the actual area cut by that cone.

There is an easier way of saying the same thing. If you look at this area, supposing the electric field, instead of being in this direction were in the direction perpendicular surface, then our E d A works out very nicely. That is what we meant by E d A. Any way, we meant the electric field is going right out of the surface, straight up; and then you just multiply E and d A you get the answer. The problem is that how much of the electric field should we use? And the answer again is obvious. You should use the projected amount of the electric field; either you project the area along the electric field or you project the electric field along the area. And we know all about projection, we have been doing it.

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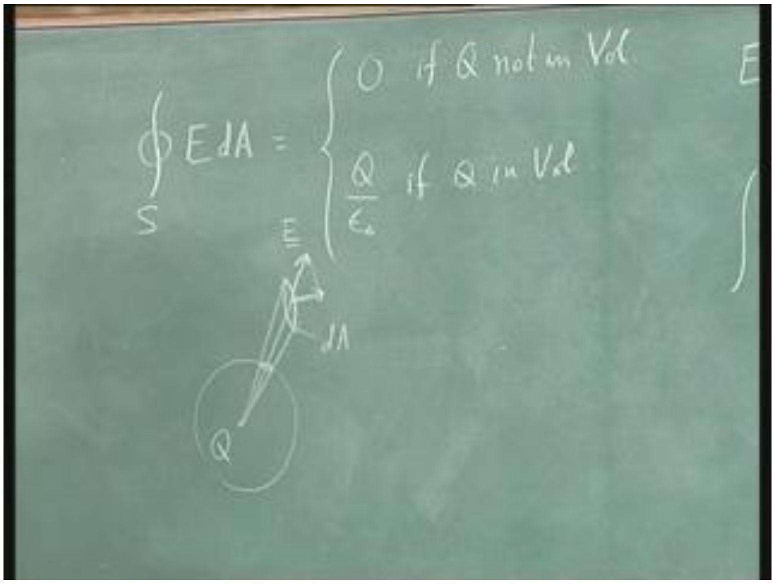

So really, this point should be really E, the electric field dot; the little bit of area taken as a vector. How can you take area as a vector? That really is quite a complicated idea. So, let us look at it. Supposing I have a square. Let us say this is x, this is y and vertically z. This distance is d x, this distance is d y. Now, if I want the normal direction to this surface, I know that this is the z direction because it is in the x y plane. z is naturally perpendicular to the x y plane but there is another way I can say it. I can say the normal direction; the direction of this d S is perpendicular to all lines in d S. By that I mean I have the square. So, I can draw any line I like that lies in the square; that is, lies in the x y plane. Whatever I choose as normal must be at 90 degrees to all of those lines, must be at 90 degrees to all of those lines. I can choose two particular lines: I can choose the x line and the y line.

Now, if I want something that is 90 degrees to two things, there is a natural operator I know off, which is the cross product.

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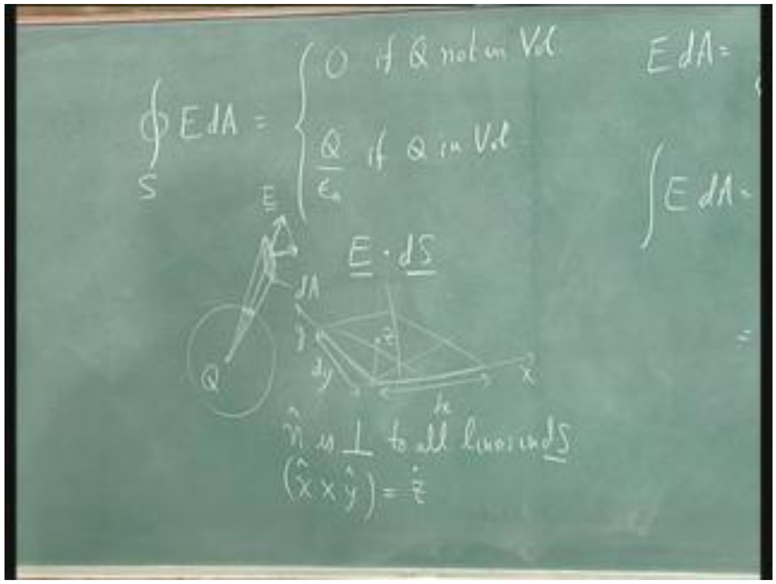

So if I take x hat cross y hat, that is the direction that is perpendicular to the x direction. It is also perpendicular to the y direction and that is why the normal direction is along the z direction.

Now the same idea can be used here, what we call the normal is nothing but the cross product of lines in that fall inside the d A; and if you take any two lines of the d A, take its cross product. That will give you a vector in the normal direction; but this is actually very useful because supposing I have a surface and in that surface I have two vectors. So my area is not at 90 degrees. My area is somewhat tilted. So, there is an angle theta here. This is my d A. Well, the area of this parallelogram is the base into height and direction is normal. That is what I want. But what do I mean by base into height? If this is l 1, this is l 2, base into height is l 1 l 2 sine theta because l 2 sine theta is the height. So base into l 2 sine theta is the area and the direction is perpendicular to both l 1 and to l 2. But if you look at l 1 cross l 2, the magnitude of that is magnitude of l 1 magnitude of l 2 sine theta and the direction is, that is 90 degrees to both l 1 and to l 2, which is outwards.

So, that is why when you look at any area and you want to mean by area, it is the cross product that comes in. The cross product naturally defines area, the magnitude of area, and it naturally points in the normal direction. So, that is how we get our integral. It says closed integral E dot d S over any surface is equal to 0 if Q outside V, Q over epsilon naught if Q in V and this is what we call the divergence theorem.

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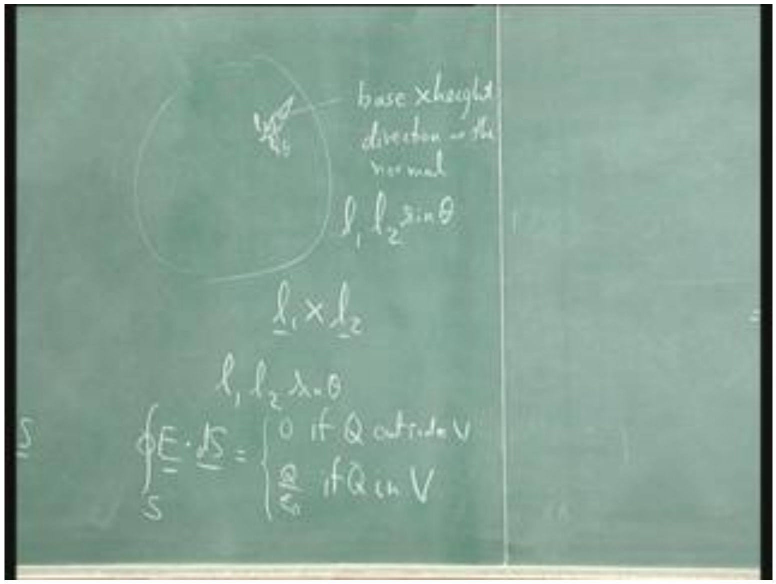

Now we have to take one more step to reach Maxwell's equation. It is not a difficult step to take but still has to be done. Now if you take any volume, first of all, I have to take actually two steps. So let me take the simple step. First, supposing I had multiple charges. So I had Q 1 Q 2 Q 3 Q N. Now I already know that for any one charge this is going to happen as volume integral E dot d S equals zero. If I will call it E 1 Q 1 in V Q 1 over epsilon naught, if Q 1 if Q 1 naught in V and in V.

Similarly E 2, similarly E 3, etcetera, etcetera, etcetera. But the electric field due to a set of charges is the sum of the electric fields. So the loop integral, sorry, the closed surface integral total electric field dot d S is really sum on all the charges surface integral, sorry, E i dot d S is equal to sum on charges in V divided by epsilon naught. By that I mean if I had a charge Q i that was outside, that particular charge would not give me any integral E dot d S. We have already worked that out because it is zero. Only those charges that are inside the sphere would contribute to this integral.

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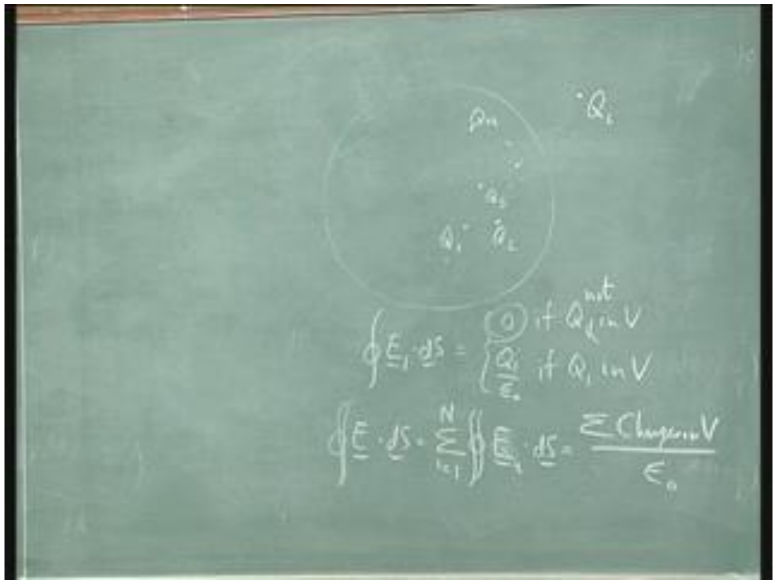

So, it is the sum of all the charges that are in volume V divided by epsilon naught. Now, this is really the useful version of Gauss' law, the divergence theorem, and we can immediately use it actually to get answers. But there is another version of this equation which is also useful and that version looks as follows. Supposing I take this charge, take this sphere that has many charges in it and let us say there is a particular Q i that is sitting somewhere. I look at a small volume that contains that particular charge. Now, what I know is if I do integral over this delta V, sorry, delta S of E dot d S, it is going to be equal to sum of charges enclosed divided by epsilon, except there is only one charge enclosed.

So, it is going to be equal to Q i over epsilon naught. That is a very funny thing. If you look at it saying that if I have lot of charge everywhere and I look at a small part of this box only, the charge inside that box is going to give me this contribution. It is tough that outside is not going to give me any effect but if it is a very small box, I should be able to write this in a different way.

Let us look at what I mean by that. I am going to go to Cartesian coordinates. So I go to a cube. When I say the integral over the cube E dot d S, I mean integral over this surface, this surface, this surface and behind the back surfaces. So there are six surfaces over which I will integrate. Let us say this is x, this is y, this is z. Now, if you look at...for example, the y z surface, we will call this surface 1 and this is surface 2. If I integrate over the y z surface, both of them, what do I get? This is the value at this point is x, the value at this point is x plus delta x. Similarly, the value is y and this is y this y plus delta y; this is z and this is z plus delta z.

So, what I get is over this surface. This integral becomes E of x y z times delta y delta z and its flux is entering. I am going to put a minus sign here. Now E is a vector. So, which component is going to enter? Well, the E x component will enter; the E y component will go parallel; the E z component will go parallel. So, E y and E z cannot contribute at all; only E x can contribute. Now, what about region surface 2? That would give me electric field leaving. So, I am going to put a plus sign again, E x; but the value of x is x plus delta x y z. Again, delta y, delta z - if I combine these two terms, I can write this as first. Let me say, plus 4 other terms, this is equal to E x of x plus delta x y z minus E x of x y z into delta x delta y delta z plus 4 other terms.

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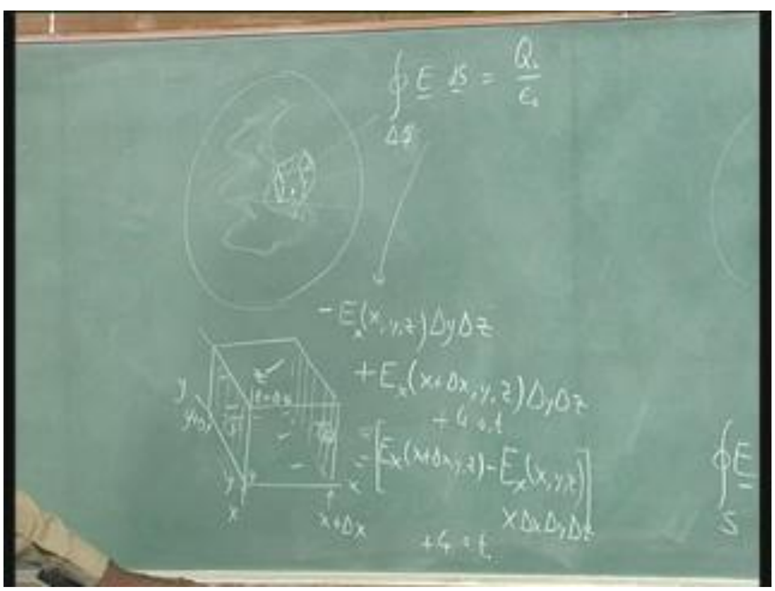

I have to divide by delta x. So, do you see what I have done? I have taken delta y, delta z, common. I have multiplied and divided by delta x and written out the other terms. What do you get? You get that integral E dot d S is equal to del E x del x times the total volume plus 4 other terms; and I am going to write out the 4 other terms. That is del E y del y delta V plus del E z del z delta V. Each of these is two terms, two surfaces.

So, there are six surfaces to a cube. This piece is the scalar quantity and it is called the divergence and in a certain sense what is measuring is amount of outward flux leaving that little volume. So, you can now relate these two quantities to the previous question we have which is loop surface integral E dot d S is equal to charge enclosed divided by epsilon naught. Now, charge enclosed in a volume is typically written as the charge density in the volume multiplied by the volume because for interesting problems in electricity and magnetism, we do not have point charges. Point charges are only where we start. What we actually have is mired out charge, many, many charges. So, we have so many charges per cubic metre multiplied by the volume.

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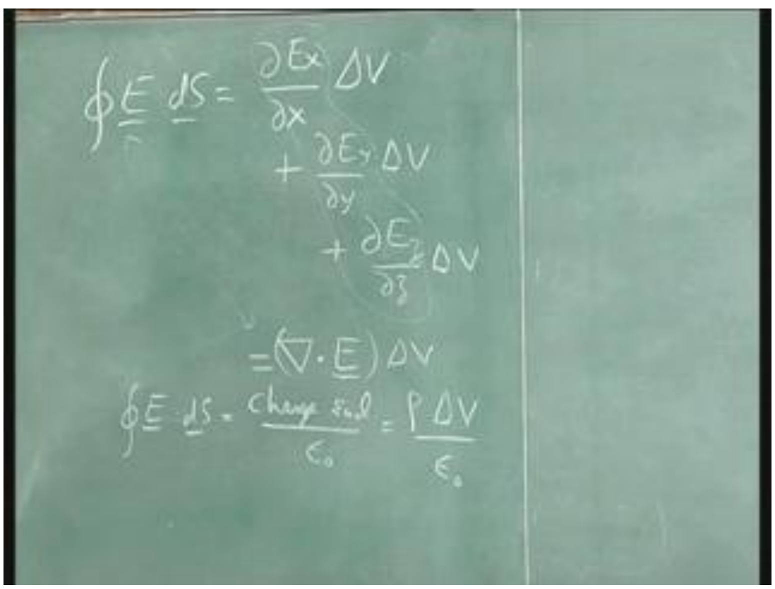

So, we now have a final equation. Divergence E is equal to rho over epsilon naught. It is the same equation as surface integral over a closed surface. E dot d S is charge enclosed divided by epsilon naught saying the same thing; but that is an integral version of the equation, this is a differential version of the equation and this is the very useful version of the equation. It is called the divergence theorem or it is called Maxwell’s first equation. It is this equation is so important that we will keep coming back to it in the next few lectures.