Electro Magnetic Field

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Lecture - 4
Electric Field

Good morning. So last time we had introduced Coulomb's law and I had also started talking about something called the electric field. I want to continue on that and I will repeat a little part of the last class so that you feel comfortable with the material.

So, we had Coulomb's law and Coulomb's law said force on a charge Q 2 due to a charge Q 1... You draw a line through Q 1 and Q 2; the force is along the line joining Q 1 and Q 2, pointing away from Q 1 and it is equal to Q 1 Q 2 over 4 pi epsilon naught times 1 over R 1 2 square, in this direction. Your textbook calls a 1 2. So, this was what Coulomb found and we worked out some examples of what happens when you use it. Now this force is a force exerted on a particle. If you ask what this force depends on, force depends on Q 1, depends on Q 2. It depends on the location of Q 1 and it depends on location of Q 2. Given these quantities, you will get a number, you will get a vector.

Now, it has been found to be very useful, if instead of talking about force, we talk about force per coulomb. You can see that the force is proportional to Q 2. So, if we replace Q 2 by twice Q 2, the force will be twice; replaced it by 4 times, the force will be 4 times. So, we could talk about a force per unit coulomb and then all we have to do is multiply by Q 2 to get the actual force. This force per coulomb is called the electric field.

It has got a symbol E. The units of force are newtons. So, the units of electric field ought to be newtons per coulomb. We actually give it a different unit, but this is correct. There is nothing wrong with this unit; but when we invent electrostatic potential, we will use that instead to give a unit for the electric field. It is defined as Q 1 over 4 pi epsilon zero 1 over R 1 2 square a 1 2. Q 2 is missing. So E is actually equal to the force F divided by Q 2 because you can see, that is the difference between this formula and this formula.

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Now, this seems rather arbitrary. I mean I could have divided by Q 1 as well and got some other vector which I then say is per coulomb, per coulomb. If I have got the force, why do I need to complicate life and create a new vector? Because there is a reason. It has to do with how we view Coulomb's law. When you look at a charge Q 1 and you look at another charge Q 2, we are saying that there is a force on Q 2. Now, how is it that Q 1 created a force on Q 2?

In the old days during Newton's time, it was thought that everything acted at a distance; that is, the Sun pull the Earth from some 200...160 million kilometres. The same way you could think of protons and electrons interacting over a distance. There is another way of looking at things. You could say that the presence of this charge has polarized this material. That is, that is around this charge. That is, the material in the presence of this charge has actually got stressed a little bit. Because of that stress, any time you place a charge there, that charge receives a kick and that kick is what we call the force.

This picture is only a picture. I mean, one could in fact do all of the electromagnetic theory without actually requiring this picture. You could work with Coulomb's law and equivalent laws that work at a distance. But modern physics has shown that there is a lot of validity to this picture of stress vectors, stress field. For example, all of special relativity and general relativity requires such a picture. All of quantum mechanics requires it. All of particle physics requires it.

So today, scientists are pretty much convinced that the picture of particles which create stresses in the space around them called fields and these fields in turn react on other particles causing force. That is the picture that is correct; and if that picture is correct, then rather than talking about fields, we can talk about...rather than talking about forces, we can talk about fields. You can ask how is this stress created. So that is this equation, which is, the electric field is equal to the originating charge divided by your normalization constant 4 pi epsilon naught zero, times...at this point there is some distance - call it d - 1 over d square, along the direction.

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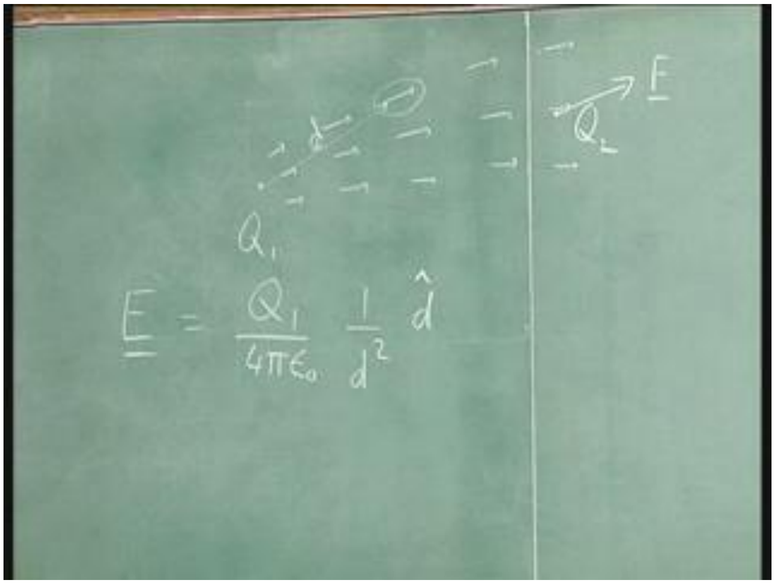

Now, there is a difference between this formula I have written down and the previous formula I wrote down here. In this formula, I was still talking about the electric field as evaluated where Q 2 is. So, this was just a short hand for Coulomb's law; but when you come to this equation, there is no charge here. The only charge is sitting here. However, I am saying that there is a potential force, a force waiting to happen at every point in space. All of space has got unhappy. All of space is stressed out because this charge is there and the moment you put any other charge there, it reacts violently and gives it a kick and that is Coulomb's law.

So now, this electric field is a function of this vector d, this vector connecting the location of charge Q 1 to any point in space. So, if you wanted to know the fields there, this vector would be d. If you wanted to know the field here, this vector would be d. So, this is now a definition of a vector quantity which is a function of a variable, independent variable. The independent variable itself happens to be a vector. We are used to seeing functions f of x. For example, f of x equals sine x. So there, x is a real number, f of x is another real number. So, if x was equal to 1, f of x was equal to sine of 1; and if x is a variable, f became a function.

Similarly, here I have a vector variable; that is d is actually d along x in the x direction plus d along y in the y direction plus d along z in the z direction. So, this is my variable. Instead of 1 dimensional, it is 3 dimensional and this vector function is a vector function of these 3 numbers: d x, d y and d z. It is more complicated than your simple function like sine x. But the equation it satisfies is a very simple equation. This is nothing but Coulomb's law, except, it is Coulomb's law applied to where there is no second charge; it is applied to any arbitrary point.

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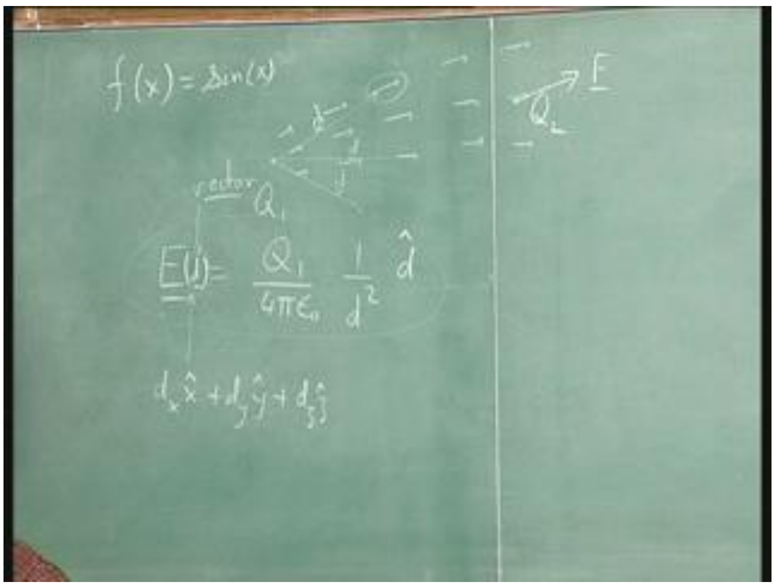

Now, last time we talked about the superposition principle and what that said was that the force at any point due to a bunch of charges...So, you had a collection of charges and you put a charge, let me call it small q. So, these were Q 1, Q 2, etcetera, up to Q N. Then I said, superposition told us that the force on this charge small q was equal to small q over 4 pi epsilon naught sum i equals 1 to capital N Q sub i divided by this distance, which are called r i. So, r i is nothing but the distance vector between the charge Q i and small q. So, this was Coulomb's law generalized for N points.

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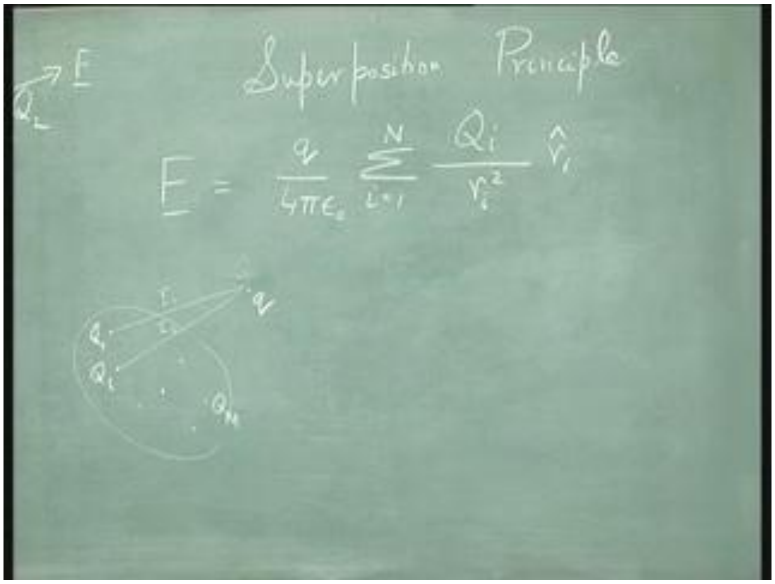

Similarly, now you can define...you can divide through by a small q and you get the electric field at any point r. So, I am choosing some point r. Let me draw a coordinate system now, this is X, this is Y, this is Z. So, this vector from the origin is r. This vector to my Q i is...well, I will call it rho i. This is not standard notation, but it does not matter. So, the charges are Q 1, Q 2, Q 3, up to Q N - all at distances from the origin given by vectors q rho - rho 1, rho 2, rho i, rho N.

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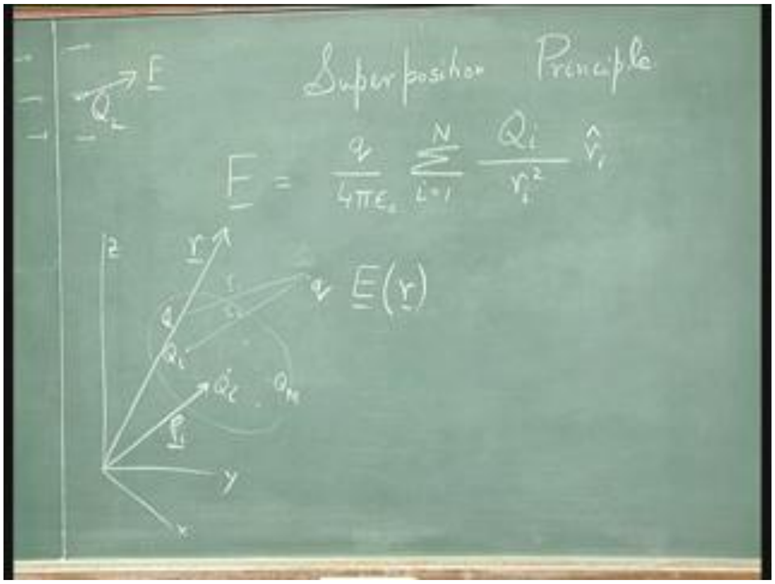

I want to know the electric field at this point. There is no charge there anymore. So, how would I get it? My answer is, I will put a charge of 1 coulomb, measure the force. Electric field is the force per coulomb, which means there is a force on 1 coulomb. What do I get? Again it is equal to 1 over 4 pi epsilon zero sum i equals 1 to N Q sub i. But I now need this distance, this vector. Now, if you look at this vector, this vector is really the vector r minus the vector rho i. That is because if this...if I call this vector s, then I have the rho i plus s is equal to r because I take this vector rho i, at its nose I place the tail of the vector s; it reaches the nose of vector r. So this must be true.

If I take s to the other side...sorry, if i take rho i to the other side, then I get s is equal to r minus rho i. So I know my distance vector. I am going to say, it is equal to s i square unit vector along s i. But if I keep inventing symbols, it does not really help us. So I instead go back to what s i is. It is equal to 1 over 4 pi epsilon naught sum i equals 1 to N Q i. When I say s i square, I really mean take the magnitude of the vector s i and take its square. So, magnitude of r minus rho i square; and this unit vector is nothing but r minus rho i, unit vector along that.

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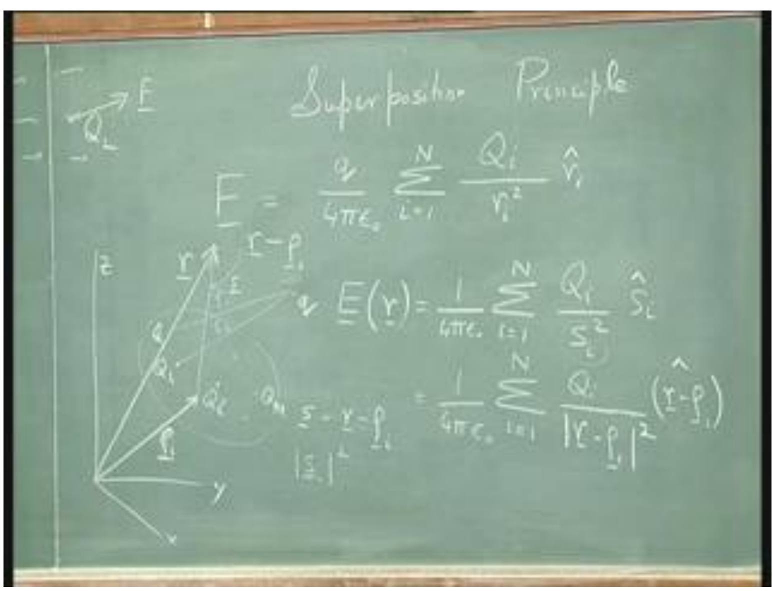

What have we gained? What we have gained is, if I have a collection of charges and I am given that collection. So the charges are fixed in location. My...I now have a quantity which is a function of one independent vector variable which gives me the potential field at any point given on these charges. It is still only Coulomb’s law because all I have to do is multiply by q on both sides and I get back force due to charges according to Coulomb’s law. Now, there is one important simplification you can make to this. Supposing I multiply and divide by magnitude of r minus rho i. What do I get? 1 over 4 pi epsilon naught sum i equals 1 to capital N Q i divided by r minus Rho i cubed.

Now out here, if I multiply by magnitude, a unit vector times the magnitude is the vector itself. If I have any vector direction and length, the unit vector along the direction has a length of 1. This has a length, say v. If I take this unit vector and multiply it by the length v, I get the vector itself. So if...since I multiplied by the length and I have the direction, this is nothing but the vector itself. So, this is a working equation and we will use quite a bit.

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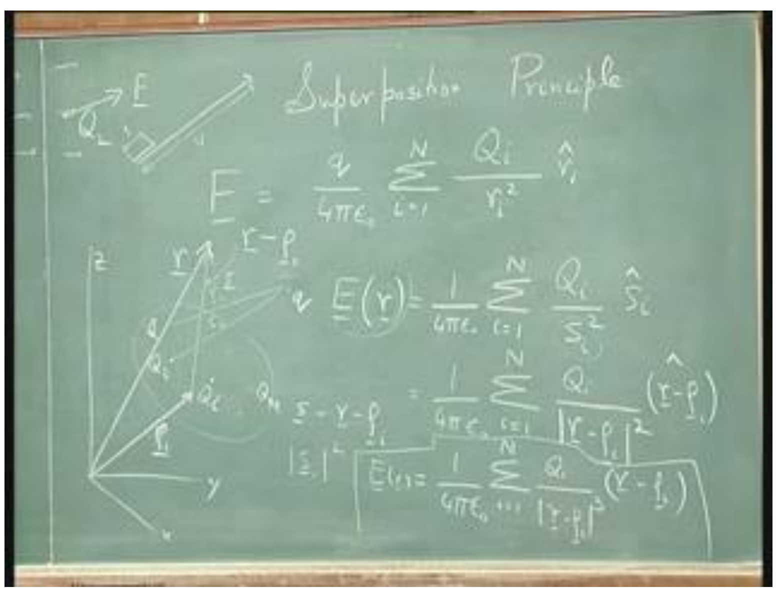

At the end of last class, I was talking about how you go from a sum of n charges to an integral. What I was saying was that if you have a cloud of charges, many charges inside your cloud, N very large, then what you could do is you could make a box inside this volume. The box could be large or the box could be small; but you keep the centre of the box fixed.

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Now for any size of box, you count the amount of charge inside the box. So, you can plot or you can make a table. Supposing the side of the...size of the block box was delta and you tabulate charge of the box. So you would have 10 centimetres, 1 centimetre, 1 millimetre, 1 micrometre, 1 nanometre and so on and so forth; and you figure out how much charge is in the box each time. Now, that of course is a varying quantity; but you can also find out how much average charge there is, which is, we divide by delta cube. And when you do this for a very large box whose size is becoming comparable to the volume, what you will find is if there are more charges here, fewer charges here, it won't...the average value won't reflect the average at the centre of the box.

So, what you will find if you plot this? If you plot Q over delta cube versus delta, you will find some trend. This reflects the fact that when your box is too large, it is not really measuring local properties. It is measuring the entire volume. But after a point, as delta becomes small, it will level off. It will level off till you start getting...start getting noise. You get noise when the number of charges in your box are very small.

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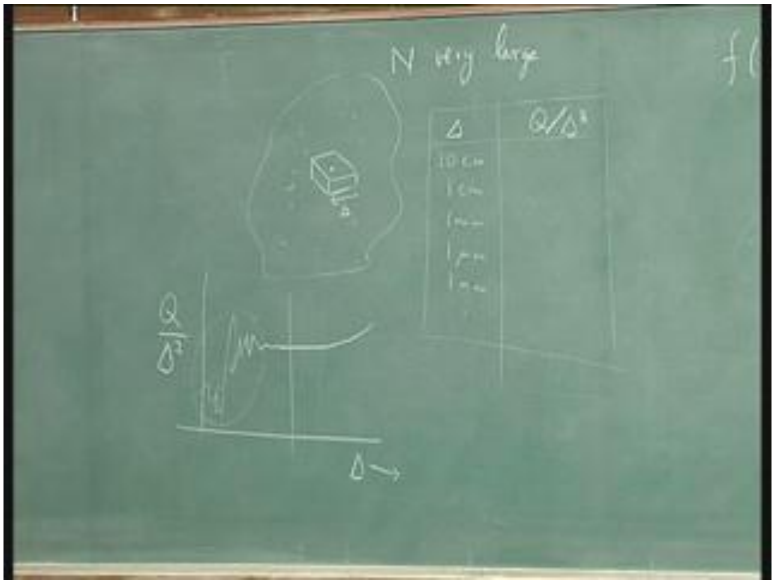

When we do any kind of measurement, whether it is electrostatics or it is mass or it is anything, when you want to talk about average quantities, we want to take a box which is small enough that is measuring local properties; which is large enough so it does not see the statistical noise. It is possible in most problems to choose such a box - especially in electromagnetics, because the building block of charge is the electron which has a charge of minus 1 point 6 into 10 to minus 19 coulombs. This is such a small number that when you take any practical problem, you have a huge number of charges inside your volume.

So, you can take a very very small box and even then have a large number of charges. So, you never reach this statistically noisy part of your measurement. But if you were doing some work in VLSI, if you were trying to do...trying to measure capacitance of a VLSI chip, then you may run into trouble because you may have only 40 electrons in your entire capacitor and you are trying to measure the amount of charge that the capacitor holds on average, it will become noisy.

Now, if you work in this region where the numbers are flat, reliable, we can tile this volume with boxes and we can fill the entire volume with boxes. Having done that, we can convert our sum. Let us see how we do that. It is equal to 1 over 4 pi epsilon zero; sum on i equals 1 to N.

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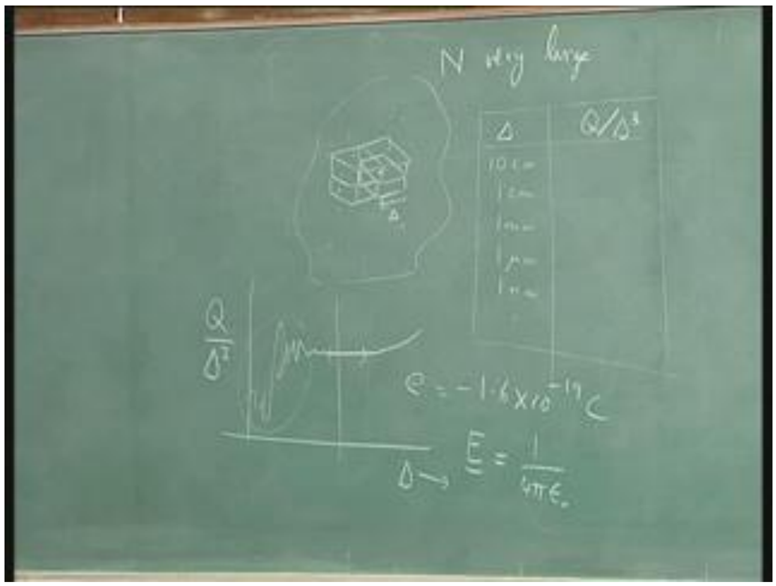

Well, I am now going to change from sum on i equals 1 to N to sum on j equals 1 to m where these are boxes; and then I say all the charges inside a box are really close to each other. So, all the various vectors, r minus Rho i are all the same vector. Because of that, I will collect all the charges, call it Q j and do r minus Rho j where Rho j is now the centre of the box, cubed times r minus Rho j.

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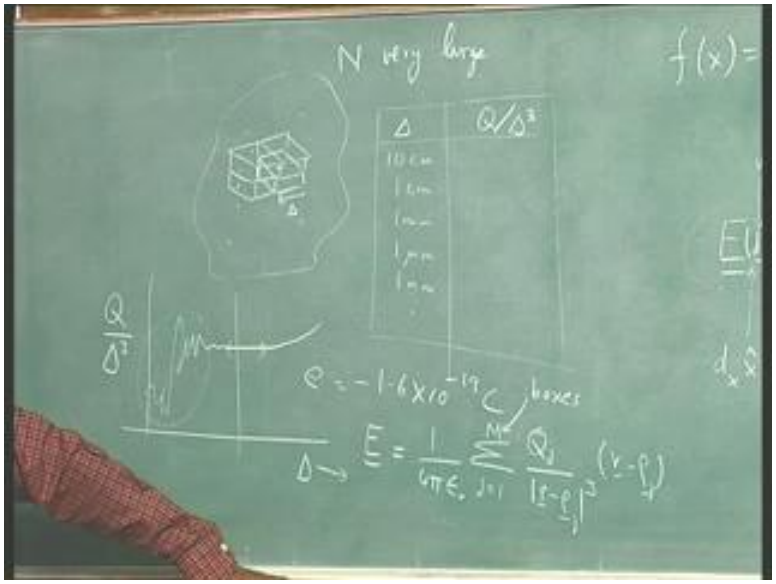

It looks like the same equation, but it isn't. There is an important change that has happened. Once I have done this, I am now summing over boxes; and if you remember the definition of integral, it is nothing more than summing over boxes. Let me repeat what I just said. I have in 1 case sum i equals 1 to N Q i over r minus Rho i cubed r minus Rho i, and the other case I have sum j equals 1 to M Q j. May be, I should give it a different symbol - r minus Rho j; again, a different symbol. They look exactly the same, but in one case, I am counting particles - particle 1 in my cloud - Q 1 may be here, Q 2 may be here, Q 3 may be here.

So when I sum them up, I cannot...I am not summing up particles that are near-by each other first and then particles that are far away. When I look at the second case, what I have done is I have taken all the particles in this region, sum them up separately and got Q tilda 1. I have summed up all the particles in this region and got Q tilda 2. I have summed up all the particles here and got Q tilda 3.

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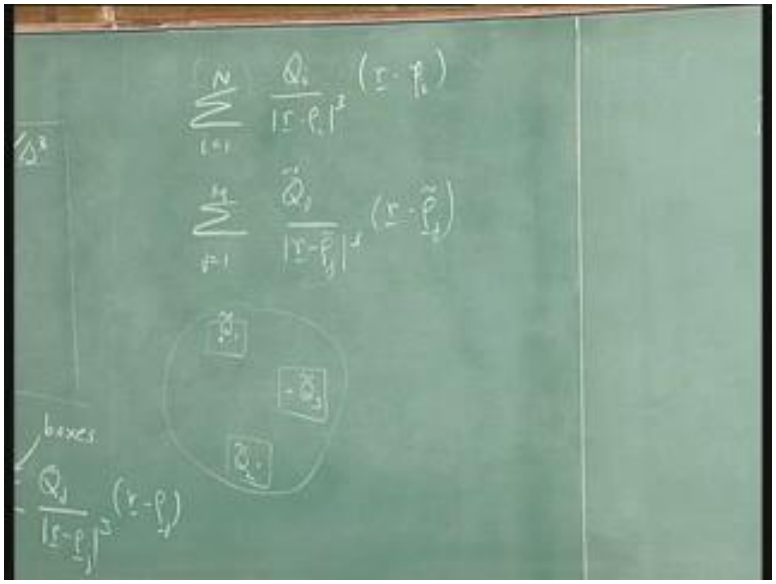

Once you have done this, then I know that as I make these boxes smaller and smaller, as I let delta go to zero....of course you have to realize that I can only let delta go to zero in this sense. If I let delta go to zero into this region, it is not correct. So, what we allow in mathematics, we say let delta go to...go all the way to zero, cannot ever happen in physics. In physics, we always have to keep in our mind - delta going to zero means delta goes to a small number, not delta goes to infinitesimally small.

So, we let delta go to zero; which means small. And this summation becomes an integral. So I get, the electric field at a point r becomes equal to 1 over 4 pi epsilon naught integral over a volume... The denominator now becomes r minus rho magnitude cubed. Numerator has r minus rho vector. But this quantity Q Q j or this quantity Q j tilda - it has to be broken up. This Q j tilda is charge in box with sides delta cubed...with sides delta. So, what does that mean? I can write this Q j as Q j tilda divided by delta cubed into delta cubed.

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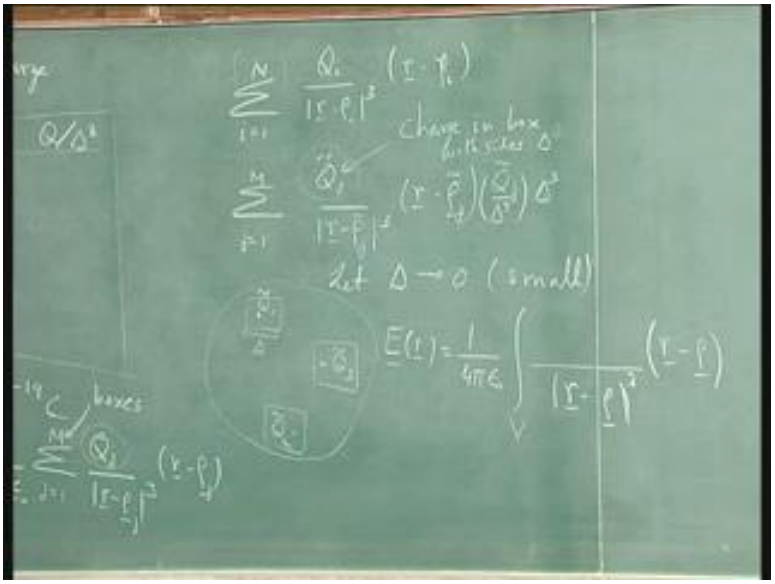

And as you remember from the previous screen, it is Q over delta cube that becomes very flat and this Q over delta cube which is the average charge per unit volume is called charge density. It has got a symbol - the symbol is rho of...where...I have used already rho, haven't I? So, I am going to...course correction...call this r prime, rho of r prime, and d V; delta cubed is nothing but volume.

So, I have two kinds of equations. Let me write the other summation right next to it. E of r equals 1 over 4 pi epsilon zero. Summation i equals 1 to capital N, Q sub i over mod r minus rho i cubed times r minus rho i. We have a summation form and we have an integral; and it is important to understand that these two things are saying different things actually. In fact, there are certain conditions in which they are saying very different things. If you calculate the energy from the integral and the energy from the sum, you will get different answers. So you have to know what you are doing when you jump back and forth from the sum and from the integral.

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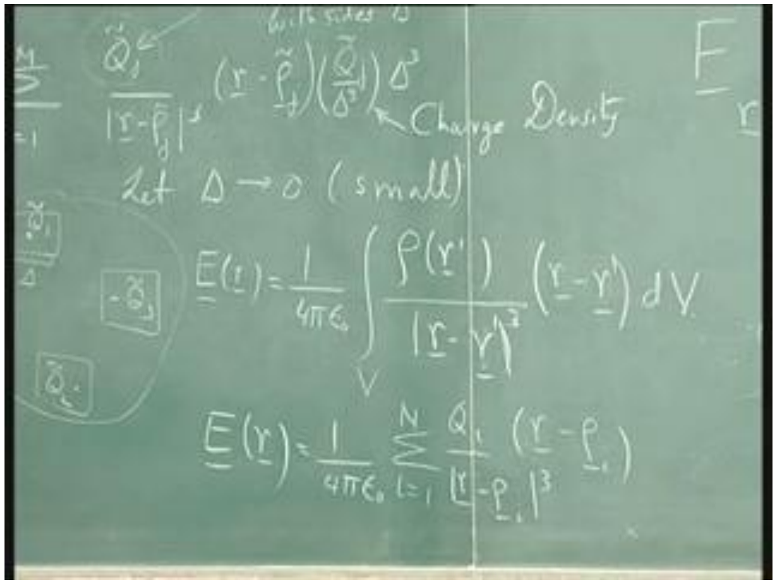

But for most interesting applications of electricity and magnetism, it is this formula that is used and for most of this course we will be using the integral. It is an extremely important set of steps you have gone through. We started with the sum; the sum went over particles that could be anywhere inside the cloud. Nobody said that when you index particles, they have to be close to each other. But then, I collected them and grouped them. I said: find all the particles that are inside this box, group them together inside this box, group them together, etcetera and then I took the total charge Q j tilda and then I said they are also close to each other; they are more or less at the same point. So I defined a common r minus r j...rho j tilda and I converted my sum to a new kind of sum.

It was a sum over boxes and this sum over boxes goes naturally into a Riemann integral; and in this Riemann integral, this Q j...Q j tilda is really charged within the box of side delta which can be written as the charge within the box divided by delta cube multiplied by delta cube. This is nothing but average charge into volume of charge of box which is what is written here. Average charge is usually given this symbol rho of a coordinate and volume of the box becomes d V.

Now, in order to fix this in your mind, I am going to do a few problems. The first problem I will take up is that of a line charge. This is given in your book. He uses a slightly different derivation, but essentially the same derivation. You have a line charge; the charge per unit length is lambda coulombs per metre. So, if I have...if I a take distance 1 metre along this line charge, the amount of charge in that part of the line charge is lambda coulombs. My co-ordinate system is like this. That is X, Y, Z. I have put the line charge along the z-axis and then some point, distance r from the line charge. I want to find the electric field. So, what do I do?

There are many different ways of doing it. But let me take a simplifying way of doing it so that you can see the symmetry of the problem. I will take a little piece whose length is d z; which means the amount of charge in there is lambda d z coulombs and this piece is a distance z above the x y plane. I take another piece which is also d z in size. Its charge is also lambda d z, except it is lying a distance z below the x y plane. Now, if I look at what kind of field these two parts of the line charge cause at this point, well, this point causes a line...causes an electric field that is pointing slightly down in the x z plane and this point causes a charge, causes a field that is pointing slightly up.

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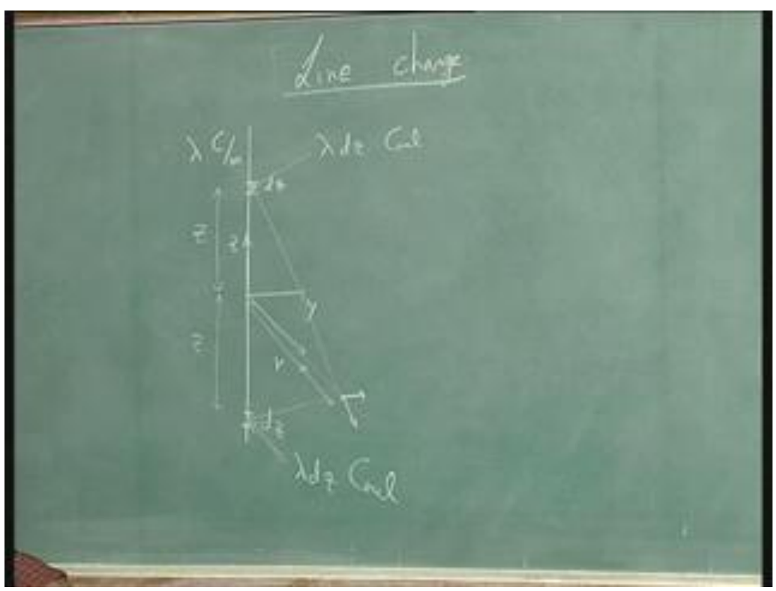

Let me rotate the problem so you can see it properly. So this is x, this is z, this is d z, this is also d z. So at this point, the distance from this charge, piece of charge to where we are - this distance is nothing but square root of this height square, z square plus the radius square. Distance from here to the bottom one is also the same thing. So, if you look at the formula for electric field, the electric field depends on the charge. It depends on the distance and it depends on the direction. So, the electric field has a different direction for each of these; but it has the same magnitude. The magnitude of the electric field is nothing but lambda d z divided by z square plus r square.

So, these are two equally long vectors. One points up, one points down. If we complete this vector addition, what we end up with is a vector that points exactly sideways. The reason is, this angle theta is the same as this angle theta. So, it is a parallelogram and the resultant is a horizontal vector. How much is the length of that horizontal vector? Well, the length of each of these was equal to lambda d z over z square plus r square. This angle is also theta. So, you have to multiply by cos theta, because cos theta is the horizontal component of these vectors. But there are two of them; so, twice lambda d z z square plus r square cos theta. So my total electric field is going to be along the x direction. It is going to be integral from zero to infinity lambda d z over 4 pi epsilon zero cos theta over z square... sorry, twice cos theta over z square plus...x square. This r square becomes x square because we are in the x coordinate.

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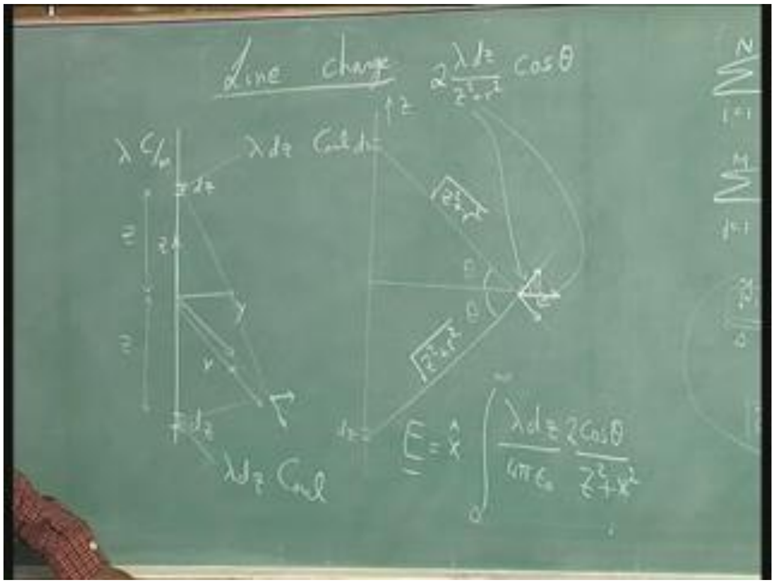

We still have not fixed what cos theta is but we can, because cos theta is nothing but the base on the hypotenuse. So, cos theta is equal to this length x divided by square root of z square plus x square. So if I write that out, I get the electric field. The electric field is along x. So, I will just write it as a scalar - E x is equal to...I will pull out all the common constants. 2 lambda over 4 pi epsilon naught times integral zero to infinity. There is a x that comes out also. d z over z square plus x square 3 halves. What I have done is I have taken cos theta x over square root of z square plus x square and put it in here. The x is not integrated over. So it came out. The square root of z square plus x square combined with what was already there to give me z square plus x square to the power of 3 halves.

This is a very standard form. Any time you see an integral that involves something like z square plus x square to the power of anything, you are immediately...to indicated that you should use a transformation that goes like x tan theta or x tan phi. Theta is already used up. Then d z becomes x secant square phi; and the denominator z square plus x square to the power of 3 halves is equal to x square tan square phi plus x square to the power of 3 halves. I have substituted for z which is x cubed to the power of 1 plus tan square phi to the power of 3 halves; but tan square phi plus 1 is nothing but secant square phi. So it becomes x cubed secant cubed phi.

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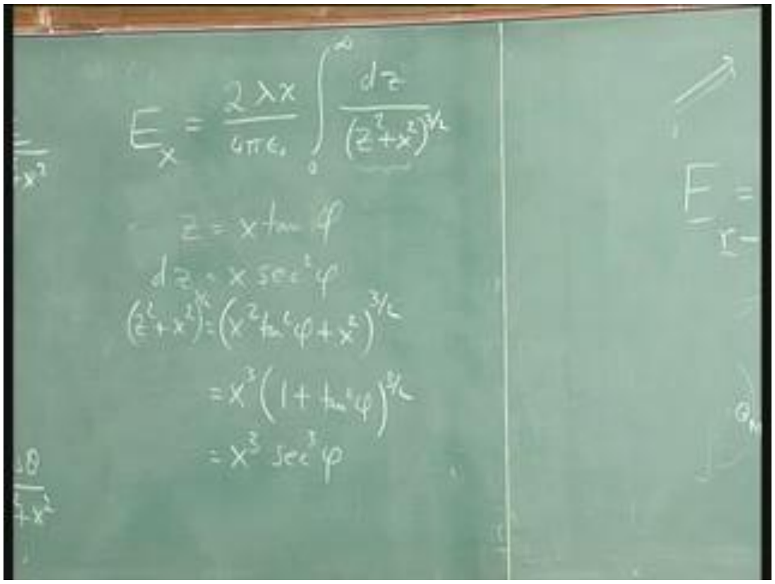

So I get terms involving x which just come out and I get terms involving secant square. I mean secant. So what happens to my electric field? Electric field becomes twice lambda x divided by 4 pi epsilon naught. I get...from d z I get one more x divided by x cubed...x over x cubed and then I get an integral d phi - secant square phi divided by secant cube phi - 1 over secant phi. Integral goes from zero to pi by 2 because when z is zero, tan phi is zero, phi is zero. When z is infinity, tan phi is infinity, phi is pi by 2. This is easily solved because 1 over secant phi is nothing but cos phi and integral of cos phi is sine phi. So this whole integral becomes 1.

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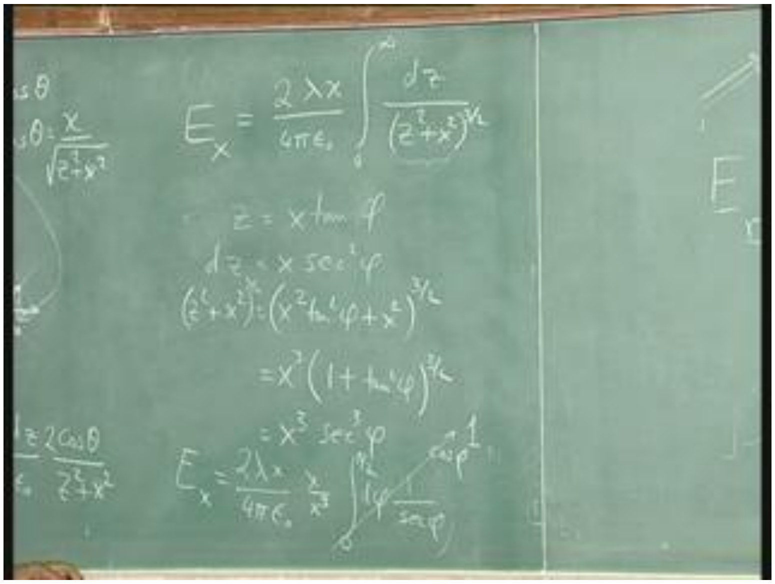

Let me make sure I am...I have got the right answer. So, what you get at the end of it all is that the electric field is along the x direction. You get an x square divided by x cube. So, lambda divided by 2 pi epsilon naught 1 over x; or if you allow the direction to change, it would be, the electric field is in the radial direction lambda over 2 pi epsilon naught 1 over r. So, if you have a line charge and you go a distance r from that line charge, the electric field falls off as 1 over r.

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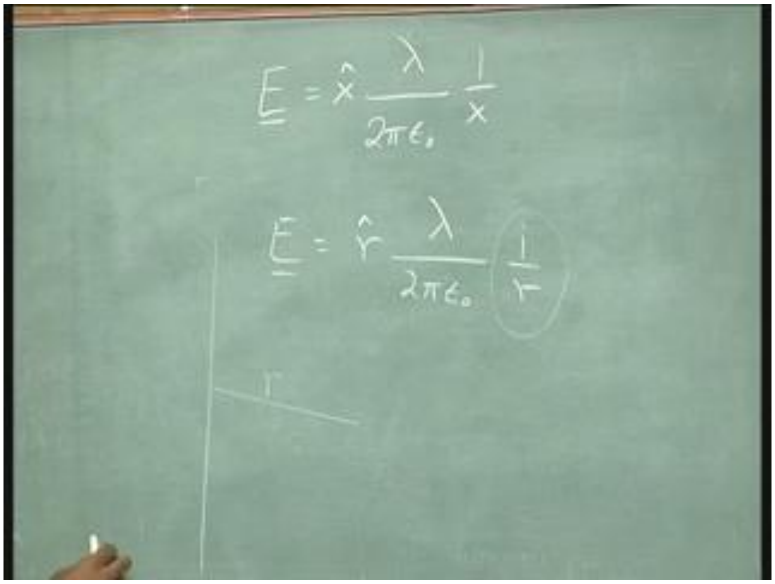

Now, 1 over r falls off much more slowly than 1 over r square. So, it is important that this is understood. What is happening is that as we go further away, we are seeing more and more portions of the line charge. We go out there, we start seeing this part. So, this is not the charge due to a point source. This is a charge due to a distributed source; very close to the line charge. Only this part of the line charge causes the electric field. These parts hardly do anything. Their electric fields are almost vertical and cancelling.

As we go far away, now it is mainly these electric fields that do the job. As we go further and further away, further and further away parts of the line charge contribute. So that is why it does not fall off as fast as 1 over r square; falls off as 1 over r. Your book has done the case of a plane and there he just uses the same formula and integrates it. One can do it in a more interesting way.

Supposing you have a plane and you want to know the electric field at some point above the plane. The plane has sigma coulombs per metre square. Now, let us take a ring centered about the x y location of the charge. So I will put the...I will put the origin so that the charge is at...is along the z axis and I look at a ring. So, you have a ring whose radius is r and the thickness of the ring is d r. The amount of charge in that ring d cube is equal to the circumference of the ring 2 pi r, times the thickness of the ring d r times, of course, the sigma.

Now, all points on this ring are equally far from this top point and for every point here, there is a point equal and opposite such that the electric fields are in opposite directions. So, only the vertical component survives. It is exactly the same idea I used for the line charge. If I take opposite parts of the circle, the x y directions of the vectors are exactly opposite. So they cancel out; but the vertical parts add. So, what you get is d E is along z. It is equal to, this charge, 2 pi r sigma d r divided by 4 pi epsilon naught times 1 over this distance times...again, if you take this angle theta, that angle is the same as this, times twice cos theta.

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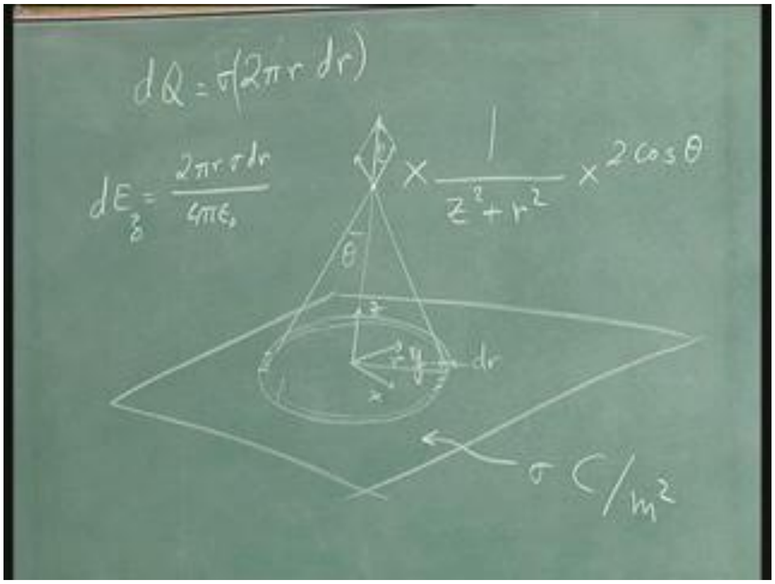

Cos theta is nothing but z over the hypotenuse. So we can replace cos theta by z over square root of z square plus r square. So the total electric field becomes 2 pi sigma divided by 4 pi epsilon naught. The r d r will go inside; this factor of 2z will come out. So, integral zero to infinity times r d r divided by z square plus r square to the 3 halves. Is that okay? I may have made a factor of 2 error here. We will check.

So, how did I reach here? For any point here, all points on a circle are equally far from this point. So, I take points that are opposite. I add up the vectors. Those vectors cancel sideways; they add vertically. So you get a vertical vector. Vertical vector means along z. How much is that vector? Well, it is equal to the circumference times the thickness times the charge; that is Q, divided by your normalization constant 4 pi epsilon divided by the distance, z square plus r square, times your cos theta, because this distance only gives you this part. The cos theta is z over z square plus r square. So, a factor of 2 may not be there; we will find out. This is a very easy integral to do. Just take z square plus r square as u; then d u is nothing but 2 r d r and since d u is equal to 2 r d r, the numerator is nothing but d u.

So you get 2 pi sigma over 4 pi epsilon naught z times integral d u over u cube...u to the power of 3 halves. The limits are: upper limit is infinity, lower limit is z square, because when r is zero, z square plus r square is z square. When r is infinity, it becomes infinity. Solving that, we get the answer and I will just write it down.

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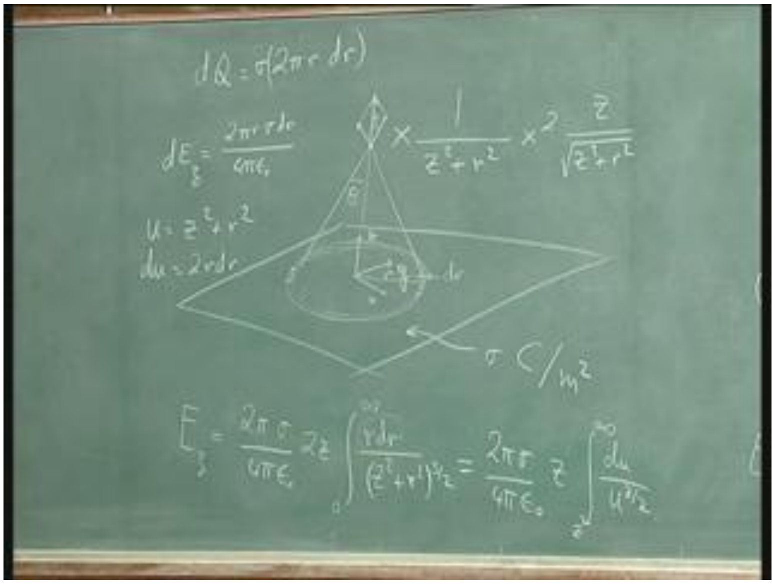

You get E z is equal to 2 pi sigma z over 4 pi epsilon naught. u to the 3 halves becomes u to the minus half over minus half evaluated from z square to infinity. 1 over square root of u at infinity goes to zero. So you are left only with this piece and so you get 2 pi sigma over 4 pi epsilon naught times the further factor of 2 times z over z which cancels out. The electric field is constant.

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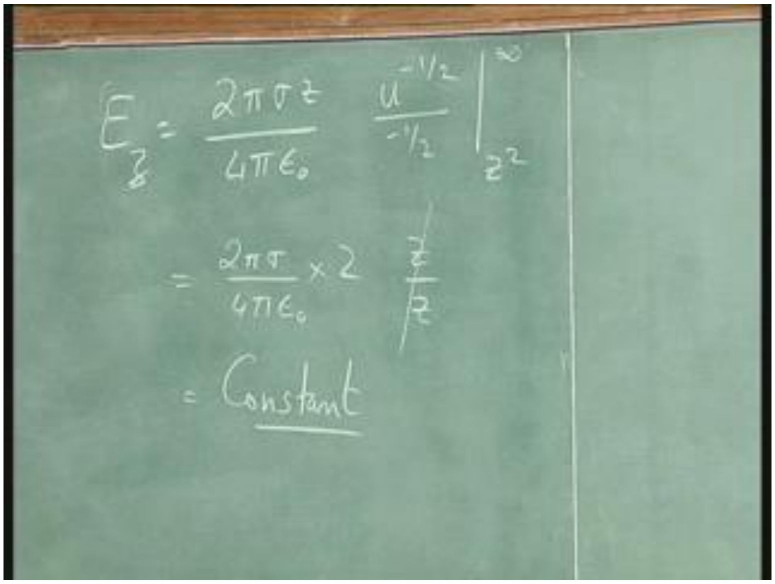

As a very strange result, we start with Poisson's equation. You have a charge. You go further away, it falls away as 1 over r square. You have a line, you go further away, follows...falls off as 1 over r. You go away from a plane and it does not fall away at all. The one kind of reason is given in your textbook - you can read it. Other kind of the reason is the further away you go, if you look at the problem within your mind, it looks as identical. There is as much of the plane available to you no matter how far away you go. So the answer can never change.

I will continue next time when I will introduce the potential and Gauss's law.