Electro Magnetic Field
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Lecture - 33
The Wave Equation - 2

Good morning. Last 2 lectures, we have been working with the wave equation. And today I am going to continue that go little bit more in detail, let me remind what you have done last time.

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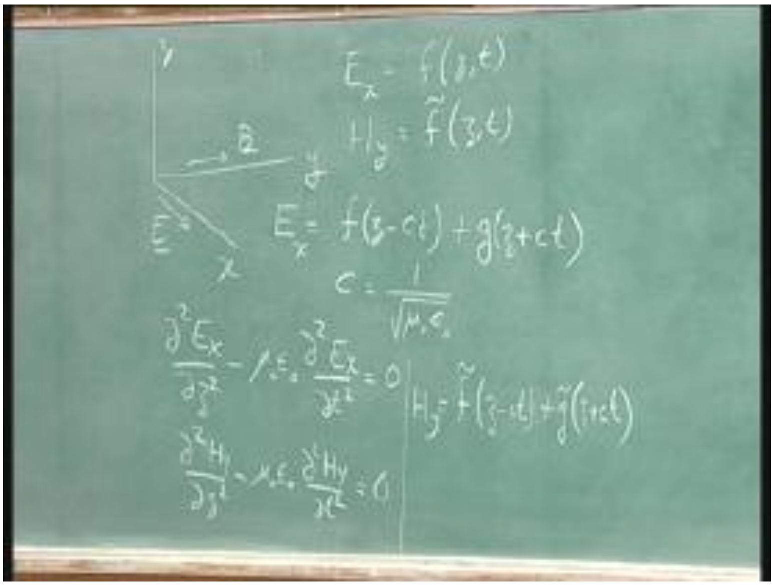

We have taken a special problem where you assume the electric field this pointing along x directions the magnetic field is pointing along y directions. And we assume that the electric field along x direction was some function of z and t that is it did not vary if moved in x, it did not vary moved in y, but it did vary if you moved in z. Similarly, Hy was some other function of z and t. And last time we worked out and we found that if you solve this problem you find that the electric field Ex actually a function of a very specific form. It is the sum of 1 arbitrary function depending on z minus ct and another arbitrary function z plus ct. This f and g are completely up to us they can be anything at all where C was equal to 1 over square root of mu naught epsilon naught this is in vacuum.

I did not prove it I gave it to as an assignment, but you can show where the equation satisfied by Ex namely del squared Ex del z square minus 1 over C squared. I should say 1 over C squared minus mu epsilon del squared Ex del t square is 0. This equation we have derived exactly the same equation can be derived for Hy I did not prove it I have left it to you work out. So, it a same equation in both cases therefore, which is in solution and you will also end up with Hy is equal to some other function f delta of z minus ct plus g delta of z plus ct. So, we have 2 solutions of course, 2 solutions in terms of 4 arbitrary functions which means that the we have tremendous amount of freedom means specifying what you want to specify. Now, in addition what I also did last time was I said supposing this is x; this is y; this is z.

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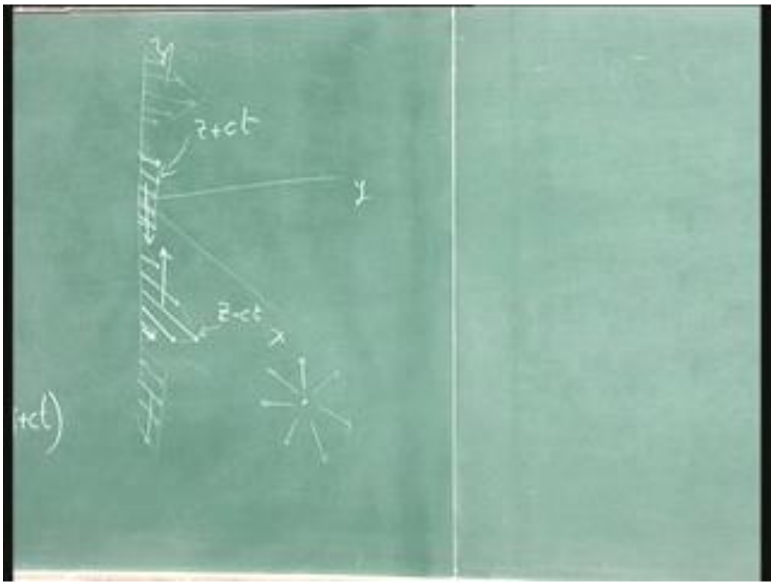

Supposing the electric field as a certain shape, that is it points along the x axis. There is a triangular profile plus there is a rectangular profile you imagine both of these are pointing along x now drawn it only on z axis. But really when I draw this arrow it mean that all x and y is the same electric field because the electric field depends only on z which uniform in x and uniform in y. So, now, this take this electric field and suppose this part of the electric field was dependent on z minus ct and this part depended on z plus ct. I think I have inverted the case the rectangle was actually here now what we saw was this was z plus ct that was z minus ct. This triangle moves in the positive z directions as you vary time and this rectangle move in a negative z direction. They meet when they meet super position means that you end up with complicated shapes. But after sufficient amount of time this triangle would have reached of here and this rectangle would have reach that down here..

So, you would have this triangle moving up and this rectangle moving down, and this is why called waves they move these are disturbances that are not like coulombs law at all it is an electric field. So, you are used to electric fields which are due to charges and therefore, this stay put you put a charge somewhere a electric around the charge is fixed to the charge. But this electric field is not fixed to a charge it is just moving there is no charge in the derivation when we derived this we assume this there is no charge in c. There is no current density just had a disturbance and the disturbance self consistently just keep moving. So, it is a very different object from what we have seen earlier coulomb law is a fixed static charges. These kinds of moving structures are coming out of Maxwell’s equations the generalized amperes law. Now, what I am going to do is to relate Ex and Hy last time I just said where Ex and Hy have same shape thatreasonable, because Ex is connected to Hy through paradise and amperes law.

So, if there is an triangular Ex there is something you do not know what does something at the same z for that t. Because otherwise you could have crul of E being related to del b del t of the same point you did not have curl of edge related to del b del t. So, at the same Z where there is a electric field there must also be magnetic field and if this electric field is moving in time like this. So, 2 must be the magnetic field and if this electric field is moving backward in Z. So, 2 must be it is magnetic field. So, we can see that there is a correspondence between whatever is the forward moving disturbance due to E and the forward moving disturbance due to h, because paradise law connects them up, amperes law connects them up. Similarly, there is a connection between the backward moving disturbance g and the backward moving disturbance c delta. Let us put all of them little more solidly.

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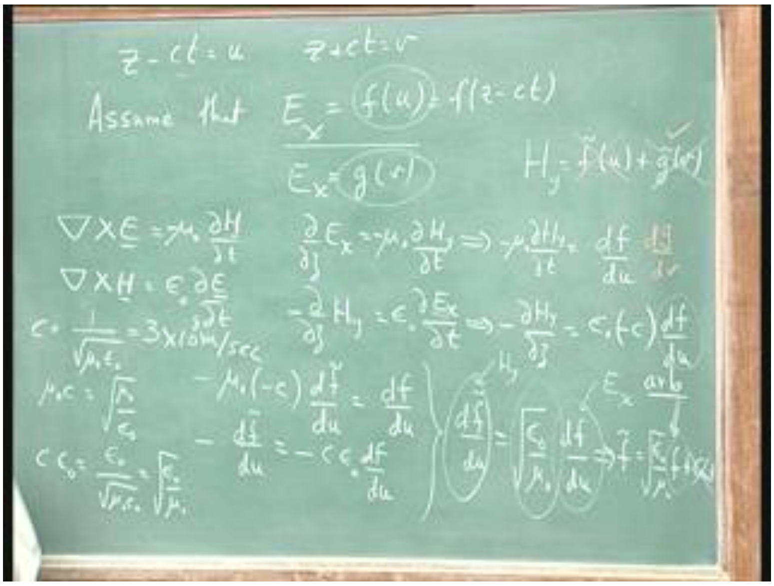

So, what I am going to say is that let z minus ct be equal to u z plus ct is equal to v alright and I am going to assume that Ex is some function of u only there is no v in it. So, I am going to try and find out what is the kind of H that comes out of such an electric field. So, that corresponds to considering only the triangle part of the disturbance. I am not doing anything terrible by assuming this. Because I can always solve this problem then I can solve a problem where Ex is equal to some g of v. And then I add up the 2 solutions that it the wave equation is linear it means that if I have solve this problem and this problem and solve the full problem alright. This is very much like a pendulum problem x double dot plus kx equal to 0 well I can study x I put k squared Cos k x I can study x equals Sin kx. I can study the properties and then I can write the general form which is A Cos kx plus B sin kx.

So, just in the same way I can look, at just E is equal to f u, and I can looks separately E is equal to g of v. And then I can add up the linear combination of u there is no need to add up linear combinations because each of this is a arbitrary function. So, it includes a scaling factor as well. So, let us see what we get I am going to use this form Ex is equal to f of u, what is paradise law? Paradise law tells us curl of E is equal to mu naught del H del t I know that this is the y direction. So, I need y components of this. So, the y component of curl of E is going to be del del Z Ex is equal to mu del H y del t, so the minus sign, because it is paradise law. What is amperes law curl of H is equal to plus epsilon naught del E del t now E is in the X direction, so del E del t is in X direction. So, I need x component of curl of H which is minus del del z Hy is equal to epsilon naught del E del Ex del t I have derived this several times. So, you should be quite comfortable with it, now I have assumed Ex is f of u which is f of z minus ct. So, I can substitute for Ex portion, so what do I get. I get that minus mu naught del Hy del t is equal to del del z of Ex which is df du, because du dz is 1. What do I do? What do I get here? I get minus del Hy del z is equal to epsilon naught times del Ex del t.

So, when I take del u del t when I get a minus C times df du now as I said H is really f delta of u plus g delta of v. But you look at these expressions you find that del H del t del H and del z of functions of u only there are not functions at v at all. You can see the right hand sides have constants multiplying functions of u which means Hy cannot depend on v. This is not saying anything surprising it is simply saying that if that the electric field disturbance is moving in this way. The magnetic field disturbance must also move in the same directions otherwise supposing they were magnetic disturbance moving other way. Then what would happen? After certain time this triangular disturbance would have reached here. But the magnetic field disturbance whatever it shape was moving in the opposite direction which would have reached some way here. But you cannot have Faradays law connecting curl of E here to del reach del t here curl of E is related to del H del t in this point. So, whatever directions the electric field is moving magnetic field must move in the same directions. So, it means there is no g of v. So, I now know that H y is f delta of u.

So, I can substitute for these derivative as well, so what do I get I get minus mu naught del f delta del t. So, when I take del del t I get a minus c d f delta du is equal to df du this 1 says minus del del z does not give me a factor df delta du is equal to minus C epsilon naught df du. Actually both of these expressions are saying the same thing because C is equal to 1 over square root of mu naught epsilon not. So, we look at this side this says mu naught C, which is equal to square root of mu naught over epsilon not, because C is 1 over square root of mu naught epsilon naught mu naught over square root of mu naught epsilon naught is going to give me a square root of mu naught the numerator. And this epsilon naught gives me epsilon naught in the denominator. This term C epsilon naught is going to be epsilon naught over square root of mu naught epsilon naught this is going to give me square root of epsilon naught over mu naught.

Now, this is on the left hand side that is on the right hand side. So, I have bring everything to the right hand side I get 1 equation out of this says df tidle du is equal to square root of epsilon naught over mu naught df du. This is just a constant this is the u derivative of the electric field this is the u derivative of the magnetic field may be integrate both sides with you, what will you get? You will get f tilde is equal to this factor square root epsilon naught over mu naught times f plus constant the constant does not matter we are interested in moving objects. So, a constant is coulombs law. So, we would not worry about constants, we are only interested in disturbances of the type which give us waves. So, it tells us that whatever shape you take for f you can look like a man you can look like a triangle you can look like a square you can look like anything at all.

The same shape is there for the magnetic field it is scaled by a factor epsilon naught over mu naught square root. But otherwise magnetic field and the electric field are the same function moving in the same direction and not only that they are even the same function. So, we write properly the magnetic field points in the y direction and it points that way. So, this electric field points in this direction magnetic field point in that direction and the both move together. It is a very important thing to understand that in coulombs law where the charges where the source of electric field. So, since the charges were stationary electric field was stationary in this case we look at this equations actually what they are saying. Let us look at this kind of equations what they are saying is that the source of the electric field is the magnetic field the source of the magnetic field is the electric field.

So, it means that the electric field causes the magnetic field which in term causes the electric field. But there is a space shift that going on, because of the space shift they are force to move in order to keep in step and that motion is nothing more than the speed of light. This is 3 into 10 to the 8 meters per second I think it is very important to understand this point. That I have not meet any assumption about what this shape was this is arbitrary can be any shape at all whatever shape I assume magnetic field of the existing machine. Now, everything as said which was for f of u can be repeated for g of v if I do that what I will get is that this df du will comes dg dv dg dv and I will be then forced throughout f and keep g g delta and I write now the equations connecting g delta and g. I will get once again the same set of 2 equations which are identical and they again will say the same thing. And I will get g delta is square root of epsilon naught over mu naught g. Once again for an arbitrary shape in electric field I get the same shape in the magnetic field scale by a fact. So, ultimately my picture looks like this is my z axis y axis x axis.

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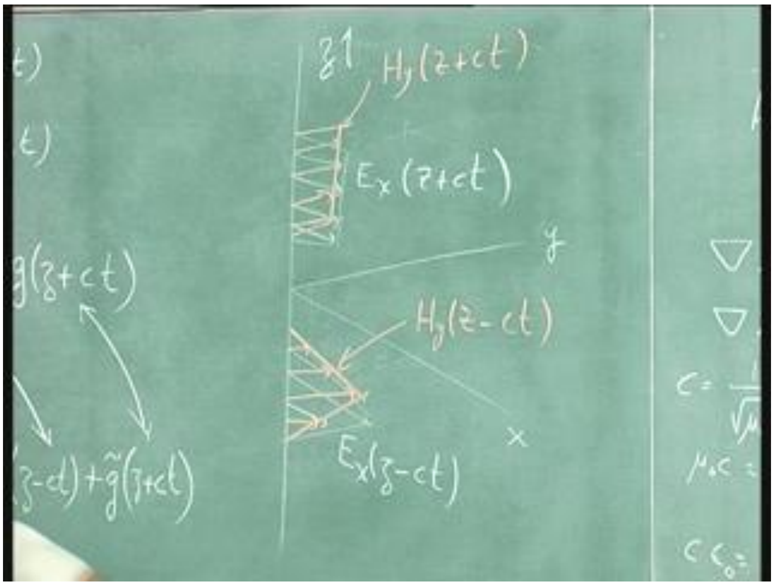

I have my electric field this part of the electric field is moving in the positive z direction this part of the electric field is moving in z negative direction. Corresponding to this electric field I have the magnetic field along the y direction and corresponding to this also I have an magnetic field pointing the y direction. And this magnetic fields Hy Z plus ct in this magnetic field Hy of z minus ct. So, both these together moving the positive z direction, both these together moving in a negative z direction and if they happen to cross each other which is very possible as in time they move in opposite direction. They will go through each other all that will happen is at every point the local electric field is the sum of this 2 fields. So, they will distort the shape will look funny for a while the go right through each other 1 step. They come through each other once again you get back this shape down here further down and this shape appear moving further up. Now, that we are worked out 1 1 dimensional problem let us do more ordinary job of the mathematics.

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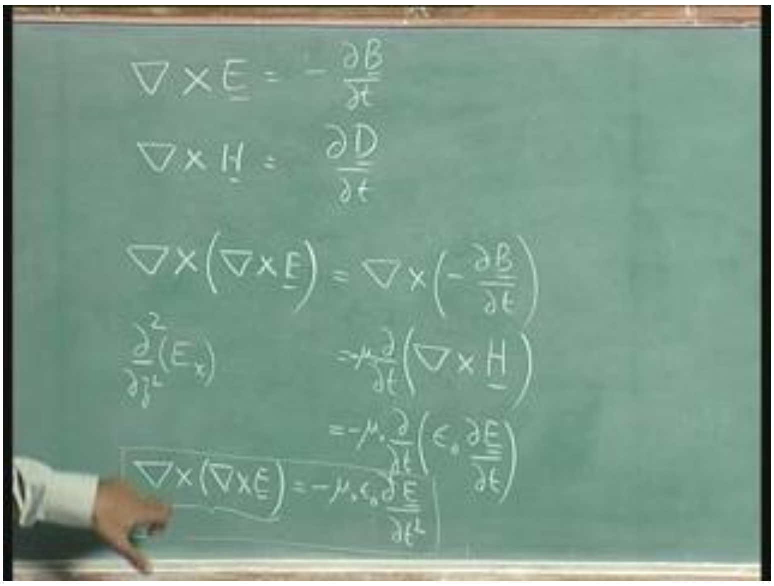

We had the vector relation curl of E equal to minus del B del t and we had curl of H is equal to an ignoring currents and charges at I do not have a j del D del t. Now, I want to do exactly what I did earlier what I did was I took this equation and notice this added time derivatives, but I needed a space derivatives. So, I took a space derivative of this equation in order to use this to eliminate B that is what I did for the 1 d case. So, I am going to do the same thing, I am going to take a space derivative of this equation. So, that I can eliminate this, but which space derivatives do I need I need curl of H. So, I need to take curl of this which means I will have to take curl of the whole equation curl of curl of E is equal to curl of minus del B del t. I have just taken this equation and taken it is curve this is acting on x y z this is acting on time. So, they are independent operations, so I can commute them. So, I get minus del del t of curl of B, but b I am going to write as mu H minus pull the mu out and H.

So, I have written B as mu naught times H, but I know what the curl of H is. So, I can write this as minus mu naught del del t of epsilon naught del E del t. So, I am following exactly the same procedure I followed to get the solution in the 1 d case, how to get the equation in a 1 d case? So, finally, I can write down an equation the equation says curl of curl of E is equal to a there is a minus here yes some minus mu naught epsilon naught del square E del t square. This is the 3 d equivalent of the 1 d equation except that there is very nasty business going on here. This does not look like del square del z squared of E x which is what we had in the 1 d case this was there, but this 1 does not at all look like what we had earlier. So, we need to do some more work we know at least in the 1 d case it simplify we can try and see and simplify even in 3 d case. Well I have thought you the tricks of how to simplify.

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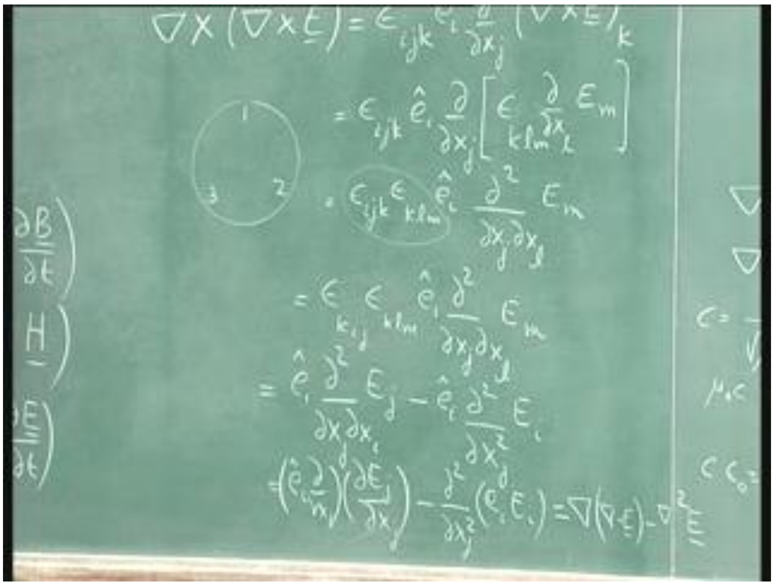

Such equations curl curl of E can be written as epsilon i j k unit vector along i derivative along j of curl of E along k that what a curl is the cross product of the gradient operation and the function here taking the curl of, but this itself a curve. So, I can write it as episilon i j k unit vector along I derivative along j of epsilon k l n I am introducing 2 dummy variables l and m del del x l. This is what the k th components of curl of E you would normally have a E sub area unit vector along k, but I have taken the dot product of that and the and this operation.

So, the unit vector is got observed you just have the k th component. So, I can write this out this epsilon i j k epsilon k l n is of I and I have del square del xj del xl of Em. If only this j and l where both equal then the beginning to look bit like del square del z squared of Ex you have to now work on this piece. As I told you epsilon i j k only cares about cyclic or anticyclic. That is you can write a circle put 1 2 3. If I j k involves going clockwise it is one if i j k involves going counter clockwise it is minus 1. So, I can always if I got 1 2 3 I can write 3 1 2 both of them involves going the same directions. So, I can write this down as epsilon k I j epsilon k l m of ei del squared del xj del xl Em.

Now, k i j must each be different k l m must be different otherwise this expressions are 0 which means either I equals l j is equals m or I equals m j equals l if I equals l j equals m. Both of this are clockwise or both of these are counter clockwise in 1 case it is 1 square the other case is minus 1 squared. So, both ways it becomes plus. So, I will try i equal l and j equals m. So, E sub I del square by del xj del xi of Ej, because i is equal to l j is equal to m minus because it i is equal to m and j is equal to l. So, one of them is clockwise one of them is counter clockwise. So, that is 1 into minus 1, so minus, so now i is now m j is now l ei del squared by del xj squared of Ei.

Now, we need to make sense of this it is equal to ei del del xi acting on Ej del xj minus del squared del xj square acting on ei Ei I just collected terms. So, del del xi are combine with Ei del del xj combine with a capital ej similarly, here the ei and capital Ei are combined together. But each of this are vector operation and let me write down the vector operation this is gradient and this is divergence at e. So, it is gradient of divergence of E minus this is del squared acting on vector E just simple as that. So, I can reduce the wave equation to a simpler form the simpler form is let me rewrite the equation.

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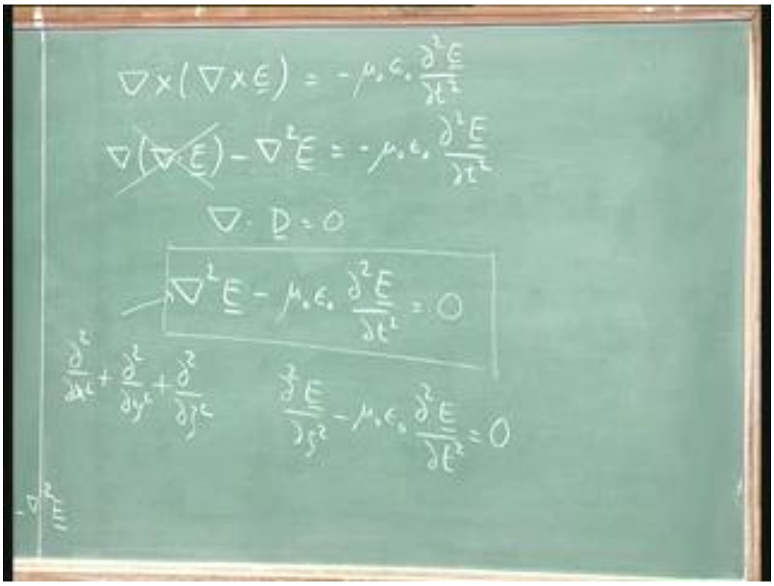

Curl curl E is equal to minus mu naught epsilon naught del squared E del t squared. This curl curl E became gradient of divergence of E minus del squared E is equal to minus mu naught epsilon naught del squared E del t square. Now, I am assuming we are working in vacuum and we have already assume divergence of D is equal to 0 no charge this D is nothing, but epsilon naught E. So, divergence of E is also 0 this goes away which leaves as finally, with the equation that is the generalization of the 1 d problem del square E minus mu naught epsilon naught del squared E del t squared is equal to 0. Supposing the electric field depended only on z. Then this del squared this del squared is del squared del x squared plus del squared del y squared plus del squared del z square. But if E depended on z only this term matters this 2 would be 0 and that would give me an equation that said del squared E del z squared minus mu naught epsilon naught del squared E del t square is equal to 0. And if further said E is nothing but Ex along x this is the equation we got earlier. So, this equation now includes this equation as a simple special case, what are the kinds of solutions? We get for this equation before I discuss that let me go back to the 1 d problem lets graph a little bit more.

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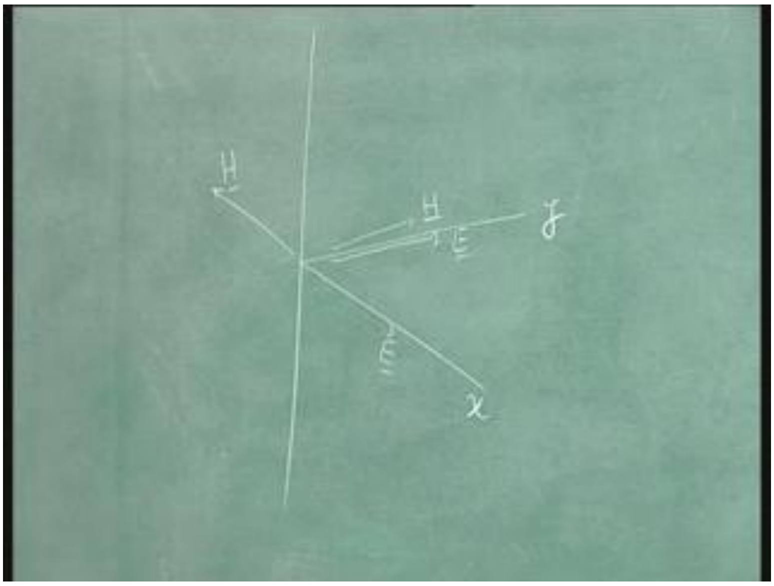

I drawn x y and I said let me guess that the electric field is in the H direction and the magnetic field in a y direction. And from that I managed to get the solution. Now, calling this x and calling this y is bit a arbitrary. I could just as well as said electric field is in this direction in which case I got the magnetic field in this direction after all if I rotate by 90 degrees it should not make really a difference. So, I am not restricted to talking about electric field along x I can also talk about the electric field along y. To be honest I can talking about the electric field along z z could not move in this direction z anymore. So, I could right an electric field that was partly in the x direction and partly in the y direction let me do that. I am going to assume I have a triangular electric field in a x direction which I know that.

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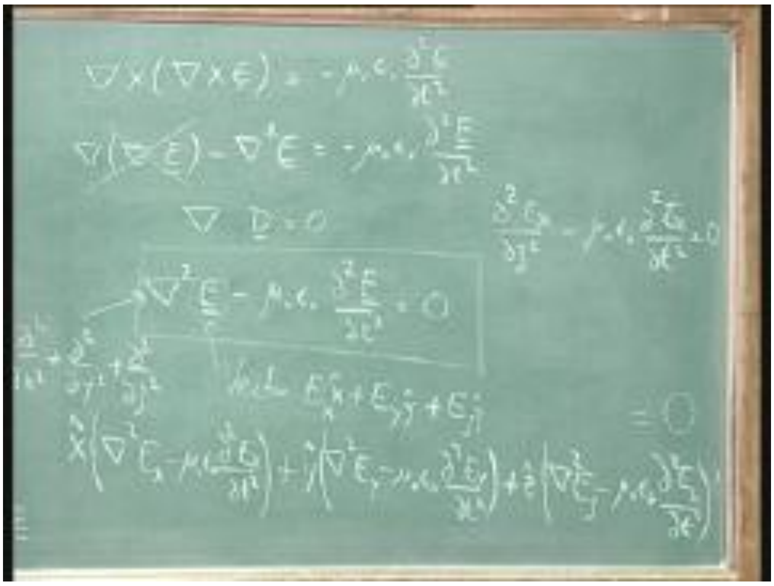

That means, I have a magnetic field that is in the y direction. Let say that I also have an electric field does in the y direction in that case my magnetic field would be in the minus x direction. So, this is Ex of let us say z minus ct this is the chalk Hy of z minus ct this is Ey of z minus ct and this is Hx in the minus sign minus Hx of z minus ct. Both of this solutions are valid and the both will travel in the positive z direction, because they depends on z minus ct. You can have 2 more waves which will depends on z plus ct 1 of which has Ex the other has Ey. And these different direction of Ex and Ey are quite arbitrary, because as I said choosing what is the direction of x and direction of y is something that we do a scientist or as the engineers. The wave does not know what is x and what is y what is z wave arbitrary said that direction in which the wave is moving called z.

But the direction x of the electric field could be anything and in fact, is anything. In general the electric field need along x nor along y it will point along some general direction. But luckily if we had these 2 kinds of solutions then we can write down the full solution. We can say the electric field vector as a function of Z minus ct is going to be equal to some function f of z minus ct along the x direction plus some other function. I am going to call it as g even though last time I was using g plus z plus ct here running out of symbols g of z minus ct along the y direction. Now, this is in general vector field this is the x component and this is y component. If as specify the x and y component I have more or less specified everything I could specify a z component. But if I specify a z component as I will show you later the wave cannot travel in the z direction. So, i by knowing the electric field along x and know the electric field along y I have covered all possible direction of the electric field.

Because the component along x the component along y if I show in the x y diagram and are some components along x some component along y I can vectorially add and I will have component in the general direction. So, this is the way of solving the full problem for any direction of the electric field what so ever. I do not have to solve the infinite number of problems I just solve 2 problems the problem along x and a problem along y. Once I have done that I got the components therefore, I can built up any electric field at all. These components the components along x and y there are called polarization, that is I can find the solution where the electric field is along x magnetic field is along y that will be called an x polarized way. But I could also find the wave where the electric field points along y magnetic field points along negative x direction that will be called y polarized way.

I could actually also find a waves where the electric field pointed either along x nor along y put pointed safe in this direction. In which case the magnetic field need a point along y not along minus x will point in some intermediate direction and it will look like that. But that wave in fact, can be broken up into combination of this wave and this wave, because any general direction of the electric field can be broken up into component along x and a component along y. So, that is a very important idea when we compute this equations because you see this equation is more complex equation in the previous equation. I write the previous equation when you look at that equation was del square Ex del z squared minus mu naught epsilon naught del squared Ex del t square is equal to 0. This part is the same this del squared now is del squared del x squared plus del squared del y squared plus del squared del z squared.

So, it also a little more complicated with the real increase in complication is here this is a vector which means it is Ex x hat plus Ey y hat plus Ez z hat. So, that is why we did lot of algebra and got rid of these particular points this is also a wave equation and this is also a wave equation. But this particular wave equation looks quite nasty we do not quite know what to do it. Whereas, this one looks sufficiently familiar to us after looking at this equation that we feel is almost like the same equation. Well there is one way of looking at this equation that makes it simpler what we can do is we can say I will take this equation and write it as 3 equations it is a vector equation. So, as the way I will write it as I will say unit vector along x of del squared Ex minus mu naught epsilon naught del squared Ex del t squared plus unit vector along y del squared Ey minus mu naught epsilon naught del squared Ey del t square plus unit vector along y del squared Ey minus mu naught epsilon naught del square Ey del t squared plus unit vector along z del squared Ez minus mu naught epsilon naught del squared Ey del t squared whole thing is equal to 0. I just written it out this has an Ex x hat plus Ey y hat plus Ez z hat.

So, does this, so the unit vector along x does not is a constant. So, it is not effected by del del t it is not effected by del squared I can pull it out. So, I have collected all terms with x hat in it all terms with y hat in it all terms with z hat in it. But this is a vector equation and if a vector at any x y z is equal to 0, which means that x component of that vector is 0. The y component of that vector is 0 and the z component of the vector is 0 which means each of this brackets is separately 0. They cannot I mean they cannot do anything in this bracket that could make this brackets 0. I cannot do anything here that would make that 0 each of them must separately go to 0 because that is what it means to say the overall thing is 0. So, that allows me to simplify things at great deal instead of having a very complex vector equation i now have 3 scalar equations.

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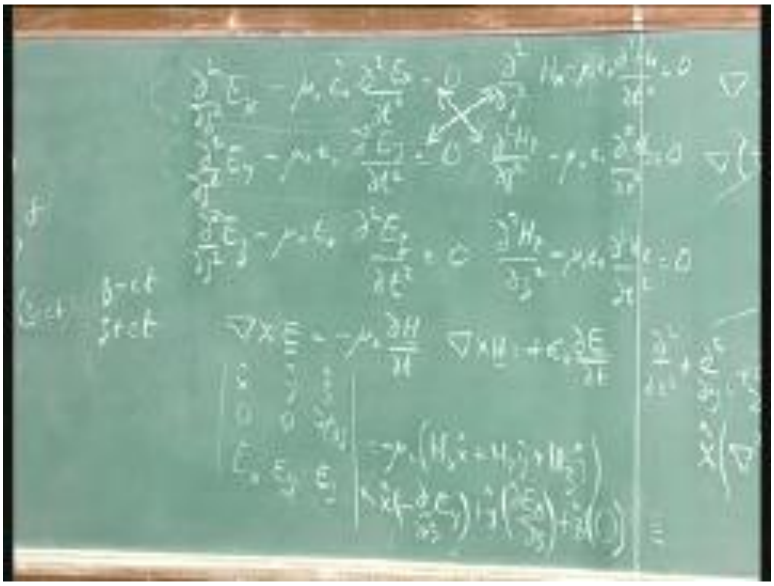

The 3 scalar equations are del squared Ex minus mu naught epsilon naught del squared Ex del t squared equal 0, that came from setting this to 0. Then I have del squared Ey minus mu naught epsilon naught is equal to mu epsilon naught del square Ey by del t squared that came from setting this to 0. And finally, del squared Ez minus mu naught epsilon naught is equal to 0. So, I now have 3 equations in 3 scalar fields and in fact, I can see a very familiar equation here now. If only I get rid of the x and y dependence I have gone back to the 1 d problem. And I have this equation which is now telling me whether y component of the electric field obeys the same equation as the x component and in fact, z component obeys the same equation as well. Now, I am going to simplify, because the whole point of this not to do a very general analysis it was to help us understand very simple.

Supposing the vector field E which is the function of x y z and t is only a function of z and t it does not depend on x and it does not depend on y. Then these equations are going to simplify I will just write them in place. So, I get 3 equations the 3 equations are del squared del z squared of Ex minus mu naught epsilon naught del squared dx del t squared similarly, dy similarly, dz. Now, these 2 equations there are quite familiar with let me write down the corresponding equation for also it will be del squared del z squared of Hx minus mu naught epsilon naught del squared Hx del t square equal to 0 del squared Hy del z square minus mu naught epsilon naught del squared Hy del t squared equal to 0 del square Hz del z squared minus mu naught epsilon naught del square Hz del t squared equals 0.

So, I have 6 equations and I would like to know what kind of solutions I can get out of this 6 equations clearly from what we have understood of 1 d problems. We know that Ex as a preference for Hy and we can guess that dy as a preference for Hx which makes as question mark at what this 2 are doing. But we have to get this relation out now out of paradise law. So, we have curl of E is equal to minus mu naught del H del t. Now, curl of E is x hat y hat z hat 0 0 del del z, because I do not have del del z del del y Ex Ey Ez which is equal to mu naught times Hx x hat plus Hy y hat plus Hz z hat. Now, if you look at this curl you can see that there are 2 zeros here. So, what; that means, is if I take the x component let me write down what this is equal to it is x component times del del y is missing. So, I do not have this term minus del del z of Ey, the y component has del del z of Ex minus del del x is missing. So, this does not have and the z components requires del del x or del del y both of which are missing 0.

So, there is no Hz the curl of E is links to Hx and Hy and how is it link Hx is connected to del del z of Ey Hy is connected to del del z of Ex. Now, we can write the same equation for amperes law except for a change in sign and this epsilon naught this is the identical equation. So, it is going to tell us that Ex is related to del del z of Hy, Ey is related to del del x del del z of Hx. So, once again we find that there is no z component. So, we think of these waves as being sources for each other. There is no source for the z component even though we have this equation here we have no way of connecting up this z components to anything else. This kinds of waves can exist they cannot exist in vacuum they can exist provided we have a media and we may look at that later on.

And those kinds of waves are what are called compressive waves and they are waves where the field is along the direction in which it is moving. Because if you have variation in only z it means the solution are going to be z minus ct and z plus ct. The wave is moving in z and the fields points along z those kinds of solutions are very rare they cannot exist in vacuum. In vacuum what you can have are Ex related to Hy and you can have Ey related to Hx you can see it coming out of these relation directly. So, it is it just coming out of the vector notation and vector relationship that you will only have solutions in this 2 combinations. That is why when I talked about polarization I said Ex will cause Hy Ey will cause minus Hx. I will go little bit more into this next lecture, but please revise and verify for yourself with these equations are the way I put in them.