Electro Magnetic Field
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Lecture - 34
Poynting Theorem

Last time we have derived the wave equation from the 3 dimensional Maxwell's wave equation. It is a Maxwell equation and derives solutions for them. Let me just remind you we started with paradise law.

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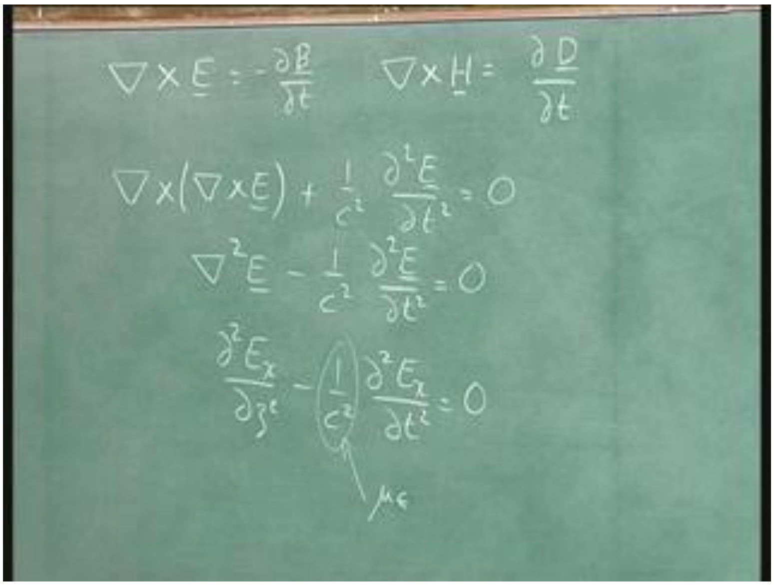

And amperes law without sources there is no j and from that we obtain the wave equation which was and in the absence of sources in vacuum this term simplify. And we got an equation which was del squared E minus 1 over C squared del squared E del t square equals 0. And this is the natural generalization of the 1 dimensional wave equation we have already derived which was del square Ex del z square minus 1 over C square del square Ex del t square equals 0. This term 1 over C square is nothing, but mu epsilon and for vacuum C is the speed of light.

Now, at the end of the last lecture I have talked about the solution of this equation and how, because of the nature of paradise law. And amperes law the electric and magnetic field are always perpendicular to z z is the direction in which the wave is going. Now, I am going to take 1 step back from all this go to some mathematical theory if will all connect up which is why I am doing this. And this is the theory of the pointing vector the purpose of doing that this 2 try and make more sense out of the solution. We have been getting for the wave equation, so to start let us look at what we have been saying up to now where said that if we guest.

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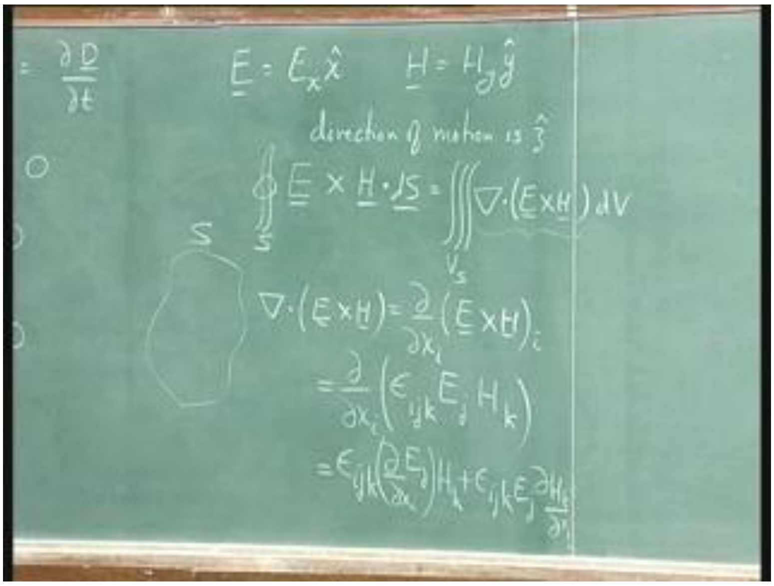

The electric field is Ex x hat the magnetic field is Hy y hat and if you assume that the direction of motion is z hat then we get solutions. So, we look at the direction z hat it is in the direction E cross H what more you have seen E cross H terms coming out of doing magnetic energy. Because we have remember we discarded a term which went like surface integral E cross H when we are trying to calculate static magnetic field energy. So, now, what I am going to do is I am going to look at this particular quantity because seem to have lot to do with what this wave is doing I am going to look at it. And ask what is that equal to well surface integral over some surface E cross H dot ds I can apply the divergence theorem.

The divergence theorem tells me that this is equal to volume integral over the volume enclosed by the surface divergence of E cross H dv. So, I have starting point here where I thing that they may be something interesting happening as for as the wave is concern but, it immediately requires me to understand what is this quantity divergence of equals H. Now, you got lot of machinery at hands I mean we have been doing for the last 4 lectures all sorts of vector identity with curves and divergences. So, let us attempt to analyze this one divergence of E cross H well divergence is nothing, but del del xi acting on E cross H i th component.

So, it is del del x of E cross H along x plus del del y of E cross H along y plus del del z of E cross H along z, but I have a nice way of representing E cross H. So, what I have to do? I am going to say this is equal to del del xi of epsilon i j k Ej Hk. So, this is just the same unit anti symmetric tenser and Ej Hk is the ith component of E cross H epsilon ijk is a constant. So, I can pull it out which means I am taking the derivative with respect to xi of product of Ej and Hk I can use my product rule for differentiation. So, I get 2 terms 1 is epsilon ijk del del xi acting on Ej multiplying Hk plus epsilon ijk Ej del Hk del xi. So, I just apply the derivative first to this term and second to this term. Now, let me write this out, so that you can see that there is a nice symmetry to it.

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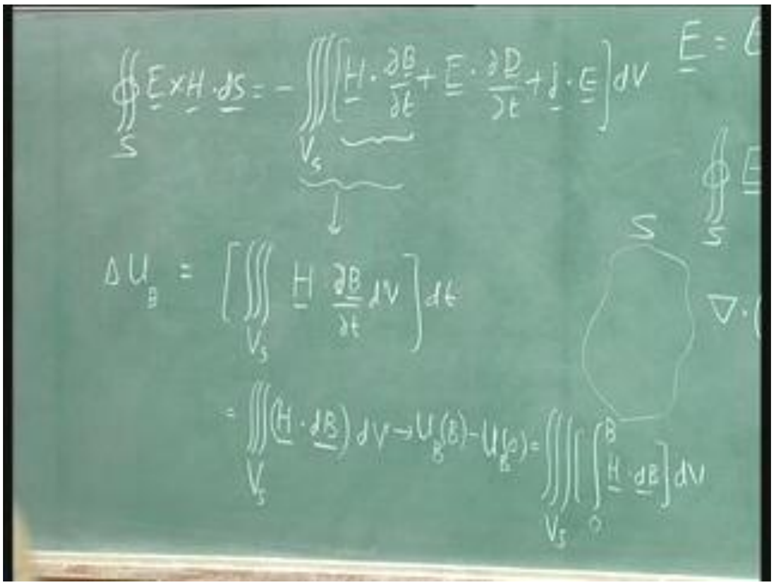

So, what we started with now I know that epsilon ijk is cyclic I keep drawing this figure well write again if I want epsilon 1 2 3 I have go clockwise. So, it is plus 1 and I want epsilon 1 3 2 I have to go counter clockwise direction, so it is minus 1 this just comes out of the theory of determinacy. So, epsilon ijk is the same thing as epsilon jki is the same thing as epsilon kij you just a matter of whether 1 2 3 2 3 1 or 3 1 2. So, I will write it out as epsilon kij no I would not sorry let me just check Hk Ekij del Ej del xi. So, what I have is taken a Hk here I have rotated this epsilon ijk as epsilon kij. And then written out this type this term what I am going to do Ej outside then I am going to do epsilon jki rotating ones del Hk del xi. So, I have done nothing except this rotation, but if you look at this term with epsilon kij del ej del xi which is nothing but curl of E k th component.

If I want curl of any vector k th component it would be epsilon kij del Aj del xi. So, you can see that is that same form this term on the other hand is again curl jki, but unfortunately k is the index second index and i is the third index. So, in other words I am going to matrix which looks like x hat y hat z hat Hx Hy Hz del del x del del y del del z. So, only in mind we should say, because still del del x acts on Hx etcetera. So; obviously, to get curl I want to switch this 2 rows and I know that i switch rows and determinant i get a minus sign. So, I can identify each of this 2 terms with a curl one is the curl of E other is the curl of H. Let me write it out my surface integral over s E cross H dot ds is. Now, a volume integral over the volume enclosed by the surface times H dot curl of E thus you can see the k has repeated.

So, it is a dot product minus E dot curl of H and just we have derive this vector identity it is that simple which is y. I never encourage any of my student to go and memories the cable of vector identity with simpler to just derive it. But now I have Maxwell's laws to help me I know curl of E is I know curl of H is we rotate out here curl of E is by faradays law minus del b del t curl of H is by amperes law plus del D del t. So, I applied here I get volume integral over v Vs minus H dot del B del t minus E dot del D del t. Now, actually this was an incomplete equation the full equation said j plus if you remember amperes law curl of H equals j. And we added del D del t to in doing this derivation I through away j, but to do this pointing theorem analysis I will keep the j also.

So, I keep the j then I will get the extra term minus E dot j let me write out this equation, because it is basically the final results surface integral E cross H dot ds over a surface is equal to the minus. The volume integral enclose by the surface of 3 terms 1 term is H dot del B del t the second term is E dot del D del t and the third term is j dot E. And this is in fact, nothing, but what we call the pointing theorem pointing was the person, who derive this theorem. And it is a crucial theorem it is probable the most important theorem in radiation. So, why is it so important? Now, what I am going to do is look at it in 2 different ways. Firstly, look at what is the same this is saying this piece we have seen already we can always look at this this part of the integral.

And say it is volume integral enclosed by the surface of I am going to say the magnetic stored energy in a time delta t is equal to H dot del B del t dv multiplied by dt. So, this is rate of change of stored energy. So, I can always multiplied by dt to get a change of stored energy in a del time delta t. Let me move that dt inside and I will write this as volume integral H dot db dv and this should be very familiar to you, because they can now integrate over time or over B. So, this will give me that the magnetic field energy had some value of B minus the magnetic field energy at say 0 is equal to volume integral over the surface another integral from 0 to B of H dot db. Now, I say this is familiar to you, because as electrical engineers you have been working with machines. You have been working with magnetic material you know that f you are dealing with a magnetic system there is something called historicizes.

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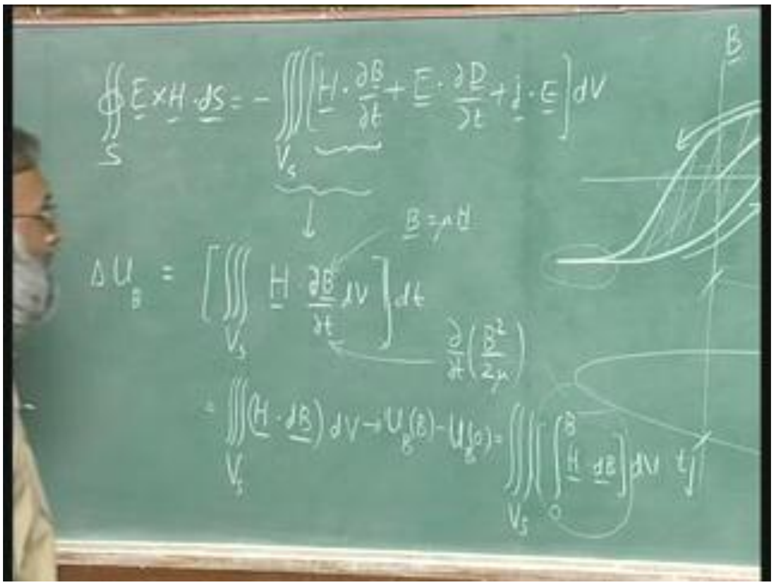

You have your H and B axis if it where a linear system you know that you what to have straight line B equals mu H, but, does not happen. In fact, if you take an unmagnetised material and you apply a magnetic field the curve that you see is actually a very non-linear it saturates. And when you try to bring V down you get a effect like this which then gives you the standard historicizes diagram. And you know; however, you want to do it whether you want to do integral H dB or integral dBH the integral of the area enclosed by the written. And the forward curves, is the amount of energy that is dissipated in any magnetic system due to a cycle of applied field.

So, it applied field B where in time doing this let say this is time this is be for H then what would happen is that as H changed magnetic field would raise and saturate then come down saturate come up in saturation. So, as this cycle went on you would get the magnetic field that did not look like H at all. In fact, it showed all this signature as saturations as well as the signature of historicizes. And you know very well that the amount of energy dissipated in a single cycle in time this given by this area of this curve BDH or HDB that is where this concept comes from.

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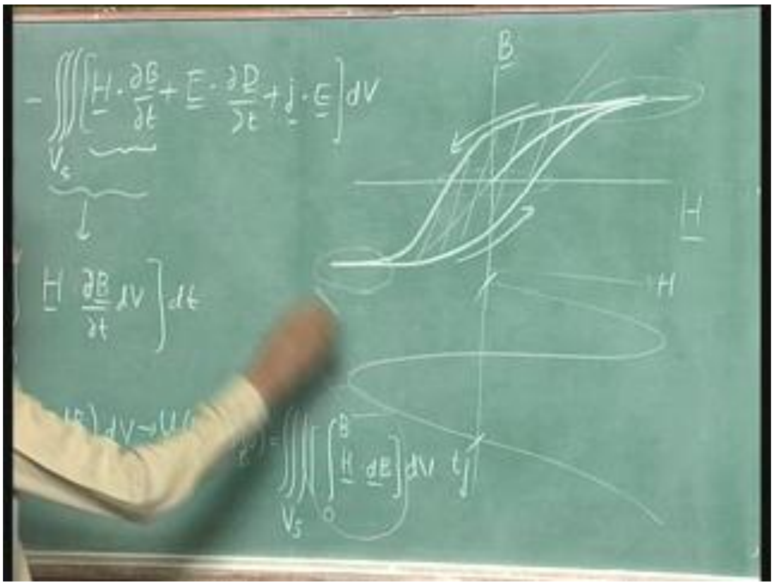

This is the general form for stored energy; however, if you can assume that B is equal to mu H and mu is a constant then you can pull that mu out. And you get this standard result which is at this whole thing will become del del t of B squared over 2 mu. So, both approaches are saying the same thing and this is what we have got earlier. So, the first term is nothing but magnetic energy the second term correspondingly is nothing but electric energy let seen this already.

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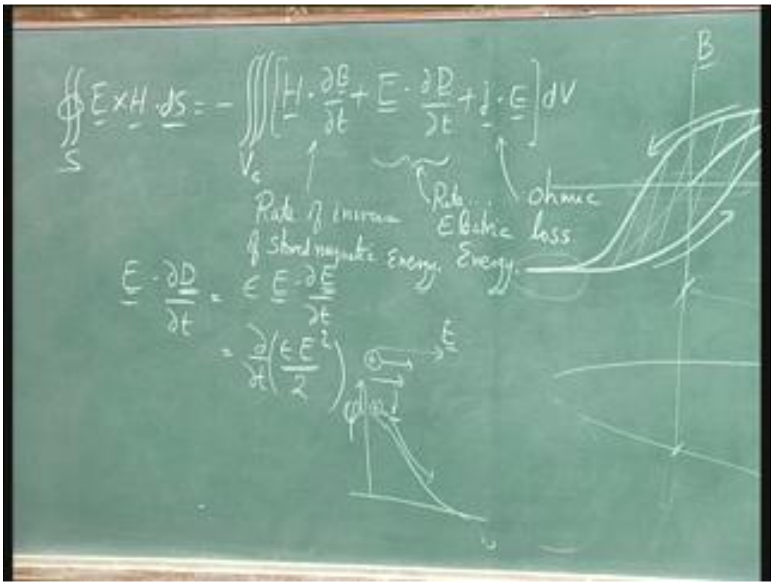

So, again E dot del D del t is equal to epsilon E dot del E del t which is equal to del del t of epsilon E squared over 2. So, this is rate of increase of stored magnetic energy and this is the corresponding rate for electric energy. So, these 2 terms are very clear and what is this term? This is ohmic laws, because if you apply a, an electric field E. And if I charge moves in the direction of the electric field then; obviously, there is a current j in the direction of the electric field. So, if j dot E is positive it means positive charges are moving in the direction of the electric field which means positive charges are going from a point of high potential energy to a point of low potential energy. So, energy is being given up by the field 1 half row pi if you like this pi exchange that what happening due to this j.

But, where is that energy going? That energy is going into the kinetic energy of the charge. The charge is falling down a potential energy well this is my file, so this positive charge is rolling down this potential well. So, you fix up kinetic energy this kinetic energy due to collusion turns into heat, so this is ohmic laws. So, the 3 terms we can see one is the rate at which stored magnetic energy is increasing the second is the rate at which stored electric energy is increasing. And the third the rate at which energy is converted into heat now actually it is not energy converted into heat. Because this same term can also represent energy being converted from electrical to mechanical energy if you have a motor this j dot E term would represent. The work being done to turn the motor axel if it where a generator you would have a negative term here. And the j dot E would represent energy coming into the field due to the mechanical energy.

So, j dot E doesnot really represent only ohmic laws it represents any kind of conversion to or from non electro magnetic energy. So, any kind of energy that takes energy in the form of heat energy in the form of mechanical energy energy in the form of chemical energy. And brings it in to a electrical energy or takes it from electrical energy they will all represented as j dot E. Because that is the only way by which work can actually get done j dot E is the only place by that it happens. So, we have this 3 terms now normally what would we expect? We would expect that the rate at which stored magnetic energy is increasing plus the rate at which stored electric energy is increasing plus the rate at which energy is getting dissipated must be 0.

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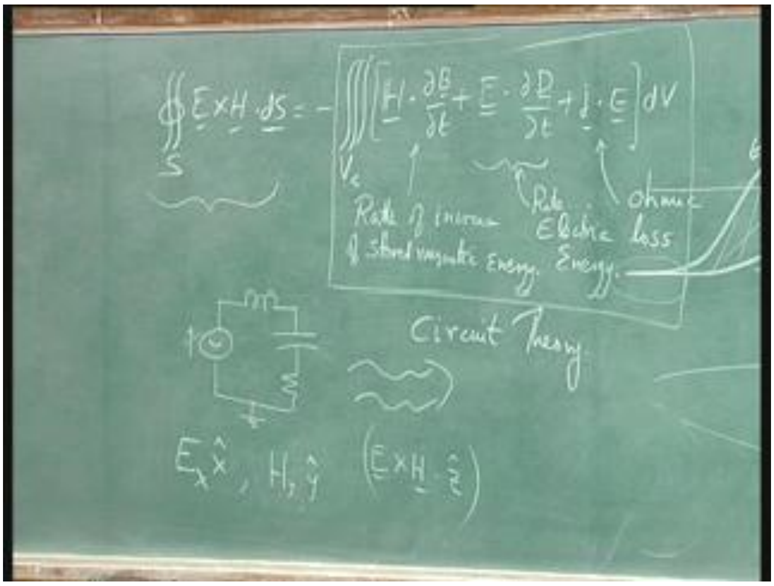

That is if you are going to dissipation energy one of the 2 must be decreased or if there is no anticipation electric energy increase means magnetic energy decrease magnetic energy increase means electrical energy decrease. So, this part is circuit theory this is what we have done for the first part of this course inductors capacitors resistors and batteries. That is all it is this is the new peace this is new peace here which says that on the surface if you happen to have E cross H dot ds. These things do not add up to 0 in other words you draw a circuit diagram. We have a battery you have inductor capacitor resistors you basically have not energy conservation equation here namely kcl kvl kcl.

And kvl are basically telling you that whatever energy is lost in the form of IV in the battery is expended in the form of IV in the resistor. And this 2 do not do anything if this where an ac source you have an accelerating voltage which causes stored energy acceleration in the inductor.

And the capacitor dissipation in the resistors all of that is covered here there is this peace inductor capacitor resistor and battery, but this peace is saying sometime it does not add up to 0. And in fact, you have the deficit or an excess of energy which is showing as E cross H dot ds. Now, if you are try to do E cross H dot ds on a dc circuit what you find is that the electric field. And the magnetic field will go in such direction that E cross H dot ds integrated over a surface gives you 0 it does not help. It does not actually violate anything circuit theory is still same the problem comes when you have base as you saw we had Ex x hat Hy y hat which meant you have E cross H dot z hat. And this is not 0 once you have this kind of possibility then there is something new that is coming into electricity and magnetism something. That is not there in the circuit theory something that kvl and kcl cannot capture. And it is this peace that is the crucial new peace which we call radiation. Now, let see where we can take this further I am going to join a circuit.

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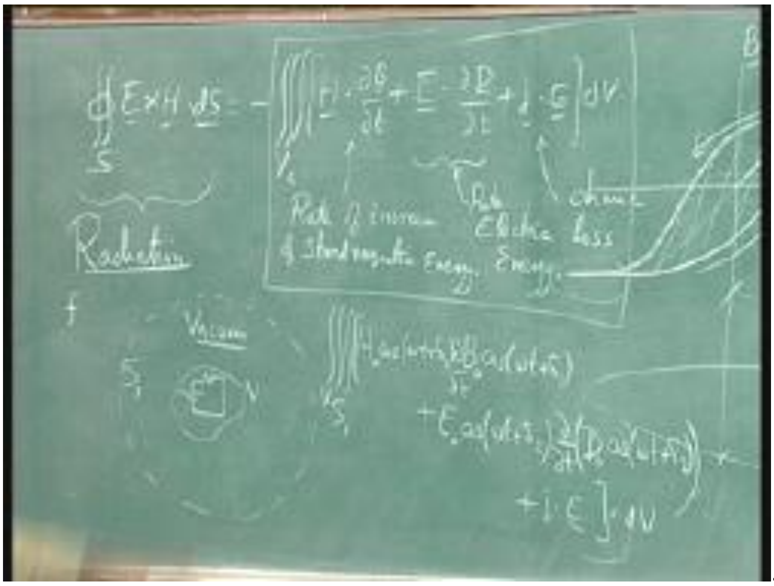

In a small region, so I have some circuit some complicated circuit if you will and it is inside a small region V I consider the nice surface I am going to call it S 1 this S 1 fully encloses V. And in fact, quite it is a fact it is bigger than V let say. And I am going to apply this theorem to S 1 in this region I am going to assume it is vacuum. Furthermore I am going to assume even though drawn the dc battery and a resistor this is an oscillating circuit. So, the voltage is oscillating as a sinusoid it has some frequency f, so whatever fields that come magnetic field or electric field there also oscillating in time. And I am going to assume this circuit is study state, so whatever currents are flowing is some I not cos omega t plus some delta no transients alright.

Now, want to look at what happen such a system well the left hand side is clear. But the right hand side if I take this B s is going to look like volume integral over S 1 sorry vs 1 sorry H is going to be some H not cos omega t plus delta 1 times B not cos omega t plus delta 1. Then I am assuming a nice simple systems to H and B are along the same direction. Then there will be del del t plus and E not cos omega t plus some delta 2 del del t of D not plus cos omega t plus delta 2 plus j dot E this whole thing dot ds I mean dv this will what I have here. Now, you look at this term it is an oscillatory term, because del del t of cos omega t will become sin omega t. So, this will become cos 2 omega t this will become this 1 half sin 2 omega t plus 2 delta 1.

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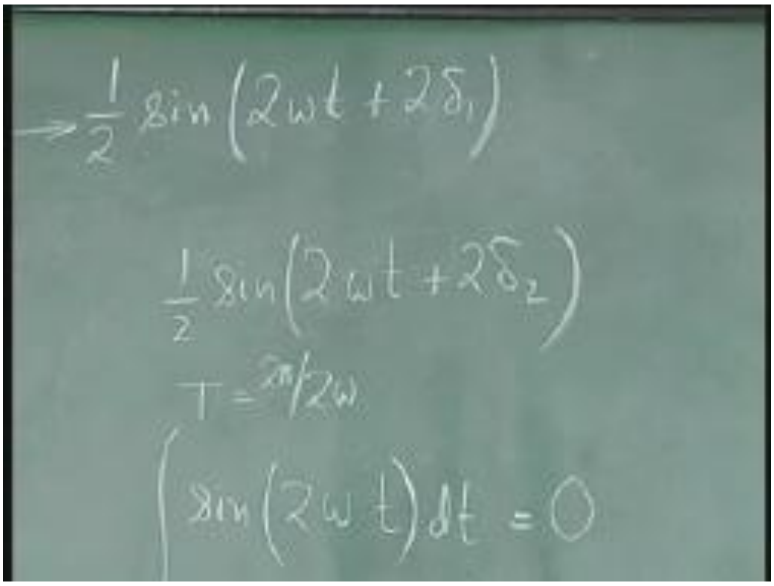

Similarly, we have here cos omega t plus delta 2 under the derivative sign, so it becomes sin omega t plus delta 2. So, we have cos omega t plus delta 2 sin omega t plus delta 2 and this in turn becomes 1 half sin twice omega t plus 2 delta 2. Now, this term involves sin of twice omega t and if you average this function sin twice omega t from over 1 period which is from 0 to t where t is 2 pi over 2 omega. We know that sin as a periodic function which means that this integral is 0. So, if you take the write hand side of the pointing equation.

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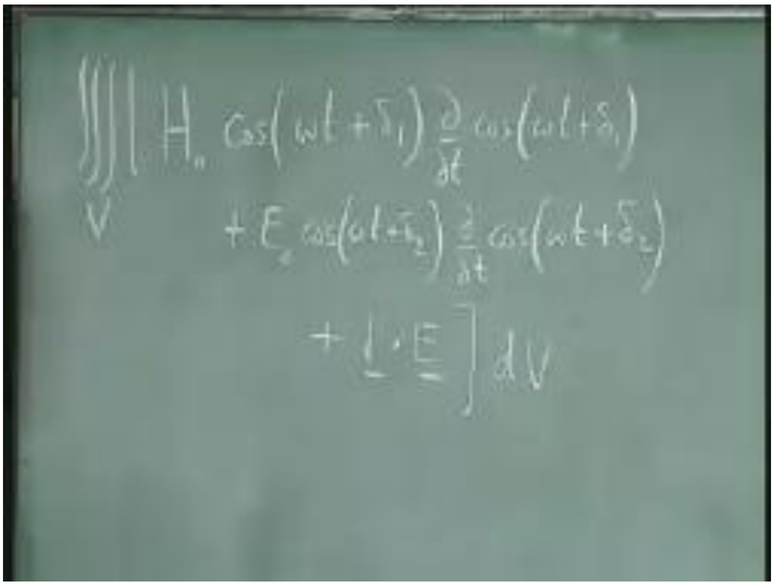

And average it over 1 period that is 2 pi over 2 omega this term goes to 0 this term goes to 0 which means that we are left only with j dot E. So, these 2 term will just vanish even though there is a term that taking about the rate of increasing of stored magnetic energy. What is happening is half the cycle the magnetic energy is increasing there is second half of the cycle. The magnetic field energy is decreasing similarly half the cycle the electric field energy is increasing half the cycle the electric field energy is decreasing. So, if I average over 1 time period both this terms go away. So, what I am left with? I am left with only this term, so this pointing theorem.

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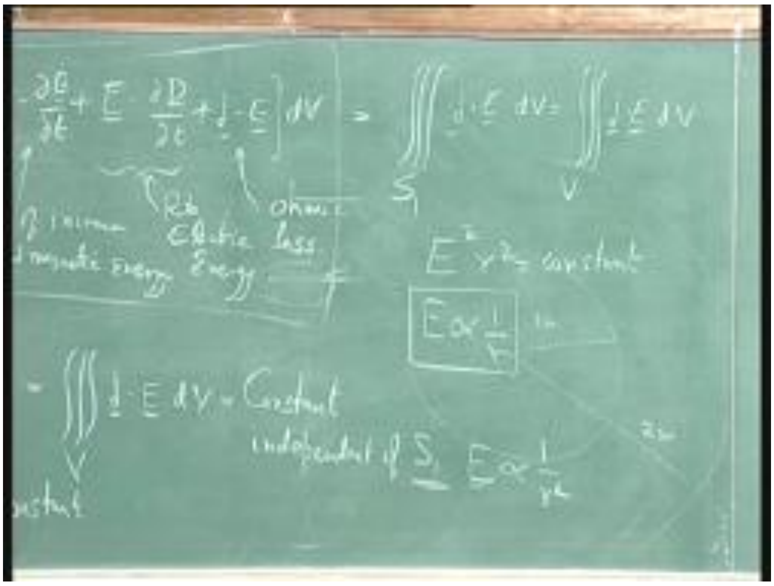

Then becomes integral over surface S 1 of j dot E dv, but if you look at the picture I assume that the, this region is vacuum and my circuit is enclosed in side a volume v. So, there are in any current except inside volume v. So, I can write this as a integral over volume v j dot E dv. Now, what interesting about this I am going to write it down again? So, that you can look at it what is interesting about it age with that the left hand side which is a surface integral over a surface S 1 E cross H dot ds is not an integral over S 1 Vs 1 at all. It is a volume integral over V j dot E dv no matter what size surface I choose this S 1 can be twice it a S 1. It can be 4 times S 1 right hand side is till integral over V j dot E; because currents are present inside only this over small region V c is equal to constant independent of S 1.

So, it is a very remarkable statement no matter how large is sphere I take to do this integration i get the same answer. So, it means if I take a sphere of radius 1 meter I get some answer i take a sphere of radius 2 meters I will get a same answer. But if I look E. and H I know that H is equal to square root of epsilon over mu E at least we work that out for 1 d case it happens it to generally. So, this E cross H is really E square times, square root of theta over epsilon sorry mu over epsilon. Now, work this out the volume in the surface integral is here is 4 pi times r squared as a surface area of a sphere multiplied by E square multiplied by a constant. So, just mu over epsilon is equal to constant or it saying something entirely remarkable it is saying E squared times r squared is constant or E is proportional to 1 over r at the beginning of this course we started everything with coulombs law. And in coulombs law we saw electric field was proportional to 1 over r squared when we added more charges together.

We got dipoles and an electric field due to dipole goes 1 over r cubed and you put more and more this charges together it only decade faster. We went to magnetic field we found that magnetic field is proportional to 1 over r square. But when you actually try to make current create magnetic field and loops it went as 1 over r cubed. But suddenly after putting all the terms that we could find in Maxwell's equations we are finding an new electric field. And this new electric field is proportional to 1 over r it is got nothing with the coulomb law. The coulomb electric field will give me 1 over r square this electric field gives me 1 over r it is a far more powerful electric field. Then the coulomb law can give me of course, we do not know such a field exist all we can say is that if it exist.

Then it goes like 1 over r, because for all we know the conclusion we can draw also is the strongest field possible is coulombs field since this goes like 1 over r this must be 0. Now, we will show that in fact, that is not true that. In fact, the special case that we already solve which is the 1 dimensional wave equation carries energy with it carries power with it. And, because it carries power with it that power must come somewhere that power. In fact, is nothing but this term let us look at just that the units for this the units for all of this are rate of change of energy rate of change of energy density integrated over volume is rate of change of energy.

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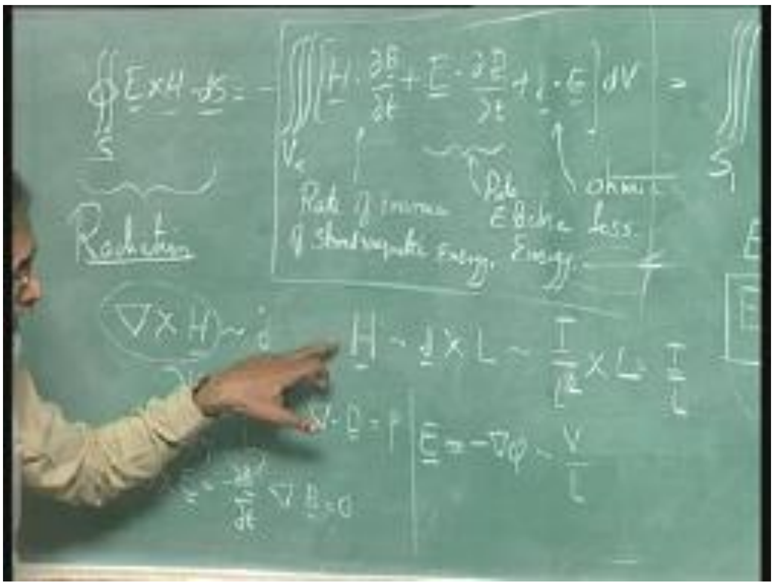

So, all of this are power therefore, surface integral of E cross H dot ds must also be power or E cross H as units of power per meter square. It is all usually called an energy plugs, because the plugs of energy means the amount of energy per second per meter square energy per second is power. So, energy per meter square will give me a, plugs, so let see what sense we make out of this units. Now, we go back to amperes law we know that curl of H is like current which means H is like current into a length current density into a length. What is current density? It is like a current divided by volume sorry current divided by area into length what I mean by that. This stepses when I do curl of H I am doing things like del Hx del y which is sort of like saying I am doing H divided by some scale length in y.

So, I can take this length to the right hand sides to the unit of H, is like current density times length current density is current per unit area. So, it is current per length squared length or current over length if you look at curl of E is equal to minus del B del t. That does not give me anything; because it just relates electric field to magnetic field similarly divergence of B is equal to 0 does not give me anything. So, we have to go to the other equation which is divergence of D is equal to row from this. We know that we obtain that electric field is equal to minus gradient of potential, and therefore electric field goes like volts per meter.

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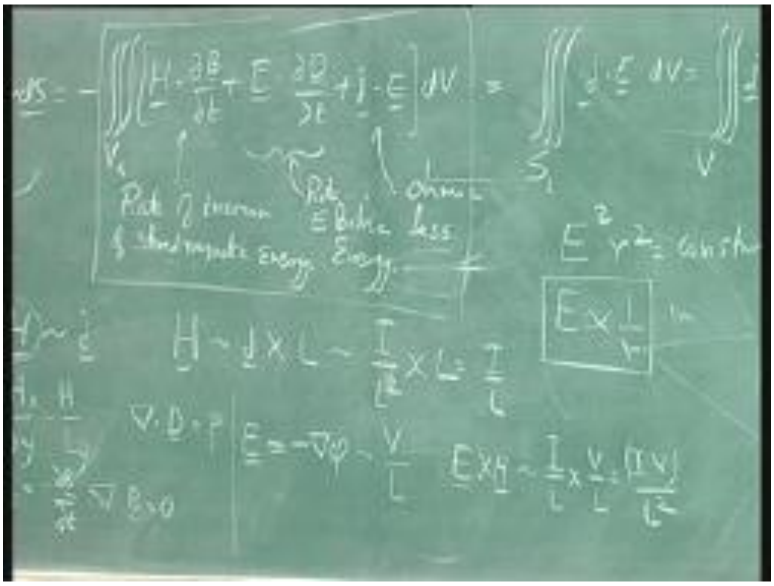

So, now if you multiply E and H what do you get E cross H as the units of I over L times V over L which is IV over L squared IV is nothing but power from our circuit theory. We know current into voltage is power, so it is power per unit area which is what it come to from this thinking as well these are all these integral all give me power. So, the total integral is power, so this term E cross H must be power divided by area. Now, there is another important thing to look at here as I told you this square box is basically circuit theory.

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And this is the failure of circuit theory radiation the question is where can the failure come from namely what generates waves and what observes waves? Now, in general if when you ask this question what we are asking is something more basic that only finite amount of energy in the magnetic field and the electric field. So, we want to have an antenna on a tower that generating waves, so that we can listen on a transistor radios. That energy on the wave is not coming from the stored magnetic energy is not coming from the stored electric energy those energy is a relatively small. And after mille seconds that energy will be reduced down if energy is going out continuously it must come from this term it must come from j dot e term.

So, the source of radiation is again j dot E like term and from the point of view of the circuit. If you want to understand what this j dot E means what it means is I have my voltage source I have various elements. And then I do my kcl and kvl I am finding that things are not adding up, but I know that whatever radiation is coming out is coming out of something. That looks like a j dot E at anything that looks like a j dot E as the form of a resistor either a battery or it is a resistor. In this case I can do better than that I can say that the current that I have is itself related to the magnetic field I will prove that later. And the magnetic field is related to the electric field therefore, this is going like E squared when the power goes like E squared or v squared it cannot be a battery it must be a resistor.

So, in any practical use of searching of radiation and circuit theory what is done is an additional resistor is added. This are the resistances capacitances inductances that you already have in our, add 1 more resistance call the radiation resistance. This radiation resistance is not a true resistance at all we just an extra term which take care of the fact that not all the energy supplied by my voltage source is getting accounted for in the inductor the capacitor. And the resistor some of it is going elsewhere and because I am able to show that it is got the behavior of a resistance I am model it as a new resistor the radiation resistance. And using it I balance my equation in other ways I make kvl and kcl comes true after all there are not true really any more the moment.

We have this term circuit theory has failed, because the circuit theory is only this side, but we fake it we say that this Bs can after all we modeled by this j dot E term. Therefore, we will just from the circuit theory point of view we will just add a new resistor it is a frequency dependent resistor the resistor with very strange properties, but we will pretend it just a normal resistor. And we will be able to balance there equations they will be able to use circuit analysis and get answers to your problems alright. So, that let me to an important question which is how do I claim that j is proportional to E before that I must go back to the wave equation that we look at…

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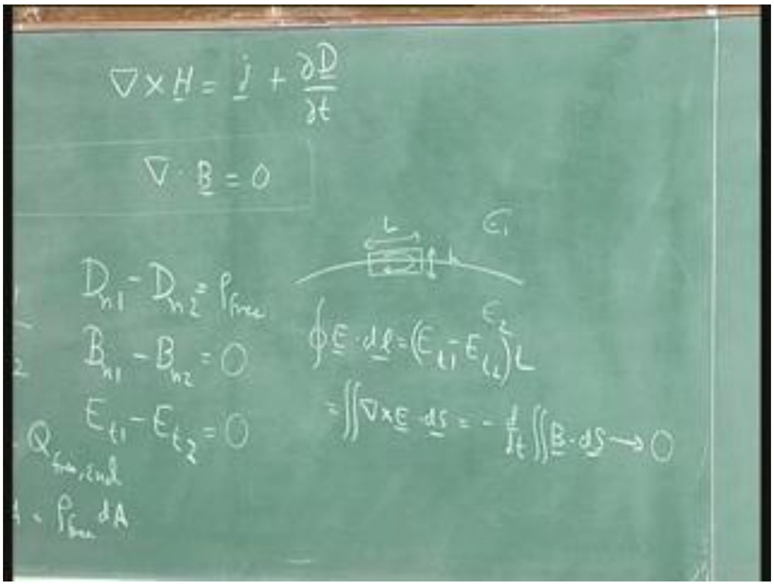

Let us right down Maxwell's equations we have paradise law then you have amperes law you have a divergence theorem for electro static. And you have the magnetic field is d divergence free now as far as we know these equations are exactly correct which means there are 2 equations which I am not changed at all from statics. These two equations are exactly what they where even under static only this 2 equations that have changed. Now, we thing back what we did in electrostatics what we said was if there is any material and it has a region where you have a change in dielectric constant. Then all I will do is I will take an imaginary surface in the shape of a cylinder and I will take the surface such that this height H is much, much smaller.

Then the radius of the top and the bottom which are pi say a squared, so I am going to assume H is much much less than a I can always do that it is my surface. So, I can create any surface I want then I look at the displacement vector on both sides and I apply the divergence theorem. So, what do I get? I get volume integral of divergence D is equal to surface integral D dot ds is equal to Q free enclosed at this boundary I do not have any free charge. So, I can set it to 0, but if I did have free charge I have keep this term. Now, this surface integral involves the integral over the top involves an integral over bottom and involves the integrals over the sides by choosing H much less than a I can make the sides negligible. So, it is only the top and the bottom which tells me that D n 1 minus D 2 1 that is this is D n 1 and this is D 2 1 that is this is D n 1 this is D 2 1 if I take this 2 quantities 1 is pointing up.

So, the positive other is pointing down, so I put minus sign to indicate it is negative this is the times the area is nothing, but the enclosed recharge. But what is the enclosed free charge? Enclosed free charge is roof free times d A. So, that give me a condition for D normal which was D normal 1 minus D normal 2 was equal to row free. The same kind of analysis for magnetic field gave me B normal 1 minus B normal 2 is equal to 0 this way done in static electrostatic and magneto static. And these arguments are still valued even for waves even for time dependent field these 2 conditions still hold. But, when we ask about the first 2 equations we have new terms present it and we have to worry about what those terms are doing. Now, if you recall we again use for electrostatic epsilon 1 epsilon 2 I used the strokes curve.

Once again the height much smaller than the length and integrated loop integral E dot dl this loop integral is equal to E tangential 1 minus E tangential 2 times L. But, it is also equal to the surface integral curl of E dot ds which is connected by paradise law. So, it is give me minus d dt of surface integral B dot ds, but this is a very tiny loop. I can make this loop get really small taking care that H is much smaller than L in which case this goes to 0 which means E tangential minus in region 1 minus E tangential in minus 2 is equal to 0. The fourth equation I will do next time, but that fourth equation is the 1 I really want for proving j dot E as the properties I said. And that will tell us that this quantity that we talking about E cross H is really an important quantity. It is not some kind of friction of the mathematic. In fact, you can have system, which generate E cross H and you can have systems that observes this E cross H. And once you have those 2 ideas then you have all the foundation required for understanding radiation and optics.