Electro Magnetic Field

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Lecture - 5
Electro Static Potential

Good morning. Today we shall complete a few more examples on electric fields and then go on to the important concept of electrostatic potential. So, let us first look at one interesting problem. Suppose you have two line charges. This is coming out of a problem in your textbook - problem 2.21. You have two line charges; each line charge has let us say, a charge of rho coulombs per metre. The line charges are at a distance d apart. Now, we would like to know what is the force per metre exerted on line charge 2; call this 2, due to line charge 1?

Well, in our previous class, we derived what the electric field was due to a line charge and the answer we got was by taking points above and points below and integrating from z equals zero to z equals infinity, and what we got was that the electric field was along the radial direction; it was proportional to this rho. It was divided by 2 pi epsilon naught and it is scaled as 1 over r. So for a distance d, this r becomes d. So, it is equal to...along the direction, radial direction, rho over 2 pi epsilon naught 1 over d. So, this is the electric field.

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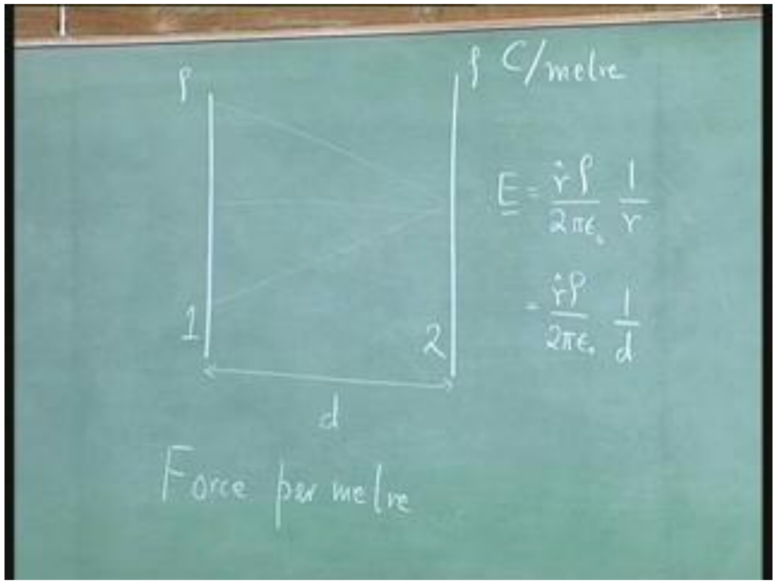

Now, this electric field acts on 1 metre of charge because I want the force per metre. How much charge is there per metre? Well, it is nothing but rho itself. So, the force per metre is equal to rho times the electric field; both are vectors. I already have the electric field. So the answer becomes...the force is along the radial direction and it is equal to rho square divided by 2 pi epsilon naught d. So, if you keep two line charges and you...each of them has a charge rho per metre, this is the kind of force per metre that they exert on each other; which means that the total force between these two line charges is infinite. So, line charges are actually very exotic things. You cannot actually create them in practical situations.

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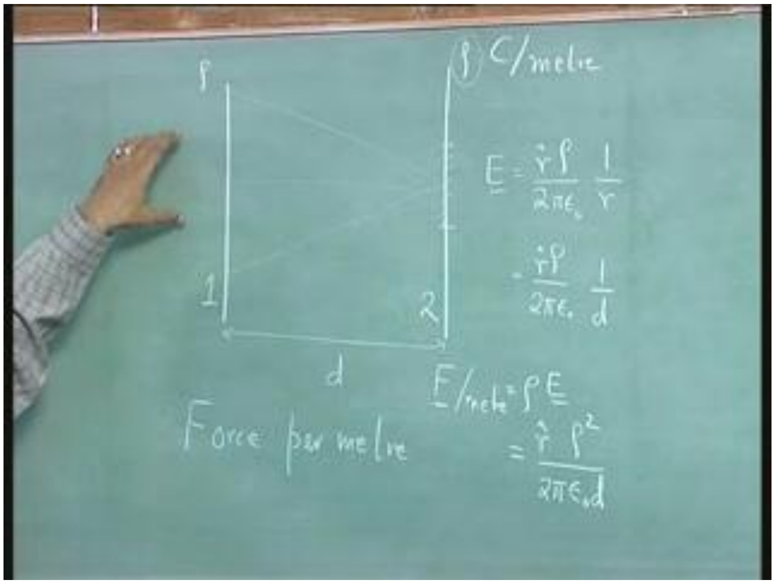

However, if you talk about wires, say an overhead wire, power line, well, those lines can be a few metres apart and distance of 100 kilometres; and therefore such very long closely placed wires are actually a very good approximation to this picture. So your overhead power lines exert force on each other when they charge up to different voltages and exert different electric fields.

Another example - I would like to know the force, the electric field due to a disc. A disc has a radius ‘a’ and it is of a height...the charge...the point where I want the electric fields at height ‘z’. The entire disc has a charge, let us say sigma coulombs metre square. And I want to know what is the electric field above this disc, z metres above this disc, and what is its direction. Last time I did half of this problem. So, let me just repeat that half. What I will do is I will take a ring. The ring has a radius r, it has a thickness d r.

Now, every point on this ring is equally distant from the point where I want the electric field, because every point has a distance that is equal to z square plus r square, square root. This is just Pythagoras theorem, because it is a right angled triangle. So, it is the...hypotenuse square is equal to base square plus height square. However, the electric field due to each of these charges is pointing in a different direction. In fact, the electric fields describe a cone. As I move the charge around in a circle, the electric field moves in a cone around this point.

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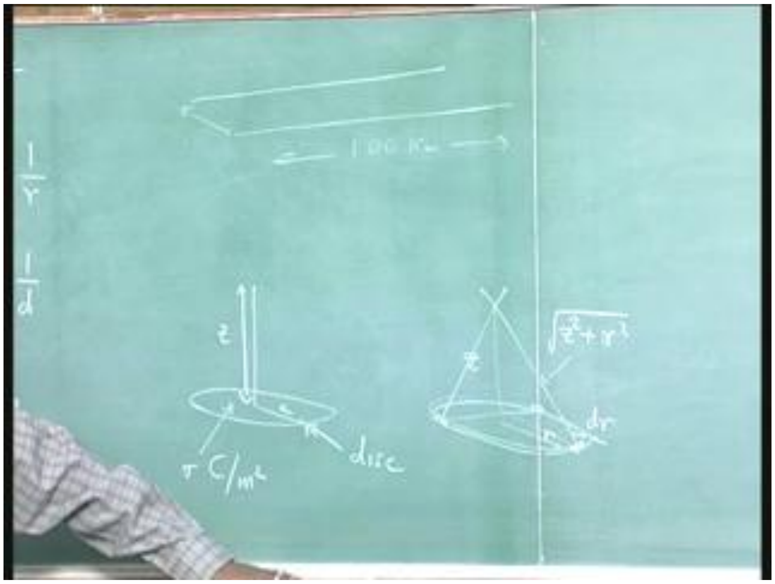

So, how do I add it up and find a net electric field? What I do is I take diagonally opposite points of charge. If I take two points of charge that are diagonally opposite, then what happens is that the two directions in which they point are...will tend to cancel in the x y plane, but will tend to add in the z plane. Let me try and make that more obvious. I am going to draw the circle looking from above. This is my ring of charge. I am considering a bit of charge here. The amount of charge is sigma times the thickness, d r times the theta direction thickness which is r d theta.

At a point up here, the field that it will cause will be actually pointing somewhere out. I am just projecting it down. So it will look like this. For example, if I took a piece of charge here, we could point in this direction. But now, if I took a point diametrically opposite on that ring, same amount of charge, sigma d r r d theta, now what is going to happen? This charge is going to produce a force. It is again up here. So, the force due to this is going to look outwards; force due to this is going to look outwards; but if I project it down, the projected parts are going to be equal and opposite. They are going to cancel.

So, the only electric field that is left will point straight out. That is, it will point in this picture in the upward direction. So, what do you get as a result for the electric field? You get d E which is in the z direction; is equal to this amount of charge sigma d r times r d theta divided by 4 pi epsilon zero. This gives me the...times 1 over z square plus a square. Radius is, well, I can say it is r square. This gives me the length of this arrow but I do not want the length of this arrow; I want this length, the vertical length. So, there is an additional cos theta.

I am afraid I am using the same symbol twice. So, let me call this cos phi. Phi is this angle, whereas theta is this angle. So, now I want to integrate this d E z around the circle. So, I get integral zero to 2 pi d E z integrated in theta alone. The theta integration gives me 2 pi times sigma d r times r divided by 4 pi epsilon naught 1 over z square plus r square times cos phi. Now, if you look at cos phi, cos phi is nothing but z over the hypotenuse. It is equal to z divided by square root of z square plus r square.

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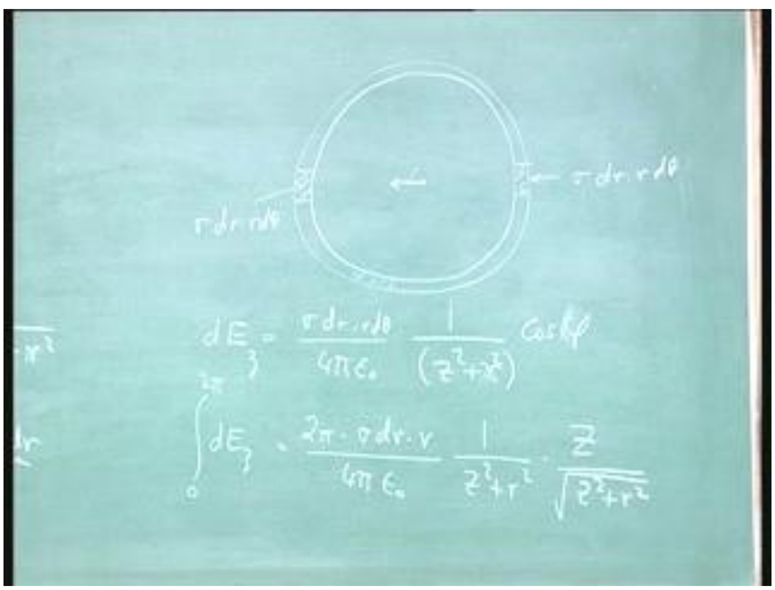

So when you put it all together, what you get is d E r...d E z is equal to sigma r z d r divided by twice epsilon naught z square plus r square to the power of 3 halves. This answer still depends on d r and we want the answer for the electric field due to a disc.

That is we want to add up rings of different values of r all the way from r equal zero to r equals a. So the electric field finally is in the z direction and it is the integral zero to a of sigma r z divided by twice epsilon zero z square plus r square to 3 halves d r. This is essentially the same equation. It is the same equation that I came up with for solving the electric field due to a plane. I ended the last lecture that way. That is because if I take the radius of this disc to infinity, the disc becomes the plane. So if I replace this a by infinity, then I have the answer for the plane.

Well you can see, this is a very simple integral to solve because you have got r d r up there and z square plus r square is your dependence. So you define u is equal to z square plus r square; d u is equal to twice r d r and you get the answer z hat. I can pull the z out. I can pull the sigma out, over twice epsilon naught; 1 more factor of 2, integral z square to z square plus a square of du divided by u to the power of 3 halves.

How I got that was I removed the common...constant pieces out, twice r d r is d u. So, r d r is d u divided by 2. The denominator has z square plus r square to the power of 3 halves. z square plus r square is nothing but u. So it became u to the power of 3 halves. This is easily solved. So you get, this is equal to z hat sigma z over twice epsilon naught, u to the power of minus half divided by minus half. The half cancels out; so you get minus u to the power of minus half going from z square to z square plus a square.

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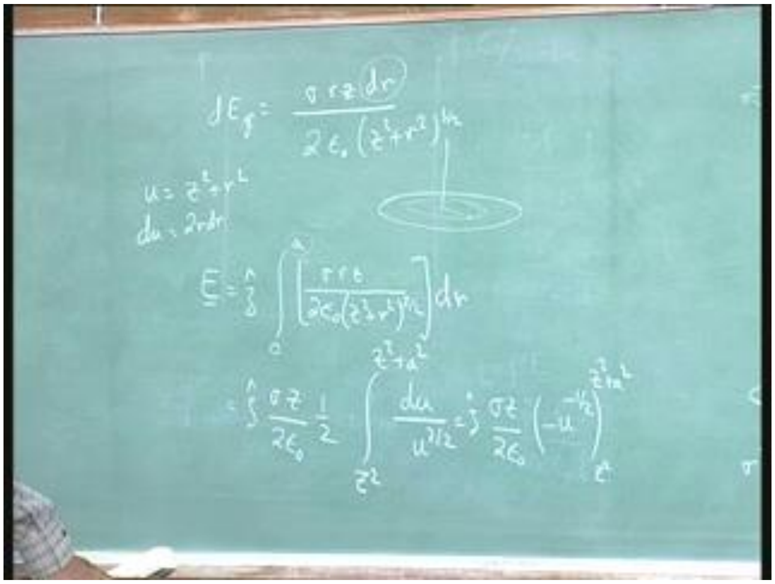

So the answer finally becomes that the electric field is in the z direction. It depends on sigma z over twice epsilon zero times 1 over z minus 1 over square root of z square plus a square. If you compare with the expression for the plane, the plane is nothing but the disc with a becoming large. When a becomes very large, z square plus a square is huge; and this huge number is in the denominator, which means 1 over z square plus a square square root goes to zero. So, this term will be missing and you will just get sigma z over 2 epsilon naught z, and z itself will cancel out.

So, this was the result I had given you earlier which was that if you have a plane, an infinite plane of charge, whose charge was sigma and you went a distance z from this plane and tried to find out what the electric field was, well, it did not matter how far you went out. However far you went out, the answer was always z hat sigma over 2 epsilon naught.

Now, how can this be? There seems to be something very wrong with this. How can it not matter how far from the plane you are? The electric field is always the same. Well, there is a reason for it. The reason is actually not that difficult. If you look at a particular theta from z, it strikes the plane on a circle; and if you ask how much charge is there between theta and theta plus d theta, the bigger angle is theta plus d theta, it will be the charge that is there in this ring. How much charge is that? Well, it is 2 pi times this radius times the thickness.

So, it will be...if this is theta and this is z, this height is z sine theta. So, it will be...the d Q would be equal to 2 pi times z sine theta times d of z sine theta. Now, what is the interesting about this is that it is proportional to z square. So the amount of charge that is present increases the further out you go; amount of charge between theta and theta plus d theta. The amount of charge is not constant. If I were 1 metre away, I get so much charge. If I am 10 metres away, between 30 and 31 degrees, there is 10 times more charge; no sorry, 100 times more charge.

However, how far away is this charge? The distance is z square plus z square sine square theta, is proportional to r square; square root is proportional to z. You can pull out z square out of the square root. So, you get an answer that is proportional to z. So the amount of force exerted by this charge force goes like d Q divided by r square. Well, d Q is proportional to z square, r is proportional to z. So, r square is proportional to z square equals constant.

So, what it means is the further out you go, the more of the charge of the plane you see. How much more? It quadruples for every doubling of your distance. At the same time, the distances increased and therefore the effect of that quadruple charge has become one-fourth. So, two effects cancel which is why no matter how far away from a plane you go, the force is the same. So the answer is, force remains constant.

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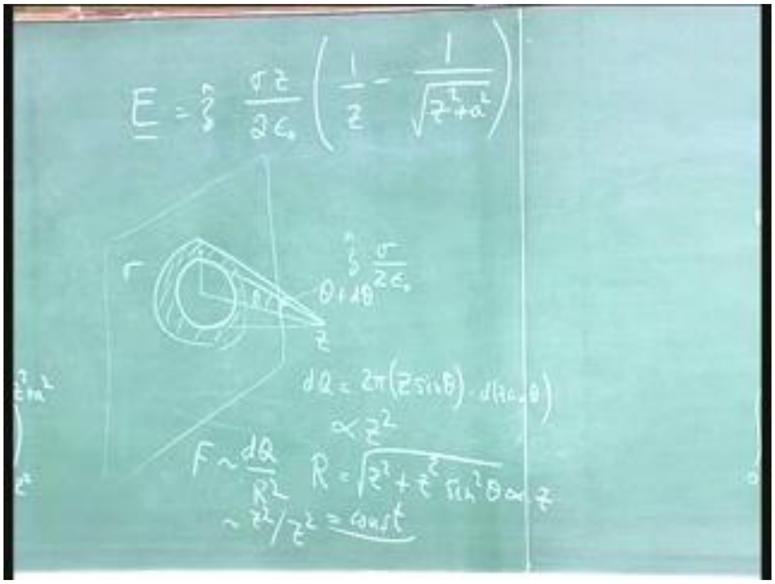

Now I hope these kinds of examples give you a taste of how one does simple integrals to calculate the electric field. Now, in a certain sense, Poisson's equation is all you require if you knew where all the charges are because...sorry, not Poisson's equation, Coulomb's law is all you require if you knew where all the charges are, because Coulomb's law tells you that the force is equal to 1 over 4 pi epsilon zero integral rho of r prime d v prime divided by...

So, if I knew exactly what charge density was everywhere, I just do this integral and I get the answer; end of electromagnetic theory. The only problem is, we usually do not know where charge is. Let me give an example. Supposing you have a metallic ball and I put 1 coulomb of charge on this ball. Well, the charges are free to move around because it is a conductor and we know that inside a conductor, charges can move, which means the charge could be anywhere on the surface and it could be anywhere inside.

So, we cannot use this formula. This formula requires us to know exactly where the charge is; whereas in practice, what we know is the total charge and we know the shape of the conductors but we do not know exactly where the charge is. See, even though we have done quite a bit in writing down Coulomb's law, we have not actually got to the really useful parts of electrical engineering. In electrical engineering, the most useful components are inductors and capacitors and a capacitor is nothing more than a metallic, pair of metallic plates which are kept close to each other.

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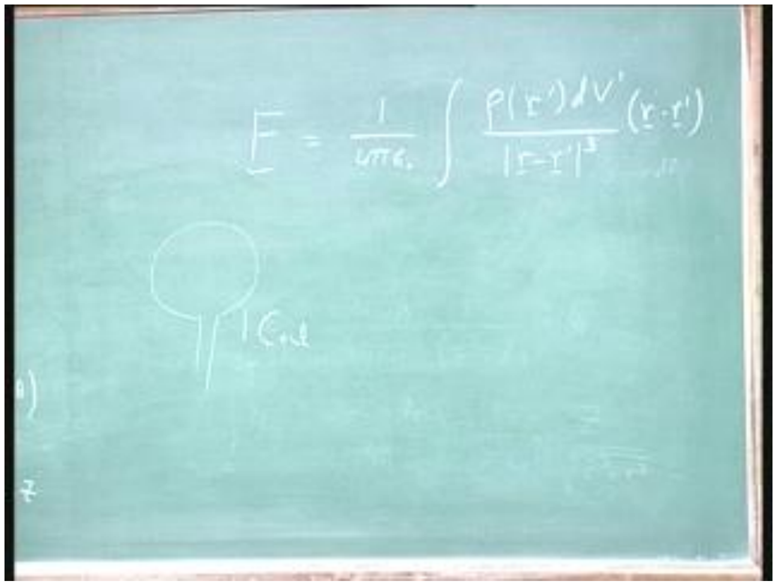

So, in order to tackle this problem, we need to introduce some new concepts. The first concept I want to introduce is work done in order to move a charge. When I was studying in college, I think work was probably the most difficult concept I ever encountered. Angular momentum was about equally hard; but those two concepts drove me quite crazy.

So, let me spend a little time trying to understand this word. You should have learned it thoroughly in mechanics, but I did mechanics too. I did not understand. See, the real confusion that comes is, as human beings, when we carry a heavy ball or we carry a weight and we just stand there...I am just holding this chalk and just standing there, we feel we are doing work. We feel our body is burning fuel with oxygen, producing carbon dioxide so that our muscles can hold up this piece of chalk. Therefore, we are doing work and we are quite correct. Our bodies are doing work.

However in physics, the word work means something else. The word work does not mean holding up a stationary object. If you have a stationary object that is not moving, no matter how much force there is on that object, no matter how heavy the object is, we are not doing work. A weightlifter, when he lifts the weight and reaches the top, is not doing any work. I mean, he will probably hit you if you told him that he is not doing any work but he, according to physics, he is not doing any work. He is just standing there. All the work he did was not lifting that weight, but keeping it stationary above his head; may have caused lot of problems to his muscles, but he was not doing any work. The weight was not going any higher.

So, work according to physics and according to engineering means motion of objects against applied fields; and the words are all important. You must be moving. If you are not moving, there is no work. It must be an applied field. For example, supposing I have an object that is under some strain. May be, I have got a compressed spring and I move the whole object. I am not doing any work. If I am...if I move the object but the object’s internal stresses are the only forces present, I am not doing any work. The forces have to be applied from outside and the motion has to be against this field.

For example, supposing force due to gravity is downwards and my weight lifter has got his weight and he starts walking. The poor fellow is doing a lot of...exerting a lot of effort, but he is not doing any work because the weight is neither going up nor down; and up and down are the only directions that are against the applied force. Going sideways does not get you anything. It may get you an Olympic medal, but it will not get you any work.

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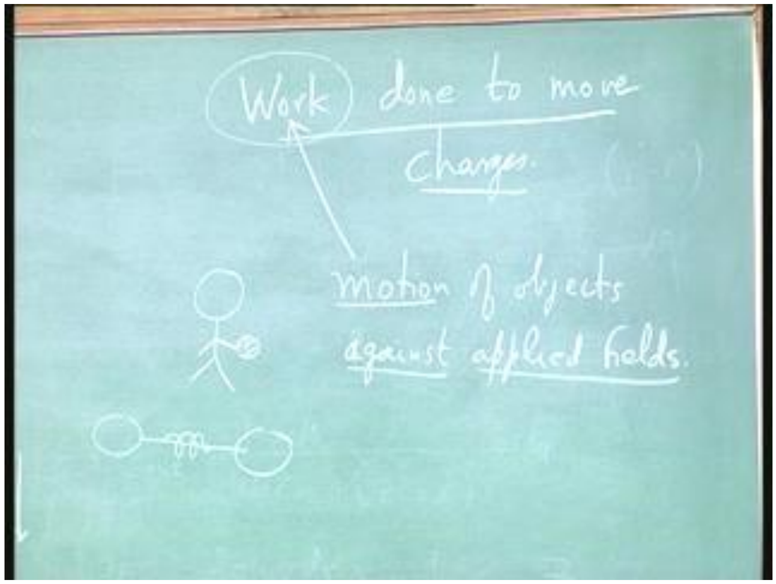

So, now that we know something about work, how do we calculate work? We have to use that definition. So, there is motion and a motion is against a field. So, if I move, if I have a gravitational field m g and I move in some direction, the part of that movement, that is sideways, does not count. That part of the motion which is like the weight lifter walking around does not do any work. It is only the part of the motion that is parallel to the direction of the field that does the work.

So, the amount of work that is done, d W is given as the force dot the vector distance travelled. So, if this is d s and this is force m g, you take the dot product of the two. If you take the dot product of the two, then, only the part that is parallel to the force counts. The part that is 90 degrees to the force does not count and that contributes to d W. This is the work done by the force on the object and when work is done on an object, usually it implies kinetic energy.

So, the object starts moving faster and faster and as you know from this concept we can say, since total energy is conserved, the object has lost potential energy and the potential energy was nothing but m g h and gained kinetic energy. So, this is exactly what we have to do in electricity and magnetism as well, because we have forces. Now the force is called the electric field, instead of gravitation and we have to find out how much work charges do when they move in an electric field – it is the same idea. But let us see how the idea works out.

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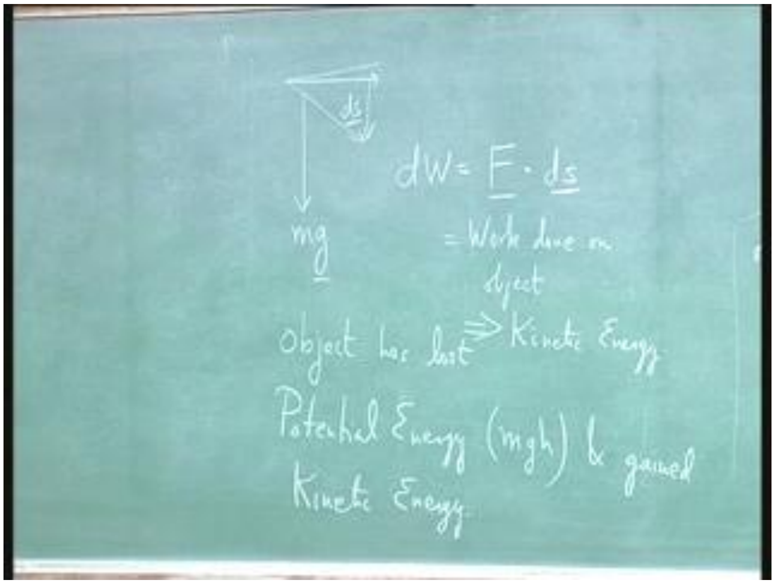

So now I have a charge capital Q. I know that if there is any other charge small q, the force on that charge is along the line joining capital Q to small q and pointing away from capital Q. Now I decide that I want to move this charge from the point A to a point B. It can be on any path, whatever other I like. I want to know, is there any concept of work done? Is there any concept of kinetic energy of this charge and does this charge gain or lose any potential energy?

We know that in gravitation the concept is there. If this was the Earth, this was the satellite, the force would point the other way and if the satellite decided to move like this, it would have picked up kinetic energy and it would have lost potential energy. Is there a similar concept in electricity and magnetism? Well, at every point on this orbit, we can work out what the electric field is. We just have to draw straight lines. The lines are getting larger as you get closer to the charge.

Now you can see that you are not, you are hardly moving in the direction of the charge of the field. So if you move on this line, the amount of work that you will do is only the dot product of the field and the distance. So, we can work out. I will call this 1, 2, 3, 4, all the way up to point n. So, total work done on the charge...that is, this is the work that would have made the charge move faster and faster. It would have given kinetic energy to the charge; is equal to r 1 2 dot q E...r 1 let us say, plus r 2 3 dot q E of r 2 plus, etcetera. I keep adding them all up till I get to r n minus 1 n dot q E of r n minus 1. I have said r 1, r 2, etcetera. I could have easily said r 2, r 3; some point, the electric field on some point along this arrow.

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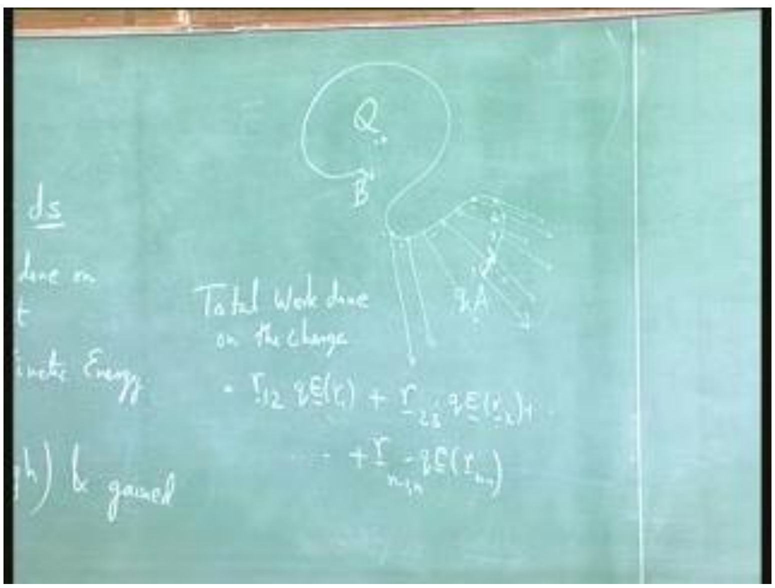

If I add this all up, this should be the total work done and there is a short hand way of writing this sum. Just as we changed Coulomb's law and made it look like an integral, we can make this also look like an integral. We can say that this whole route that we went on, we are going to call it a path and typically symbol c is used – contour, c for contour. So I will say that I am going to do an integral. It is a vector integral along this root c and on this root I am going to do q times the electric field. That is the force, dot d r. Why do I say this? If I take r 1 2, r 2 3, r 3 4 up to r n minus 1 n, I have actually built up this entire curve. So in a certain sense, if I want to know how much work I have done by moving on this curve, it is like summing up all these little pieces; and you know that a sum consisting of very small steps is nothing but an integral. But it is a different kind of integral. It is what is called a line integral. It is not your standard kind of integral. Let us write one just to see what we mean by the difference.

For example, if I have x and y and I wanted to know going from zero to 1, what is the value of f of x, which is equal to integral zero to 1 sine x d x. That is the kind of integral we have learned in mathematics, very straight forward. You learned what it is. It is integral of sine x of minus cos x and you put it between limits and you get that this is equal to 1 minus cos of 1.

So, this is an integral along the real line but I could equally well ask, I want to know the integral of going around a circle. I want to know the integral as I go along the circle of integral, let us say, sine of x square plus y square d theta. Or, if you like...let us leave it like that - d theta. Now, what am I doing? I am saying that I have a circle and I am dividing the circle into lots of little pieces. Each of these lengths is nothing but r d theta which is equal to d theta because r is 1. It is a circle with radius 1 and I want to integrate a quantity called sine of x, square root of x square plus y square. Well, square root of x square plus y square itself is nothing but r square which is equal to 1. It is a constant. So, I have taken a trivial example.

The point is this also is an integral but it is not an integral on a real line. It is an integral on some curve in x and y and more generally, it could be a curving x, y, z, t. It could be any general curve. If you talk about intervals of general curves, those are what we call line integrals and the particular example that I gave which is q E dot the variable which I choose d l I say is an example of line integral; and what it means is if I have a curve, I break the curve into many small pieces. On each piece I find the vector corresponding to d l. At each piece there is also an electric field. So, I can form E dot d l. That is now a number and I will just add it up. Sum I equals 1 to N q E i dot d l i; and if I make these d ls very very small, this sum becomes an integral. That is what it means and this is so important. I think it is important that you think about it and you properly get comfortable with it. We will keep coming back to this. It will come back to haunt us unless we are quite comfortable with the idea.

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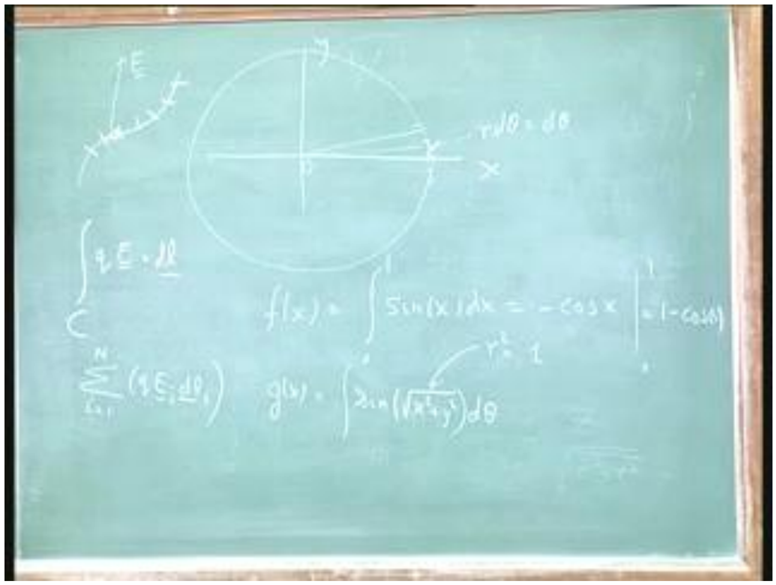

But what is the result now? The total work done on the charge which is in a certain sense equal to minus of change...sorry, in potential energy, we assume total energy is conserved. The energy gained, kinetic energy gained by the charge must be the loss of potential energy of the charge. It is equal to the integral from the starting point, the ending point of q E dot d l.

Now that is interesting. It is just a definition. There is one more interesting thing about it. If I have a charge Q and I have a curve A to B, at any point the electric field points in the radial direction, it points away from the charge. The distance I move d l does not necessarily point in the radial direction but you can always break d l into two parts. There is a part which I will call d r and a part that is the rest of it; and its obvious that no matter what we do, it is only the part d r that can do any contribution to this integral because dot product only counts the portion of d l that is parallel to E.

So, you can rewrite this equation as going from integral A to B but q E r d r. That is at each point, instead of keeping the full direction of l, I am taking advantage of the knowledge that E is always pointing in the r direction. So the only...the r part of l matters. So I only keep d r. What does this do for me? Well, if I have got rid of all the other directions, I do not have to keep A and B as points; I can say going from r A to r B.

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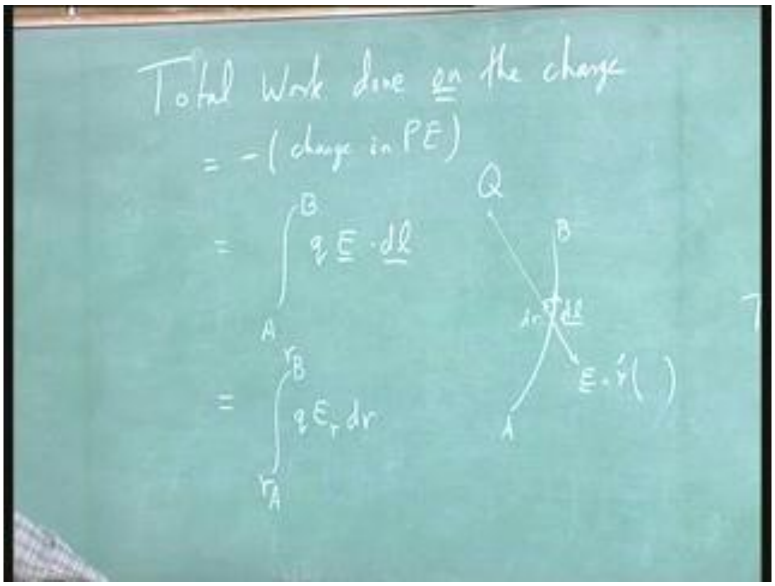

And furthermore, I know an expression for E r. So, let me write that out. It is r A to r B q times capital Q over 4 pi epsilon naught. E r is nothing but 1 over r square d r. Integral of 1 over r square is minus 1 over r. So it becomes q, capital Q, over 4 pi epsilon naught times 1 over r A minus 1 over r B.

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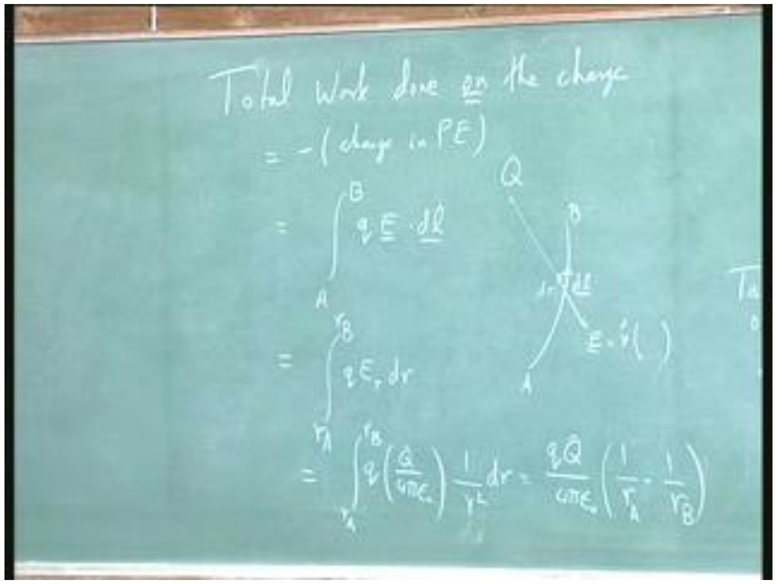

It is a very strange result. What it says is, supposing instead of going this way, I had done this. I had looped around this Q many times. I had gone way out, come back, I had gone this way and come back. No matter what I did, if I finally landed up in B and if I initially started from A, the answer does not change. The answer only depends on where I started and where I ended. It does not depend on how I got them; and we have meant such kinds of integrals before. Or, if we have not, we should have.

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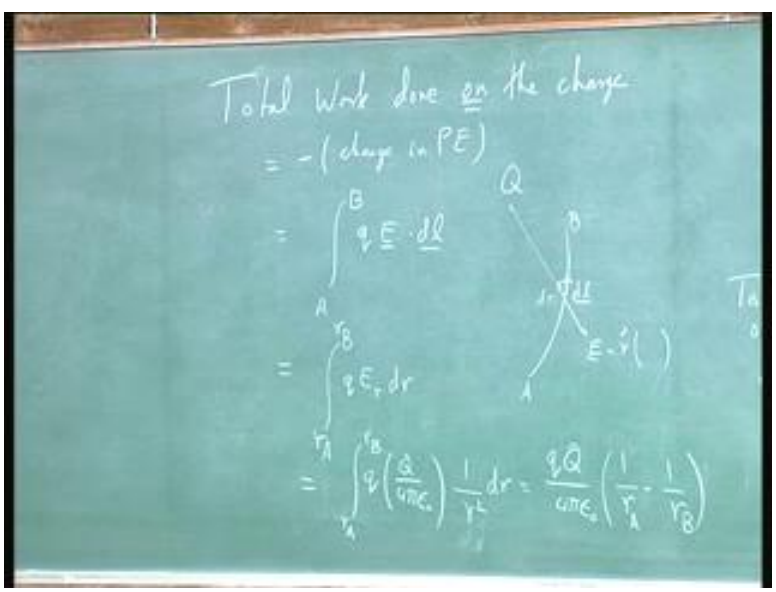

If you have done any course in thermodynamics, you know of things called state variables entropy and variables like that that you will encounter when you do the second law of thermodynamics. They do not depend on the path either. When you are actually changing the state of a thermodynamic system, you go through some complicated path. But the initial and final states are defined based on where you landed up. What we are saying here is something similar.

There is some quantity here that...if the amount of work done on the charge which did not care how you got from point A to point B; only where you started and where you ended. Now, as it turns out, since we are doing electricity and magnetism, we are not so much interested on the total work done on the charge but we are interested in potential energy gained by the charge which is equal to minus of this quantity, because it is the total work done by the charge; this will give you kinetic energy. So, if you want potential energy, you have to take minus of that. So it gives me q Q over 4 pi epsilon naught 1 over r B minus 1 over r A. And typically when we are talking about point charges, we have a point charge Q and we want to use this formula, we will take the other point, starting point, very far away. If you take it very far away, then 1 over r A goes to zero. So you can say, this is equal to q Q over 4 pi epsilon naught 1 over r B.

So, you have now got an expression for a state that depends only on the final location of the charge and it represents potential energy gained by the charge, when you come from very far away to that point. As before, this is potential energy. We would like to know potential energy per coulomb. So we define a special variable, a special field called electrostatic potential. The symbol we use is the Greek letter phi and the Greek letter phi is nothing but potential energy gained divided by q itself. That is the potential energy gained per coulomb and it is defined for a point charge source Q over 4 pi epsilon naught 1 over r.

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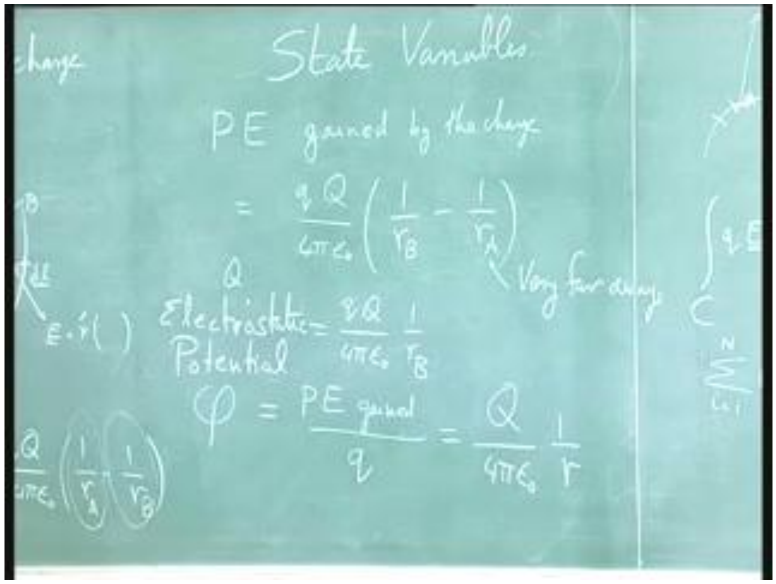

Let me recapitulate. The important thing about this potential function is precisely that it does not depend how you got to the point r. However you got to it, this potential function is the same. So, we have...we originally had electric field which was Q over 4 pi epsilon naught 1 over r cube r. But now, you have come to a different function phi. It is equal to minus integral to r from far away E dot d l which is equal to Q over 4 pi epsilon naught 1 over r. This is Coulomb’s law and this is what we have just worked out. Now, there is a huge advantage to this function. We may have invented it but this is a vector, this is a scalar, which means that this quantity involves 3 separate numbers for every point; this quantity involves only 1 number.

So, phi is much simpler than E. The other thing is phi represents something. phi represents potential energy of charge, potential energy per coulomb of charge. Now you worked all this out for a single charge Q but we know that any interesting problem is going to have many charges and in fact if you have a metal we do not even know where the charges are. So what use is this?

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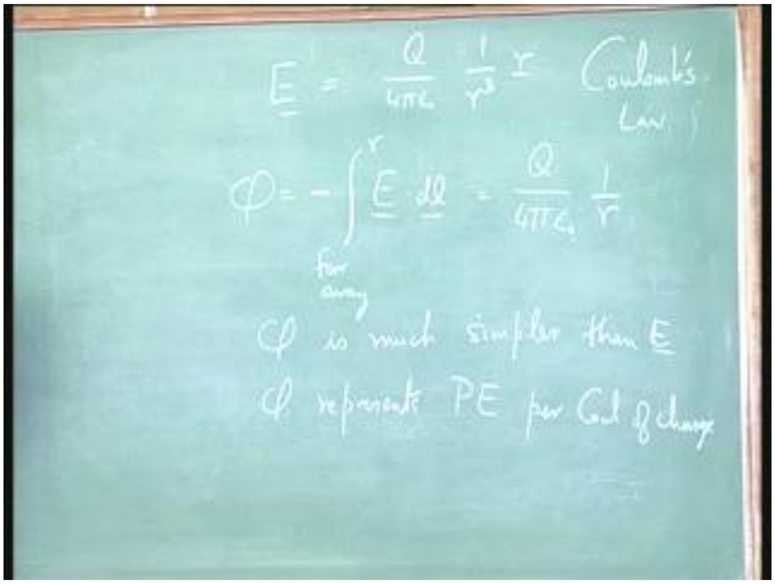

First things first. We know that electric field is equal to 1 over 4 pi epsilon zero integral rho of r prime d v prime divided by r minus r prime cubed and multiplied by r minus r prime. So, there is superposition working for us; for every little piece of rho d v prime, I can calculate potential, which means that the potential due to all these different pieces of charge phi of r must be equal to 1 over 4 pi epsilon naught integral rho of r prime d v prime divided by...

Once again, potential is a much simpler function. Electric field involves a vector and it involves 1 over r cube times r. It is going as 1 over r square; potential just involves a simple 1 over r. It is a very very simple function and in a certain sense, this potential contains everything because if you knew this potential, then if you went to a nearby point and measured potential and took the difference, then it must be true that this is equal to E dot d r with a minus sign. It is just coming from the definition because this is integral from far away to r plus d r and this is integral from far away to r. And you know, if you take two integrals, you can take the difference by same r to r plus d r of E dot d l with the minus sign; and for very very small lengths of integral, it is nothing but the integrand multiplied by the difference of the limits. So, that is this.

So it means that if I know this potential function and I take the potential function at very nearby points, I have effectively got a feel for electric field because the difference between nearby points of potential is nothing but the electric field itself. Of course there is a d r in there but it is nothing but the electric field.

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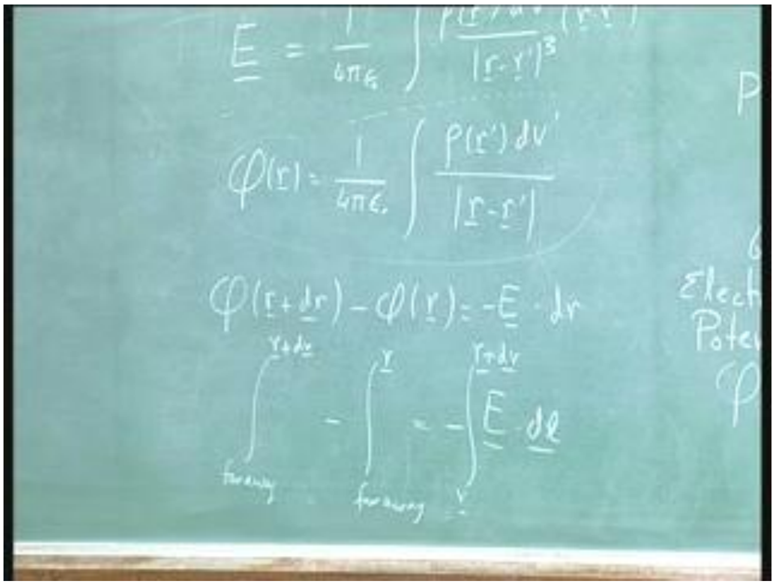

So, you have got a case where you have simplified the problem. We started with an electric field which was a complicated vector field and we have now devised a scalar field, much simpler field. It has got a physical meaning to it. It is potential energy of the charge and it does...it gives you the electric field, after all.

I will continue next time.