Electro Magnetic Field

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Lecture - 38
Phasor Form of Poynting Theorem

Last time we discussed how the poynting theorem and that the skin effect, that we had being talking about for the last few weeks as a can be applied to circuits. So, I took 3 cases one I looked at a current flowing through a sheet the 2 dimensional wires. And looked at how the current actually flows and what we found was that the current loads primarily at the walls.

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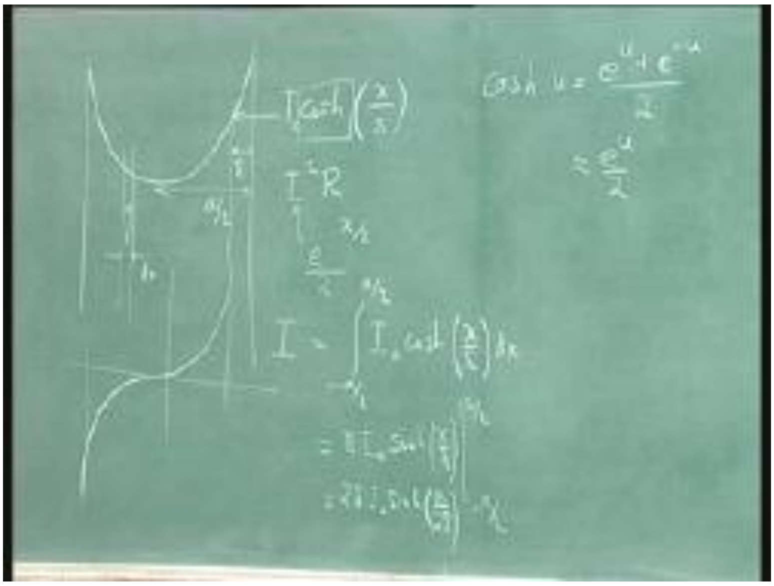

This curve being cosh of that is, this distance is a by 2 x over delta some basic I naught. So, it is basically, an exponential provided a by 2 is much greater than delta. If a over 2 is not much greater than delta, it has the more complicated shape, which is given by the cosh hyperbolic. What this means is the following; the amount of power that is send through this portion of the wire, which is 1 delta y has to do with I squared R. And if the current is basically is proportional to e to the minus x over delta, which is what it is cosh is this divided by 2 for large arguments. That is to say cosh of u is equal to e to the u plus e to the minus u over 2. So, if u is large it becomes e to the u over 2, so if I substitute that x is much larger than delta, I can replace the cosh sorry e to the x over delta divided by 2.

Now, of course, there is an I naught, there this I naught is given by the total current is equal to integral minus a by 2 a by 2 of I naught cosh x over delta dx. That is, current everywhere, that the current in here with the width dx is I not cosh x over delta dx. So, if I integrate from the left boundary to the right boundary, I get the total current now, integral of cosh is nothing but sin hyperbolic. So, it becomes delta I naught sign hyperbolic of x over delta between a over 2 times and minus a over 2 and if you look at sin hyperbolic sin hyperbolic is an odd function. So, function like this, so a over 2 it is large and positive that minus a over 2, it is large and negative. So, I subtract a large and negative number from and large and positive number, I get twice the large and positive number.

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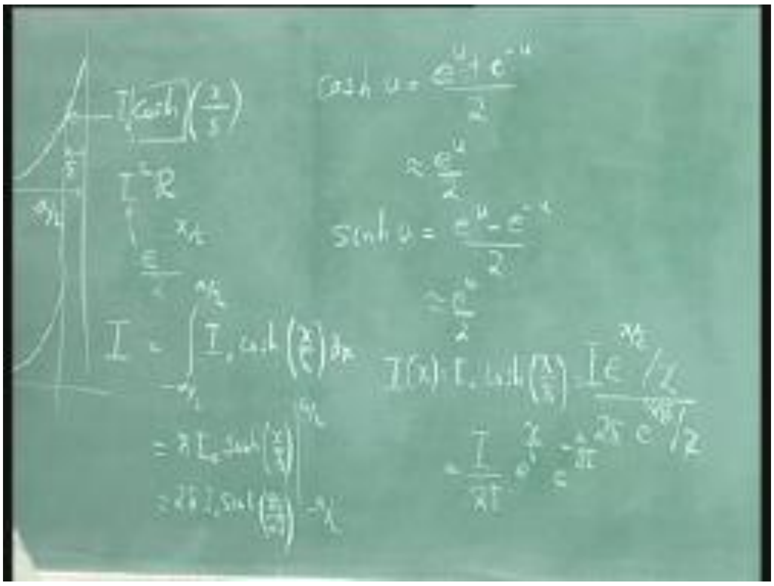

So, it becomes twice delta, I not sin hyperbolic of a over 2 delta, once again sin u can be approximated sin hyperbolic of u is e to the u minus e to the minus u over 2, which is approximately equal to e to the u over 2. So, if I want to know, what is the current at the edge? What I have the current as the function of x is equal to I not cosh hyperbolic of x over delta, I can substitute for cosh and substitute for I not from here. So, I get it is equal to e to the power of x over delta divided by 2 gets from here divided by this, I naught is given by taking all of this taken to the denominator there.

So, I divided by twice delta times sin hyperbolic of a over 2 delta, which is again e to the x over delta divided by 2 sorry k omega. So, you can see that the exponential variation of that current, but this current is always going to be smaller than e to the power of a over delta, let me take the exponential over the enumerator. So, what do I get? This 2 cancels this 2, so I get that the current is equal to I over 2 delta times the exponential of x over delta. And the exponential of minus a over 2 delta, what I have done is I have taken a over 2 delta to the numerator, where it become exponential of minus a over 2 delta, this factor of 2 get cancel. So, I over 2 delta times this now I can combine exponents.

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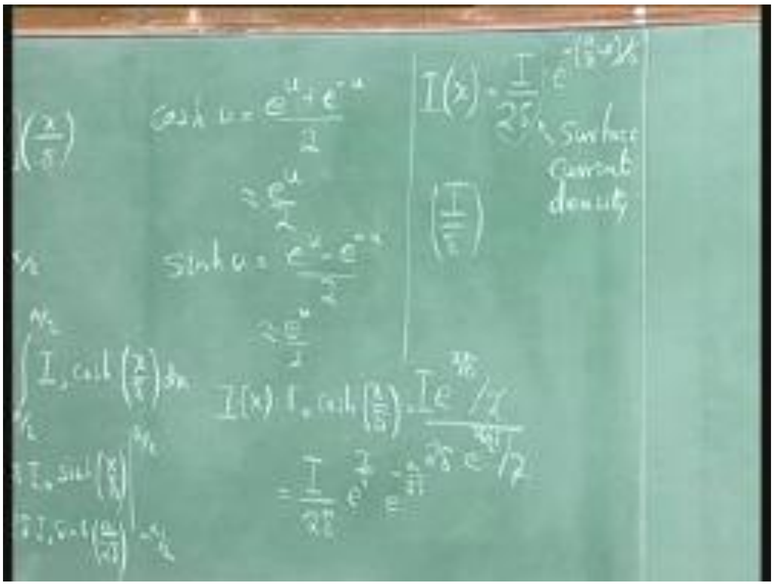

And therefore, I get my answer that I of x in a wire is equal to the total current in the wire divided by 2 delta times e to the minus a over 2 minus x divided by delta. So, it is an exponentially falling function, it falls to 1 over e of it is value over a distance delta. And the current density at the surface is given by the surface current density is given by I over 2 delta. Now, usually if you look at any of this treatment you will see I over delta as the surface density the reason, why we have I over 2 delta is, because the current not only flows in this surface. It flows on this surface since that there are 2 surfaces flows on, which it flows the amount of current flowing through this surface is I over 2. So, that current divided by the delta is the current density. So, there is a very high current density, even though the current may not be very high the current density is very high. Now, if you look at how much of the current flowing through which part that comes from taking an integral over different parts of this wire, but we know about exponentials.

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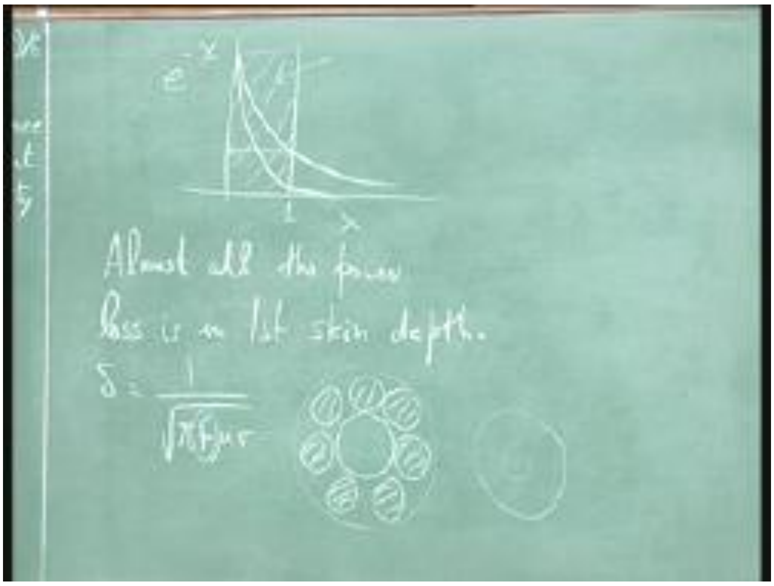

So, if you have a exponential e to the minus x and you plot it versus x, you know that over a distance 0 to 1 the amount of the exponential is more or less 67 percent of the total value. In fact, if you take area of the rectangle, it is equal to the integral of e to the x all the way to infinity. And you can easily see that, the piece that, we have left out is smaller in piece you have taken. So, more than half of the current is in this first delta region, if you take 2 delta that becomes even more. Now, you ask of how much energy dissipation is happening, then that comes to I squared R and when you look at I squared R all that energy dissipation very much is happening here. Because I squared goes us E to the minus twice, this and if you look at the e to the 2 x the curve would look like this, they would almost nothing left beyond the beyond the skin depth.

So that as far as the power goes almost all the power loss is in first skin depth. This is why skin depth is such an important concept the current is there the power dissipation is there. And therefore, we really need to understand this skin depth and how it applies to our problem very carefully. And as we saw 2 lectures ago the skin depth in different materials depends on it is frequency, because delta is equal to 1 over square root of pi f mu sigma. So, it depends on F, so the larger the frequency the smaller the skin depth for copper at 60 hertz of 50 hertz. This skin depth is 1 centimeter, which is y power cable you have the ability to avoid putting unnecessary copper in a middle of the wire. So, you have seen this, I have drawn it 2 lectures ago and I am sure, you seen it in your power lectures by packing copper cables around the central health member.

And having current carrying through, these the result is the full copper is used for current conduction the central strength members, that gives you gives the cable the ability to avoid being distorted. And snubbed hardly taken a single copper cable of the same total gauge. Then what would have happened is that the central portion of this copper would not carried a any current. Only the outside would carry the current and rid of wasted the copper as, it is by putting all this copper around another member we are. In fact, saving on copper cost and it is a clear cut case of skin depth in use. I want to do things now, in this lecture. And the first is I want to take the poynting theorem, that I have been taking about and make it more useful.

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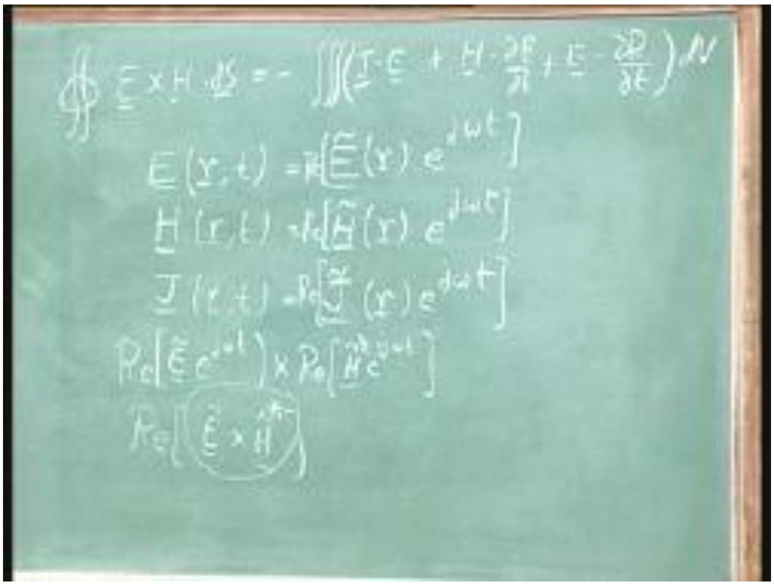

Let me remind you what we have derived? We derived that the surface integral E cross H dot ds, which is the radiation plex out of a surface E is equal to the minus the volume integral of energy sources of the flex, which is j dot E plus H dot del B del t plus E dot del D del t. Now, this is conceptually very satisfying, but it is very general also, it is meant for arbitrary fields E and H. But in electrical engineering regardless of the field whether it is a power or it is a power system or it is communication, we are always dealing with faces of electrical and magnetic fields. So, we would like to translate this, poynting theorem to a phasor equation by that, I mean I would like to represent the electric fields, which is the function of position and time. I would like to replace, it with a phasor, which is the function of position time e to the jth omega t now, just as last time we have in confusion between j and square root of minus 1.

So, I am going to call this capital j, but typically current density is represent as small j, but to avoid the confusion I am calling it capital j. So, your electric field is the phasor multiplied by e to the j omega t your magnetic field is also a phasor against, function of position e to the j omega t similarly, b similarly t and similarly tJ. So, all of these are now, represented as phasors, now what you know that means, is that really, we take the real part of these expressions and knowing that this is going to represent. We want the equation not for E H and j, but for the faces E H and J. Now, if you look at this, left hand side I am talking about E cross H. So, E cross H would really be really be real part of the phasor E e to the j omega t cross Real part of phasor H e to the j omega t.

Now, if I want to make this useful, I would like to say that this is equal to the real part of E phasor cross H phasor. Otherwise; this representation is of no use to me, but if I do if I combine this 2 expressions I do not get this, because I have E to the j omega t times E to the omega t to the 2 t omega, obvious; since I wanted in without E to the j omega T. I take the complex conjugate of the second term; if I take the complex conjugate of the second term, then clearly this will become complex conjugate minus j omega t. So, each the j omega t will cancel out and I will have this quantity now, I do not know what use it is? But at least it does not have the phasor exponential. So, I am going to try and work out, what is the poynting theorem? When you apply it to the phasor E cross H star, it may seem rather artificial. What we are doing? But it is actually, very useful. So, let me write out the Faradays law and Amperes law.

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Curl of the phasor E is equal to minus del B del t, but now del del t is j omega. So, minus j omega phasor B curl of phasor H is equal to the phasor J plus j omega the phasor D. So, I have this 2 equations and to compare with them. Let me write down the original equation, which where curl of E is equal to minus del B del t and curl of H is equal to J plus del D del t. So, the del del t is became J omega now, as before I want to work with E cross H, but you can see that, I want to work with E cross H star. So, let me write the complex conjugate of that equation curl of H star is equal to J star plus complex conjugate of J omega D the complex conjugate will become the complex conjugate each of this multiply together. So, minus j omega just in case you did not understand, that let say this is D this is the real and the imaginary axis when I multiply by j omega effectively, I am rotating by 90 degrees. The reason is the real part multiplied by j becomes, imaginary part the imaginary part multiplied by j becomes minus real part.

So, the entire vectors, rotates by 90 degrees and of course, is scaled by omega. Now, I want to take the complex conjugate, which means, that the real part is unchanged the imaginary part goes down. Now, how can I do that? Well, if I want to reach this vector, because if j omega D is poynting this way, I should leave the real part unchanged, I should make the imaginary part negative. So, what I do? I say this vector is equal to the complex conjugate of D, this is D star and j omega is poynting this way. Then this way would be j omega complex conjugate, because again imaginary part is change the sign, the product of these 2 will be a rotation by minus 90 degree, it brings me here. So, j omega D complex conjugate is equal to J omega complex conjugate D complex conjugate and j omega complex conjugate is nothing but minus j omega time is complex conjugate of D. So, that is what I have written there.

So, I have the 2 equations, I want now, I want to work with E cross H as before, I want to look at divergence of E cross H star. And it is equal to del del xi of epsilon ijk E tilde j H tilde star k, where I use the epsilon ijk notation to represent cross product. I can pull this out and write this, as epsilon ijk del Ej del xi Hk plus Ej del Hk del xi product rule, when I take a derivative of the product, I take the derivative of each in term. So, now, I just identify this, as suitable curl operation. So, the first one is I rotate 1s, so it is equal to epsilon kij del Ej del xi this is nothing but curl of E times H star the second term, I want I and k. So, j as the first index Ejki del Hk conjugate del xi Ej, if I look at this, you can see that there is second index not the third index. So, it is equal to minus curl of H tilde star, I am going through this quickly, because exactly the same thing was done for poynting theorem given, that you done you got this curl the rest is easy.

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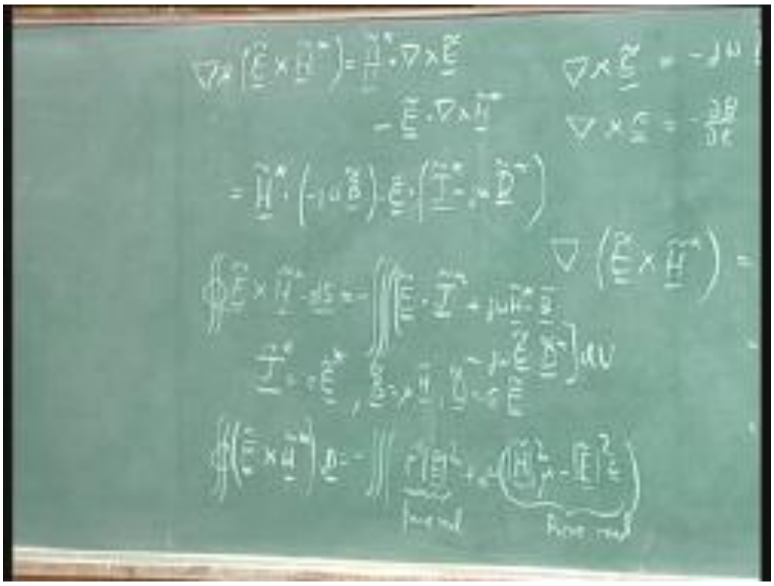

So, my curl of E cross H star sorry divergence of E cross H star is equal to 2 terms. The first term is the conjugate of H dot the curl of E; let us this, term plus sorry minus E dot the curl of H conjugate. But now, I can apply Faradays theorem and Amperes theorem Amperes law, I can apply this equation, I can apply this equation. So, what I get is complex conjugate of H dot minus j omega B tilde minus E dot cross product, I mean the curl of H given by this expression? So, j complex conjugate minus j omega D complex, so I have a expression for the divergence. So, I can apply the divergence theorem and I get surface integral phasor E cross phasor H conjugate dot ds is equal to I can see a minus sign everywhere. Some I will take a minus sign out volume integral E dot j conjugate that, is take care of this term plus j omega H conjugate dot B. And since you pulled out a minus sign, I have to keep this minus sign minus j omega E dot D conjugate.

Now, we can apply your knowledge that J is equal to sigma E B equals mu H and D equals epsilon E. So, we put those end you get that 1, this is equal to minus volume integral sigma E squared, because E dot E conjugate. So, we conjugate, so E dot E conjugate is the sigma E squared actually. This should be sigma star come back to that, then plus j omega times H squared to the mu minus E squared epsilon. So, this is the new equation, we have the phasor E cross H with H conjugated is equal to minus of volume integral of sigma E squared plus j omega mu H squared or till I have B squared. So, mu minus j omega epsilon E squared now you can look at this and I remove this complex conjugate.

We will assume that sigma and epsilon and mu are real quantities, this is pure real and this is also pure real. So, sigma E squared is pure real, because whatever E may be e may be a complex number, but magnitude of E squared is the real number. Similarly, magnitude of H squared is a real number magnitude of E square is a real number. So, this quantity is real, so inside this integral is a real number plus j omega times, another real number. So, this is the real part this, is the imaginary part is equal to this E cross H. So, we can say, the real part of this surface integral is equal to the real part of this, volume integral. The imaginary part of this surface integral is equal to the imaginary part of this volume integral where does that lead us, you can look here and this is the time non phasor version of poynting theorem.

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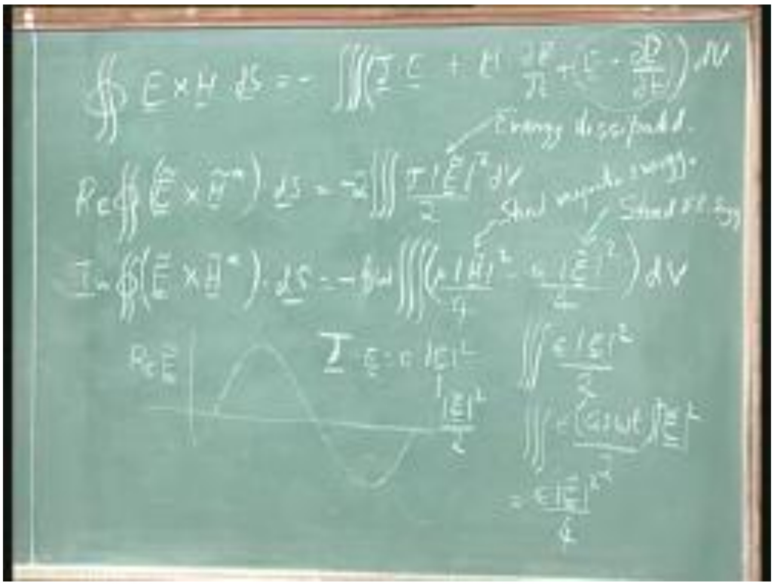

Now, I am going to write out the phasor version underneath, it the phasor version is actually 2 equations. It is says real part of phasor E cross phasor H star dot ds is equal to minus the volume integral, sigma E squared dv and imaginary part of E cross H is a star dot ds is equal to again minus sign j omega sorry j goes out. Because it is imaginary part, volume integral mu H squared minus epsilon E squared dv it is does not look anything like these equations. You got 2 equations, instead of 1 and this quite a bit of difference between these equations. And this equations, let us see we can understand, what they are saying and why they are saying the same thing. If I look at the time dependence of the phasor in time does not e to the j omega t.

So, the real part of phasor E is going to be make a sinusoidal, because it is going to be cosh omega t. Now, if you take cosh omega t and take absolute value squared, because you want epsilon E squared look at this piece, that piece is saying volume integral epsilon E squared over 2. Now, this epsilon E squared over 2 is really, volume integral epsilon cos omega t squared times this, magnitude of the phasor squared. If you now, average this cos omega t squared over time, this average is half, because we take square of this number. You get an expression that looks like this and as you have seen before the average of this expression is half, because this piece exactly fixed here. So, there are equal bits above half and equal bits below half.

So, the average is half, so this becomes, epsilon mod E squared over 4 what we have here is epsilon E squared and then we have an omega, let us time for the D dt. So, we can put this, 4 inside and we can put this, 4 inside and pull the four out. Then this part represents represents, stored electric energy and this part represents stored magnetic energy.

Now, what about this piece? Well, this piece you look at this function, we know that J dot E is really a sigma E squared, but E is cos omega t. So, we know that E squared is really this phasor amplitude square, but average delta and gives you the factor of 2. So, this quantity, I will divide by 2 and multiply by 2 this piece volume integral of sigma E phasor squared over 2 is energy dissipated. So, now, let us try and read this, new phasor equation the original equation said that whatever, energy is leaving a surface through radiation is because there is a battery inside. So, negative J dot E or there is reduction in stored magnetic energy or there is reduction in stored electric energy those, where the things that give me outward radiation. Now, when I have a phasor it means, that the time dependence is cos omega t.

So, on average the stored magnetic energy and the stored electric energy cannot change, because after the period 2 pi over omega will come back to, it is old value to the stored magnetic energy and the stored electric energy are just cycling. They are going to your maximum going back to 0 going to negative maximum, going to a 0; these 2 quantity do not change on average. However, this can the real part of the phasor poynting theorem is, only talking about the part that survives averaging in time. And it says that the real part of E cross H is equal to twice minus twice the dissipated energy. What about the imaginary part? Imaginary part does not carry about the dissipated energy at all. It is only looks at the electric and magnetic energies and what it saying, is that since the energy in the magnetic and the electric fields are periodic. They can be only 1 thing that is happens, half the cycle energy is in the magnetic field the remaining half of the cycle energy is in the electric field.

So, the electric field and the magnetic field are exchanging energies and that is, what this represents the imaginary part represents the fact that the energy is shifting back and forth between electric and magnetic field. The rate at which the shifting comes is important, because we are talking about del del t and this is a factor of 4. Because of that way that, we are not talking about dc fields; we are talking about ac fields. So, the imaginary part is what is called reactive energy and yet is talking about that the rearrangement of energy inside the volume. And this rearrangement shows up as the flexivating E cross B at the surface, there is no net the time average the energy going out energy goes out for half the cycle comes back in.

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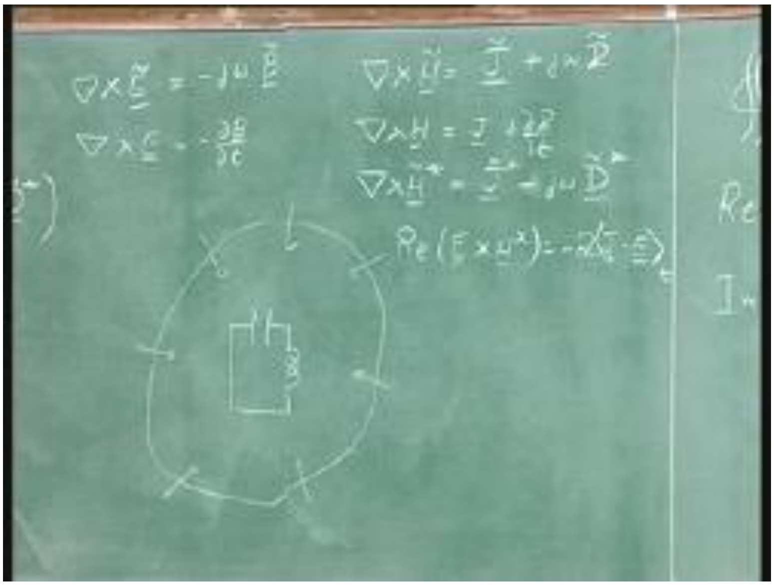

So, if you looked out a pictorial representation of the phasor equation, it will be like this, I have the surface inside the surface, I have field is currents. These currents for example, could be dissipating energy in, which case I would have energy coming into the system from all direction. So, this time dependent are at least time aver gable term in the E cross H is the real part is equal to minus twice, J dot E average, averaged in time and also of course, volume integrated. So, the real part is referring to dissipation are generation of energy, what is the imaginary part representing. For example, supposing I had the circuit, where the capacitor and an inductor are exchanging energy and let us say this is not a very well insulated system. So, the flex lines both the electric flex lines and the magnetic flex lines are leaking in that case, there will be a electric fields due to the capacitor, which will be oscillating in time. And there will be magnetic fields flexivating at the surface, because of the induct ant, this flexivating electric and magnetic field half the cycle, will give me a E cross V out.

But the remaining half of the cycle, that will be giving E cross B in, because there will be in a general direction. So, they might be giving a E cross out this way and this way, but whatever, it is it does not lead to an time averaged E cross H. So, if I time average, this equation and required that B and H are B and E are sinusoidal periodic function. They will just vanish, because they will not survive time averages only this term would survive. But within each period the amount by, which E cross H differs from it is time average will be basically due to this. And that is what you see here the imaginary part of E cross H star is equal to minus 4 omega multiplied by average stored, magnetic energy, minus average stored, electric energy and all this averages having time. So, the phasor form of the poynting theorem tells us the great deal and therefore, it is quite important to use it in your problem. Because it is really giving you the information that you specifically, want this equation is fully correct this nothing wrong with it contain everything.

It contains, much more then these equations, because it is valid even if the fields are not phasor, it is valid for the fields that slowly going from 0. But these are the useful expressions for most of the electrical engineering that is why it is important to know that, before I leave this topic, let me just go back. And revisit stored electric and stored magnetic energy, if you look at the phasor expression either of this. What do you see? Let us, go back here, because you will meet to use this form for this derivation. What do you see is that you have terms, which involves J dot E terms in involve del del t of d squared over mu and del del of mu squared times epsilon over 2. Now, before I go there, let me point out one important point, when we did this? Derivation in phasors, you will noticed, we did not get a factor of 2 the moment. We work on this, we find that we get del del t of B squared over 2 mu, but there is no factor of 2 coming out of the phasor form. The reason is that the factor of 2 came, because we try to push the electric field in the time derivative and the magnetic field into the time derivative.

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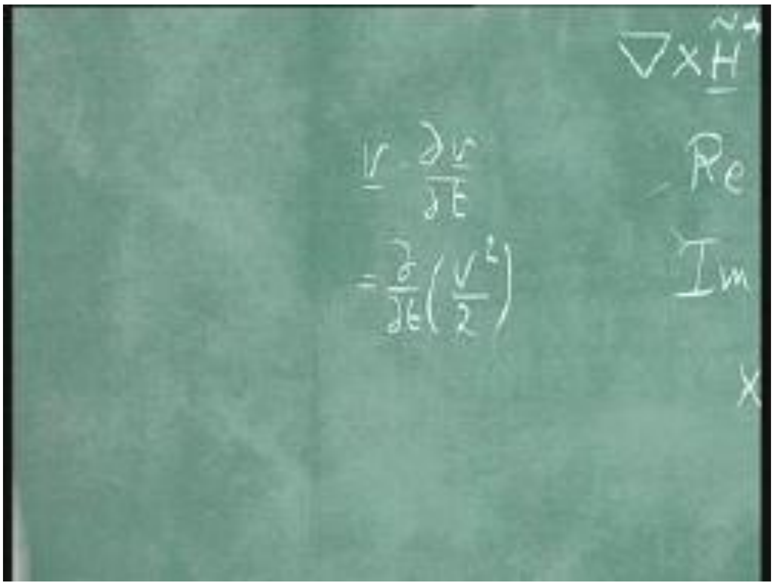

We were using the fact, that you have any vector V dot del v del t then this is equal to del del t of V squared over 2; however, when you work with phasors with derivative as, become a multiplication. And is no lower true, that in the factor 2 is requires, but the factor of 2 is still there and that is why we have taken it in account by putting a 1 over 4 here 1 over 2. Because of cos squared and 1 over 2, because really the fact of 2 is therefore, a stored energy and that is why we came to 4 omega factor in these expressions. Now, let us look at this expression and let us try and understand where stored magnetic energy came from. You have already done this, derivation before, but I like to show, it as the trivial consequent of poynting theorem. I am going to say, I am looking at paradise law my amperes theorem is just the curl of H is equal to j. So, I do not have radiation must not yet happen, before Maxwell.

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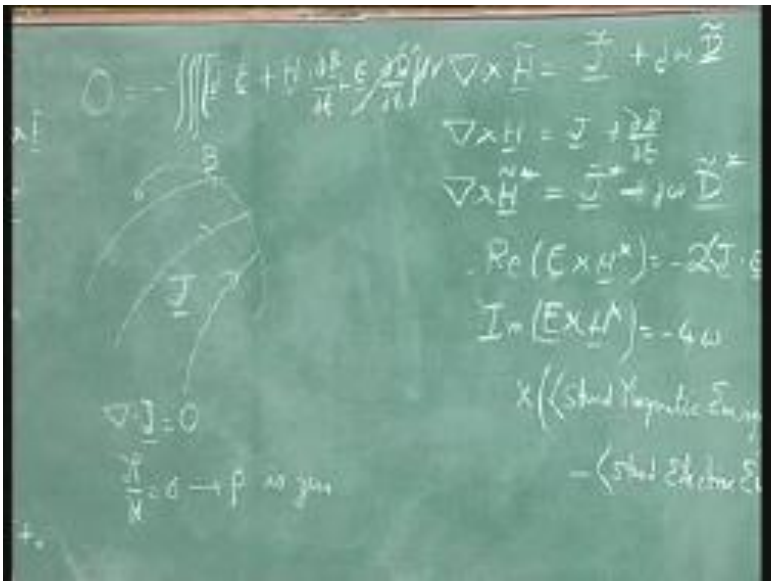

So, my poynting theorem says 0 is equal to minus volume integral j dot E plus H dot del B del t plus E dot del D dv. Now, I can choose supposing, I want to know I have some currents J and I want to know what this stored magnetic field energy, that threads this J. So, I can carefully introduce the J such that the divergence J always remains 0, if divergence J remains 0. Then del row del t is 0, which means, the row is 0 the electric field does not have any source. So, I can drop this term, there is no electric field, what is that giving me? It tells me that whatever amount of energy that coming is through j dot E must be coming out of minus H dot dB dt.

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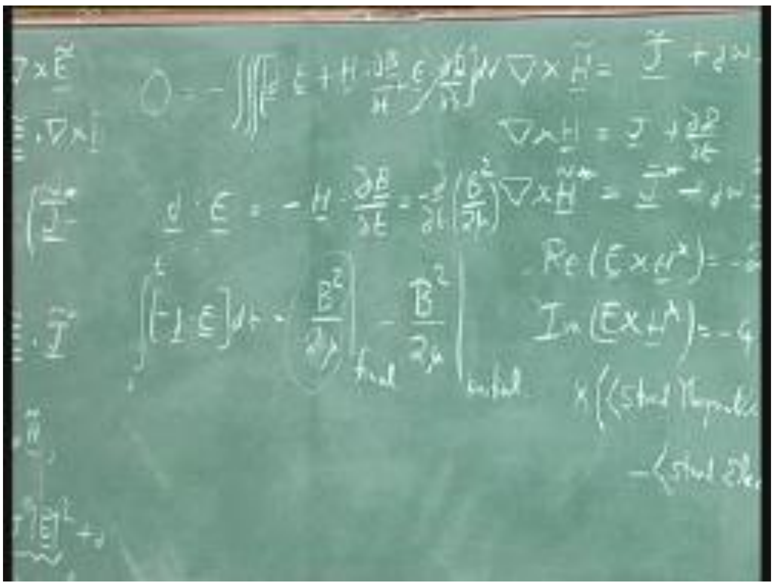

That is j dot E is equal to minus H dot del B del t if I assume, this mu H that is saying minus del del t of B squared over 2 mu. This is the energy dissipated; this is the expression in terms of the magnetic field. So, I can write integral 0 to t of minus j dot E dt is equal to B squared over mu, final minus B squared over 2 mu initially. So, that is saying that the amount by, which the magnetic field energy is increased is nothing but j dot E, which is the energy I putting. So, it is a confirmation that, this expression for magnetic field energy is correct poynting theorem essentially, includes all the earlier work. We have done it automatically, includes stored magnetic energy stored electric energy dissipation and energy in the forms of the battery. Then all of circuit theory is here and therefore, naturally stored magnetic energy is one of the thing that is way the last topic that, I want to talk up in waves is the problem of boundary conditions.

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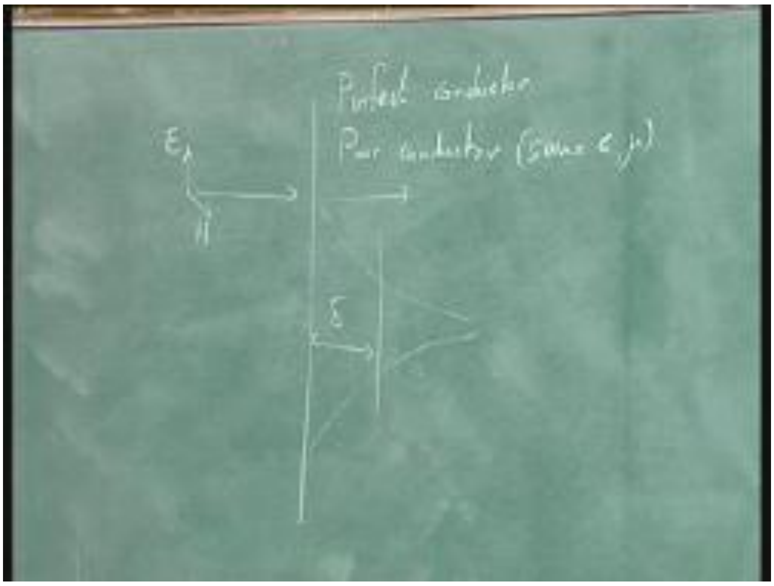

We have already looked at 2 such cases one is perfect conductor and we found that, the wave arrived at the perfect conductor. And reflect it no energy went into the perfect conductance, we also looked at the case of poor conductor. And we assume that the conductor was at the same epsilon and mu as the other side. And there we said there is no reflected waves, it is all transmit in which case, we found that we had the skin effect energy decade over a distance delta and the wave propagate into the material. Now, I would like to generalize this, I like to talk about the general material and ask what is happening in it.

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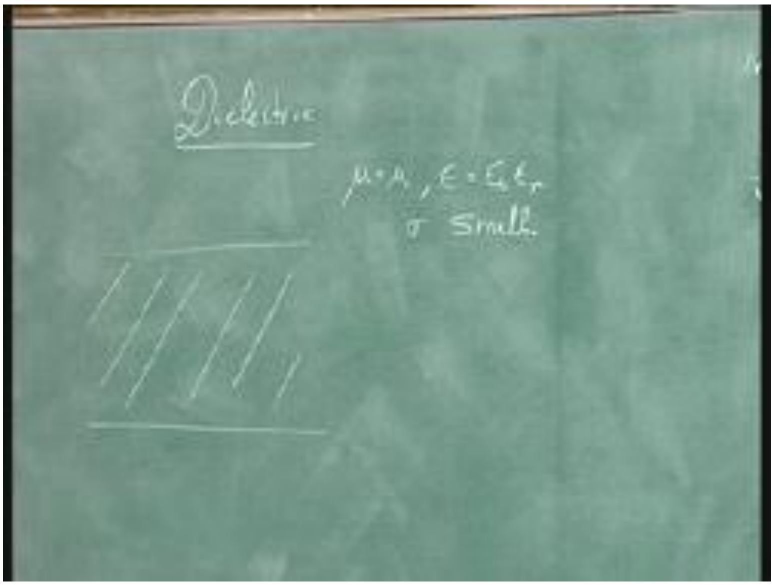

First of all let us, look at the dielectrics so I have a material, let say mu is equal to mu naught epsilon is equal to some epsilon naught epsilon r. And there is a small signal this is typical for example, if you have a capacitor plate with dielectric in the middle there, will be a very small leakage current is. So, small let me take capacitor a day to discharge, but there is going to be a small amount of leakage current, which means sigma is not 0, it is small now what is the wave equation in this system?

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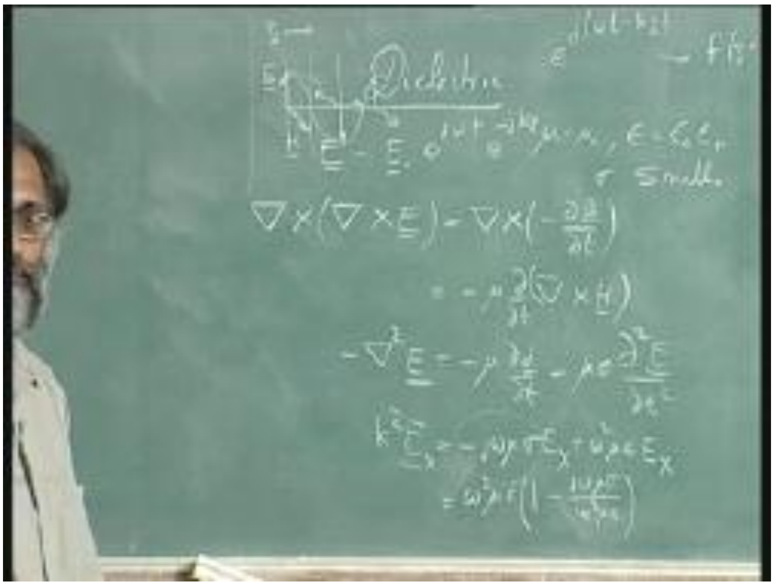

Well, it is a curl of curl of E is equal to curl of minus del B del t minus mu del del t of curl of H, which is minus mu del j del t minus mu epsilon del squared E by del t squared on the left hand side, this becomes minus del squared E you have derived this many times now. So, there is no need to spend time on the equation now, I am going to immediately assume phasors. So, I am going to assume that, my electric field is E not e to the j omega t further more since, I am interested in looking at plane waves, I am going to assume e to the minus j kz.

So, I have the material this is z and the plane wave is propagating in z. So, the electric field is in the x direction magnetic field is in the y direction since it looks like into minus jkz, what it really means is it like a cosine. So, after a distance z, it goes to constant, then it is poynting downwards plane it go back to constant, then it goes back to poynting of E. So, the electric field is a sinusoidal function of z, which also sinusoidal function of t. So, it has the same form e to the j omega t minus kz the same form is f of z minus ct this is, where we started from some lectures ago, we are still there.

So, if I substitute all that in here, I get k squared E Ex is equal to minus j omega mu sigma Ex plus omega squared mu epsilon Ex. Now, mu is mu naught, but epsilon is epsilon naught, epsilon r and I have this additional term earlier on, we looked at this term say this is dominating this side. But I am looking at a dielectric in a dielectric sigma is very small. So, I cannot assume that, this term dominates omega squared mu epsilon. In fact, just the other way around omega squared mu epsilon dominates minus j omega mu sigma. So, what I am going to do is. So, I am going to combine these 2 terms, if I combine these 2 terms, if I combine these 2 terms, I get omega squared mu epsilon times, 1 minus j omega mu sigma divided by omega squared mu epsilon. I can now, take this piece as a complex correction to my epsilon and if I do that and I treat the dielectric as the system, which has some real and some parts to it is dielectric constant. You find that all of plane wave theory that we have already derived applies to a dielectric. I will complete the derivation next time and you will see how it works out.