Electro Magnetic Field
Dr. Harishankar Ramachandran
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture - 32
The Wave Equation

In the last lecture, we finally completed Maxwell's is 4 equations, we are introduce the last correction to this equations, namely the displacement vector term after now, we had been working in the area of statics and motional emf. And that in a certain sense covers all of the theory of machines transformers generators, but it misses a very, very important part of electromagnetic theory. And it was this part that Maxwell identified, when he completed this equation. So, the first part of the lectures, I am going to just review, because these kind of ideas. It is good to hear them twice, whatever you miss the first time you get the second times around. Then I will go on and look a little bit more into what we call as the wave equation.

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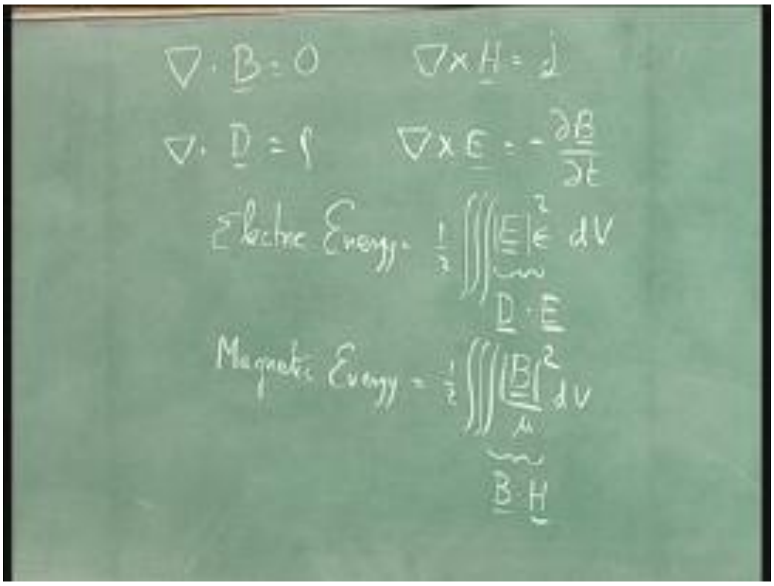

So, let me review, what we had up to the beginning of last lecture, we know that the magnetic field satisfied. Divergent B is equal to 0 and we also had that curl of H is equal to j the current density, we know that divergent of the displacement vector was equal to charge density. And we knew that curl of the electric field is equal to minus del B by del t. So, this was due to the fact that B was only due to the currents and there is no such thing as a magnetic charge. This was from guesses law, this came from Amperes law and I never really proved it, but I saw by motivated it by showing that it have little loops curl of H is 0 except near the loop.

And this is paradise law and using these 4 laws we also showed that you can prove that electric energy is equal to one half volume integral. Epsilon is a epsilon naught or epsilon into e square dv actually, more accurately this is displacement vector dot electric vector and since displacement vector is epsilon E. That is where this comes from; you also have shown the last few lectures magnetic energy is equal to one half volume integral B squared over mu dv. And this came from B dot H actually, it came from E dot del D del t and H dot del B del t. So, these concepts we had,, so what was the problem? Well, there were 3 problems actually. And you all have to do with this equation, a first problem was, there if you looked at where B came from

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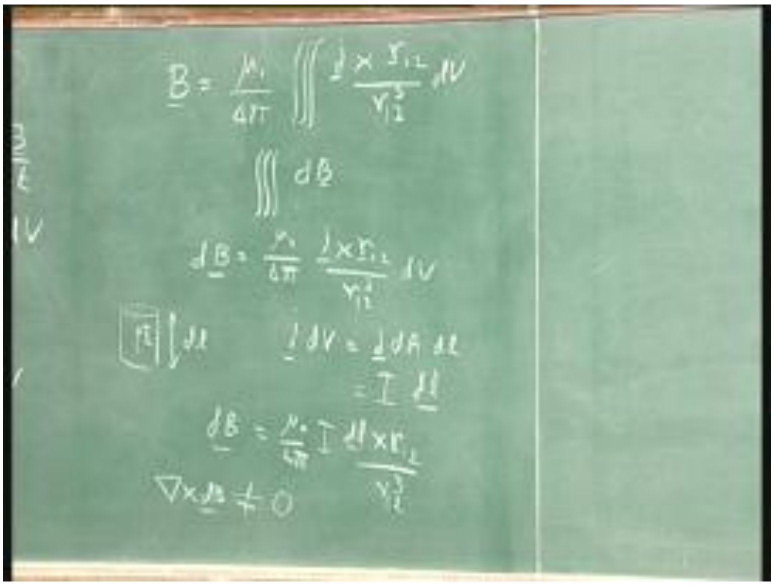

We know that the magnetic field is equal to mu not over 4 pi volume integral j cross r 1 2 over r 1 2 cube dv. So, what you could do? You could consider this as some kind of a integral dB, that is microscopic amounts of magnetic fields due to these pieces. So, then that would say dB is equal to mu not over by 4 pi j cross r 1 2 over r 1 2 cube, dv. And we can we are basically, taking a reverse path you can now, say j dv is nothing but j times dA times d l. And then we can say that is nothing but the current times dl. So, this tell us, where the magnetic field is built up out of tiny pieces of magnetic field, which are mu not over by 4 pi I dl cross r 1 2 over r 1 2 cubed, which is where we started actually, because if you remember right. This is by Biverse Edward law well we have a little bit of current in a wire. This is the little bit of magnetic field, if reduces that is only 1 problem with this supposing; we assumed that we had a wire whose length was dl and have it current I then this is, in fact the magnetic field, you will get for it. So, we can take the curl of the magnetic field, we can see curl of db and if you take the curl of this.

In fact, you find its not 0 that is a big problem. This is the very building block out of which all of magnetic statics come and magnetic statics is proven that Amperes law, wholes and yet a very building block of magnetic statics says curl of d is naught equal to 0. So, how can, we have an equation for all of arbitrary fields, yet the building block field does not satisfy, this same equation does not make sense. I mean, after all what prevents me having little piece of wire carrying a little current a current answer to that is very, obvious; answer to that is the current represents moving charge.

So, if the study current here, it means there is charge building up at this top of this wire and this charge duplicating at the bottom of the wire. So, as we saw last time, this little piece of wire implies at time dependent electric field the time dependent came. Because Q is the function of time that constant current Q is linear in time sturdily building up in time which means, the di pole field is due to a stronger. And stronger di pole which means, the electric field is dependent on time and if you take the derivative in time of the electric, field it is independent of time.

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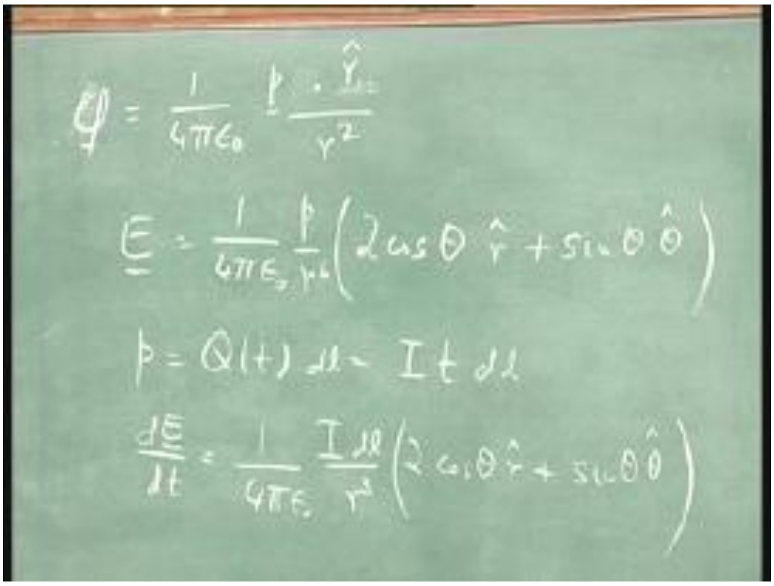

Because the electric field E is equal to 1 over 4 pi epsilon not p cross I guess P dot, I can never remember the formula, we can check for me P dot r 1 2 over r cubed. So, you get this is the potential is p dot r over r squared the electric fied will, become 1 over 4 pi epsilon naught p over r cubed 10 twice cos theta of r hat the sine theta hat I am not sure exactly, how to put it as a vector operator. So, I will do it, this way now, this p equal to the charge as a function of time times the distance dl and this charge is nothing but the current times time dl. So, this electric field is the function of time, but if I take it the time derivative of the electric field where I should put partial derivative, but I am being sloppy here right. Now, 1 over 4 pi epsilon not I dl over r cube times twice cos theta r hat plus sin theta theta hat the details do not matters. What matters is if I take the time derivative of this dipole electric field it does not depend on time. It is a constant it, depends on space of course, but it does not depends on time.

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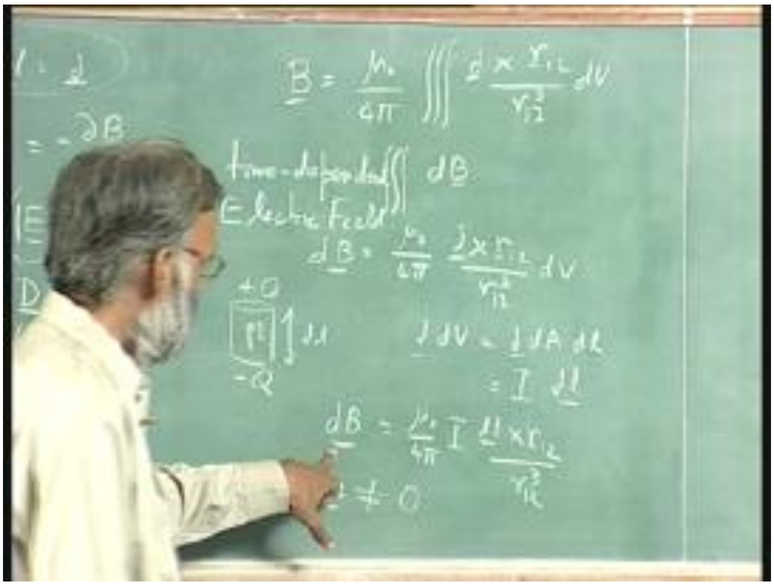

Now, if you look at this expression, this is also independent of time, but unfortunately the curl of B is not equal to 0. I am not going to prove it, because there are better way of proving it, this B s dE dt is exactly the correction. You need to add to this piece, new times, that is required mu epsilon times that to correct this B s and make the curl of this 0.

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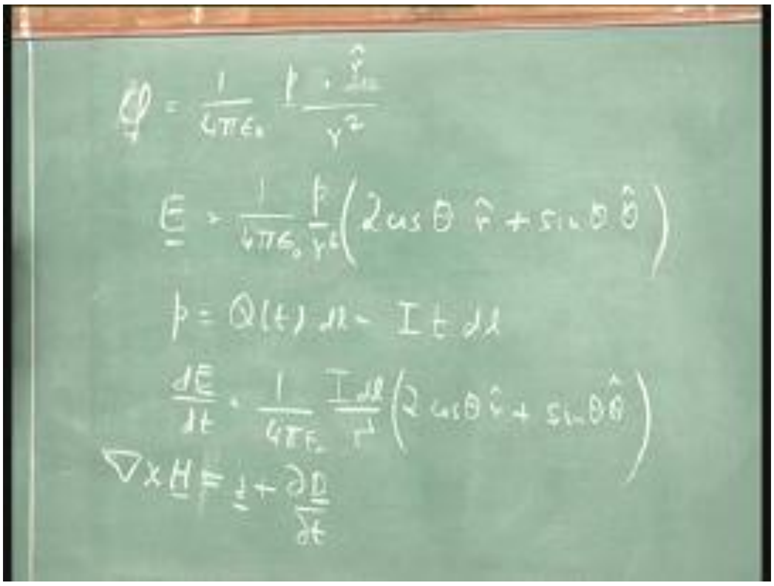

So, it turns out that really this equation. Curl of H equals j is wrong is not equal to 0 it is actually, equal to something more. And if you ask, what is that requires to make curl of the little piece of B equal to 0? You find that, it is equal to j plus del D del t I mean proved it. But I have motivated in that sense with a difference between a loop carrying current I and a little piece of current carrying current I is that charge with them in this case. We already proved curl of d is 0 away, from the loop this case curl of V is not 0. And what we find is? If you add dE dt it makes, curl of B 0 since there is no dE dt in this case there is dE dt in this case that solve both problems. And, so you suggest, that it should have curl of edge equal to j plus del d del t now, there are 2 other ways of looking at this they are both given in a text book whereas, this approach is not given. Whereas, I think this is the fundamental reason ultimately after all bias award law building blocks and the bias award law does not obeys ambient law. You have to worry the second approach, that we talked about was you take this equation and take the divergence.

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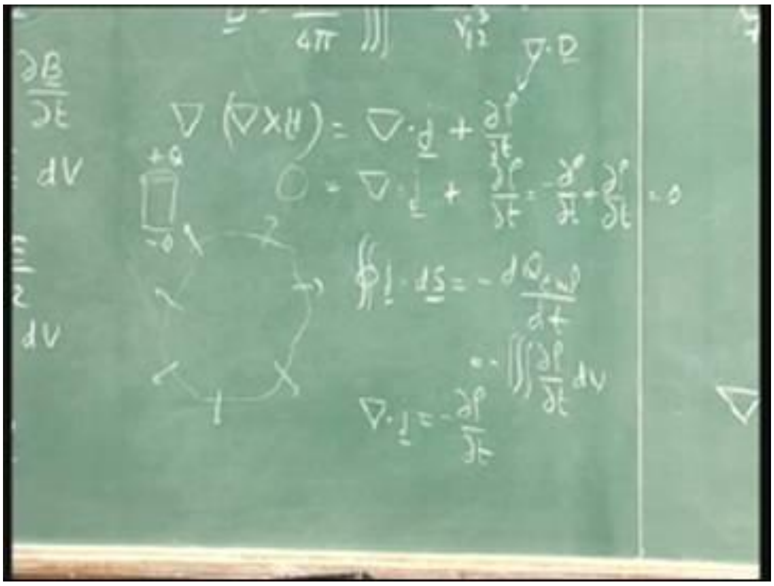

Shall I got divergence of curl of edge equals divergence of j, but the left hand side is 0 divergence of any curve is 0; 0 is equal to divergence of j. But you already, know that for any volume that current density leaving at each point integrated over the surface is that current leaving total current leaving. So, surface integral j dot ds the total current leaving must be equal to minus the rate at which the current is charge inside is dropping minus dQ dt, but I can write this as a volume integral. That is the charge decreasing means, charge density at every point inside must be decreasing and this is sum of the decreases in the charge densities. That is the total decreasing charge apply, guesses law that gives you divergence at j is equal to minus del row del t. So, I can apply that here.

So, Amperes law is telling as something is telling as that charge density is not changing that is not true. I mean Maxwell's equations must apply, everywhere including places, where I am charging up or charging down an object. That is in the presence of our capacitors Maxwell's equation must be correct. And in fact, it must therefore, hold for the case of bias award law, where I am having plus Q and minus Q building up clearly here del row del t is not 0 yet Amperes law requires del row del t to be 0. So, I need a corrective term, what corrective term can I have it must be. But I must add a B s it goes like, del row del t in which case my answer would be divergence j plus del row del t it would become, minus del row del t plus del row del t to be 0. So, 0 equal to 0. So, I need a term that's it look like plus del row del t row is nothing but divergence of D that is one of the Maxwell's equations. So, immediately tells us what we need to do as a text book point it is not a proof, but is being proven.

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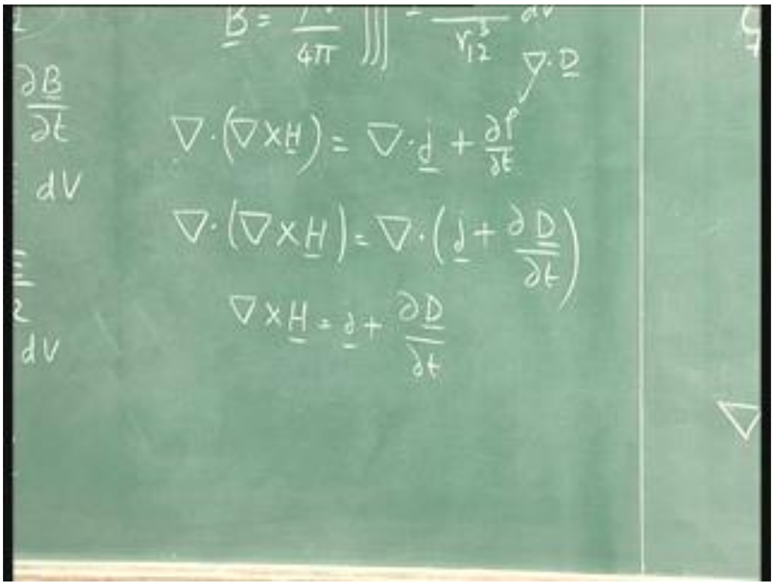

So, we can just write this, whole equation as the divergence of curl of H equals divergence of j plus del D del t which suggest that curl of H is equal to j plus del D del t. And this is the new Amperes law that includes an electric field term the third way of looking at it. I want to repeat that was to look at stokes theorem and look at what stokes theorem tells us, when its cuts a wire and when it goes between the plates of the capacitor. That is also very well covered in most textbooks, so that gave us a correction to Amperes law which say that is equal to j plus del d del t.

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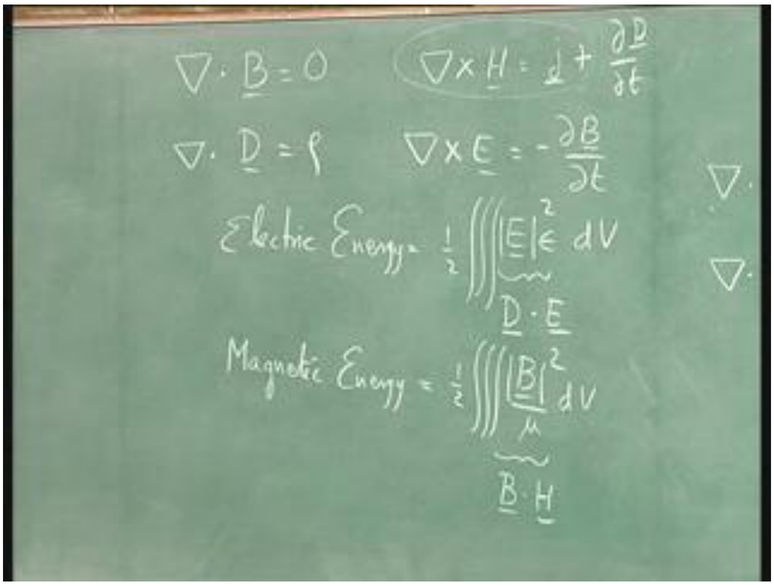

All right, so now if you got this equation we can now look for solutions of it. And what I did last time was to look for a particular kind of solution.

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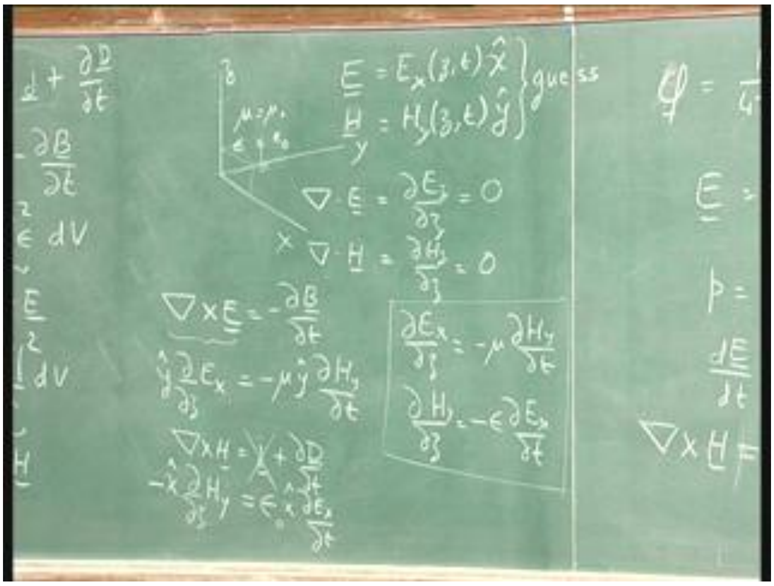

I say it supposing I have x y z and supposing, I was looking for a particular kind of electric and magnetic field. I was looking for an electric field E, which is equal to E x of z and t. Then depend on x and y along the x direction a magnetic field I will call it H, which was H y which is also a function of z and t along the y direction. Now, I can apply the various laws and look at it first apply the divergence laws, what do you get divergence of E since only z is the dependence. There is no derivative with respect to x there is no derivative with respect to y is equal to del Ez del z, but z is 0 divergence of H del Hz del z equal to 0, I am going to look in a case where it is vacuums. So, mu is equals mu not epsilon equals to epsilon naught. So, really when we talk about an electric and E and H I am really, talking about d and B.

Now, let us go to the curl equations curl of E is equal to minus del B del t. Now, if you look at these from, which I just guess this is the guess I do not know if a solution x is of this type, but I am going to look for it in my equations. If this is the form for electric field the electric field is along x and it varies along y I mean along z. So, I want a curl I must draw my strokes curve in the x z plane, because one of the arms of the strokes curve must be along x. And the other arm must be in the direction, in which the electric field is changing and therefore, the normal to that strokes curve stroke surface points along y. So, if I take the curl of E for this particular problem, what will I get I will get y hat del del z of Ex and of course, there will be other terms like del z which is 0 and then the term involving x has ten z hat. But you can easily show that, this is the only term that present, because if you draw the strokes surface any other way we will not get any contributions.

So, this is along y and magnetic field is any way along y, its equal to minus mu along the y direction del H y del t. So, this gives as one equations, which says del Ex del z is equal to minus mu del Hy del t the second equation, we had was curl of H is equal to j plus del D del t. I am working in vacuum mu not epsilon naught no materials no current. So, it looks similar to Faradays law, in fact, the options of current the generalized Amperes law and faradays law look very similar except for the sign. So, I apply that same argument, if I have magnetic field in a y direction variation in z direction. So, my strokes curve must be in a yz plane, because one arm must point in the direction of the field otherwise, you do not get a loop integral H dot dl. And the other arm must point in the direction, which its vary otherwise the 2 arm that point along H will cancel out and if you look at this 3 points in the x direction.

So, we expect an x component out of this and the electric field is any way in the x component. So, what do we get? We get x hat del del z of Hy with the minus sign thus, because if we take the curl x hat del del y of Hz minus del del z of Hy Hz is 0. So, the only term present is minus x hat of del del z of Hy is equal to epsilon not x hat del Ex del

t. So, again the x hats are common, you can remove them and you get del Hy del z is equal to minus epsilon del Ex del t is a are very important equations come in many areas especially they come in transmission line theory. You will see similar equations in power systems, when we do transmission line equations. So, I am going to take this 2 equations and combine them and when I combine them all I do is I take the z derivative of this.

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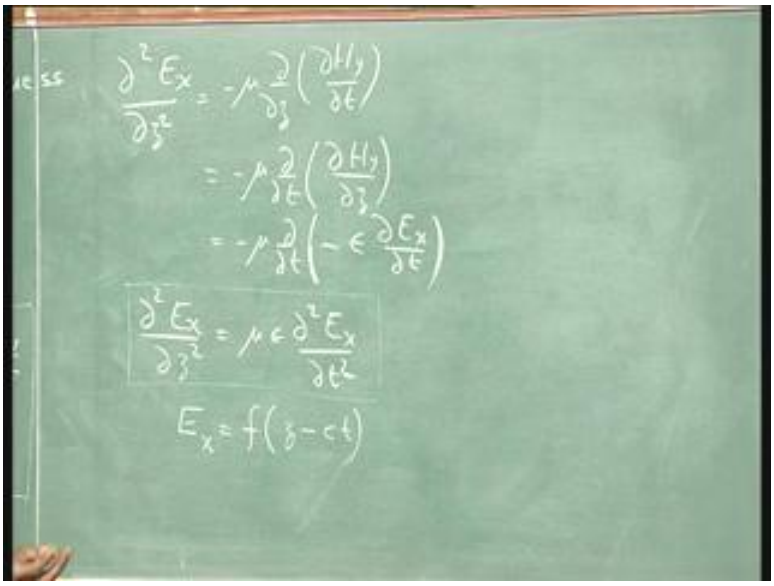

So, I get del square Ex del z square, which will be minus mu del del z of del Hy del t lets taking this equations. And taking the z derivative of it, but because of second partial derivatives, it does not matter, which order you do it in this will be equal to minus mu del del t of del Hy del z, but then I apply the second equation. Because del Hy del Z have an equation for that, so let me use that, it give me minus mu del del t of minus epsilon del Ex del t. So, that gives me, my final equation the minus sign cancel out mu epsilon again pull out that gives me del squared Ex del z squared is equal to mu epsilon del squared Ex del t squared it is a single equation in Ex.

But all of it came from a guess it came from the guess, that my field varied in the direction 90 degrees to the direction in which it is pointing both case and if varied in the common direction z. So, it does not mean just, because I got an equation there are solutions. So, we have to try and solve this problem again. I am going to guess, I can see that I basically want the same behavior with respect to z where I have with respect to t otherwise the second derivatives cannot be proportional to each other. So, I can think a one way, which is happen which is I can say supposing Ex is equal to some function of z minus ct. So, in that case if I take a derivative with respect to z or if I take a derivative with respect to c. So, I am taking basically a derivative with respect to f, so the second partial derivative will be the same function. So, let us try it out, well I will take this function and take its derivative with respect to z.

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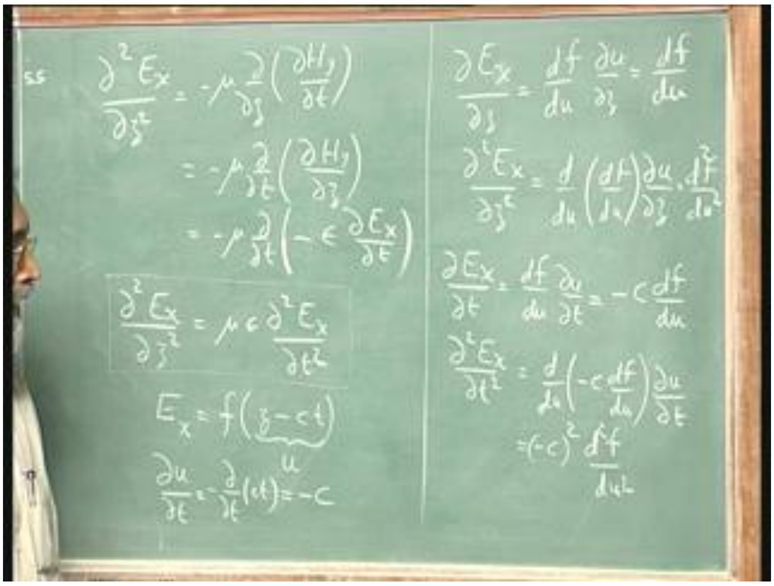

So, I get del Ez del Ex with respect to del z is equal to let me call this z minus ct as u. So, it will become, df du del u del z it is that chain rule, if I have a function, which it depends on some variable and that variable depends on wonder. I am differentiating with respect to first I can take the derivative with respect to u and then take the derivative of u with respect to z. So, is that it is just the chain rule, but del u and del z is 1,, so is equal to df du if I take the second derivative del squared Ex del z square is equal to d du of df du hence del u del z. Because del squared del z squared is del del z of this quantity again del u del z is 1,, so its equal to d squared f du square.

Now, what about this respective time? Well, I get del Ex with respect to time is again, df du del u del t what is del u del t u is z minus ct. So, del u del t is equal to del del t of z minus ct z does not depends on t. So, it does not give you anything, but it minus del del t of ct which is minus C C is a constant I assume here. So, del u del t is minus C and get minus C df du, let me take the second derivative del square Ex del t square is equal to d du of minus C df du del u del t del u del t is once again minus C this minus C does not depend on u. So, it can be pulled, so I get minus C squared d squared f du squares. So, I can take all this and put it in to this equation going to do it on this screen itself. But you can see all the equations in 1 go del squared Ex del z square comes from here.

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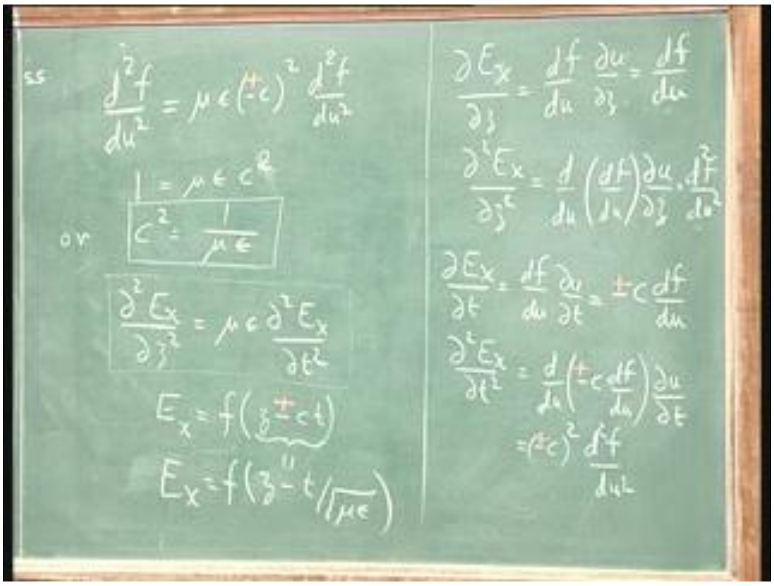

It is d squared f du squared is equal to mu epsilon del squared Ex del t square comes, from here. It is minus C whole square d squared f du squared well d squared f du square cancels out same on both sides what I get? 1 is equal to minus C squared is just C squared mu epsilon C squared or C square is equal to 1 over mu E another way of stating the same thing is going back here. I can write this, as Ex is equal to f of z minus t over square root of mu epsilon. I had made no assumption that what so ever about, what f is you look at, it f can be anything, f can be a triangle or f can be a square, f can be a trigonometric function, f can be a exponential f can be anything. All I have assume this whatever, the function f it depends on z and t through this combination z minus ct. And I guessed that, because I could see if I wanted z derivative proportional to d derivatives. So, that was to happen I should have the dependence through a combination is not just this that, we can do we could have actually chosen to make this minus plus.

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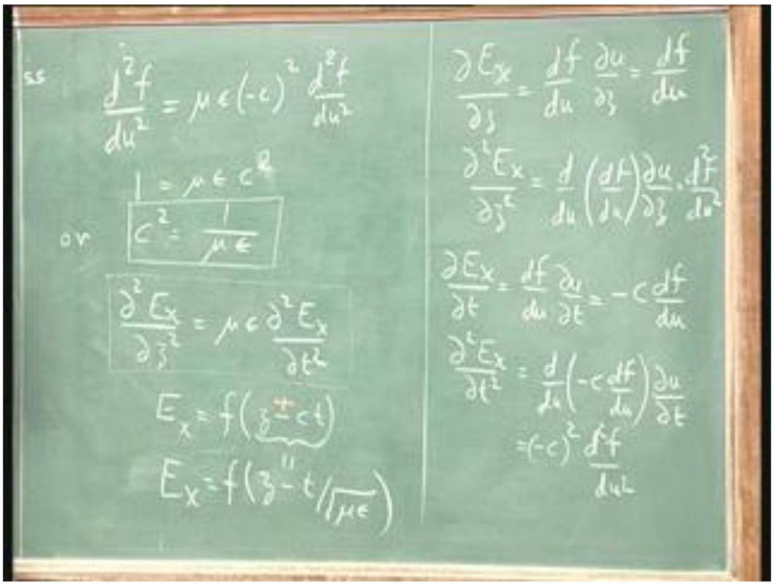

If you made this minus plus, what would change this would still be true del u del z is still one this would still be true, but here the minus C becomes plus C.

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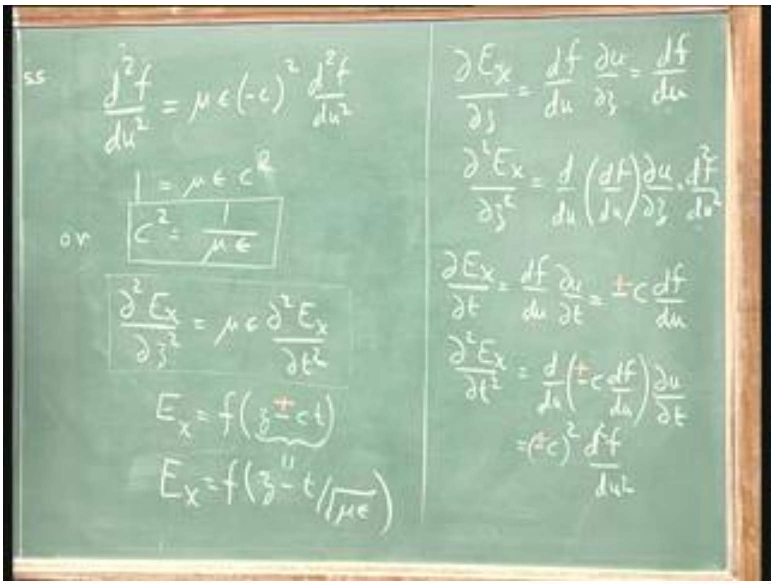

And here minus C becomes plus C, again therefore this become plus C here.

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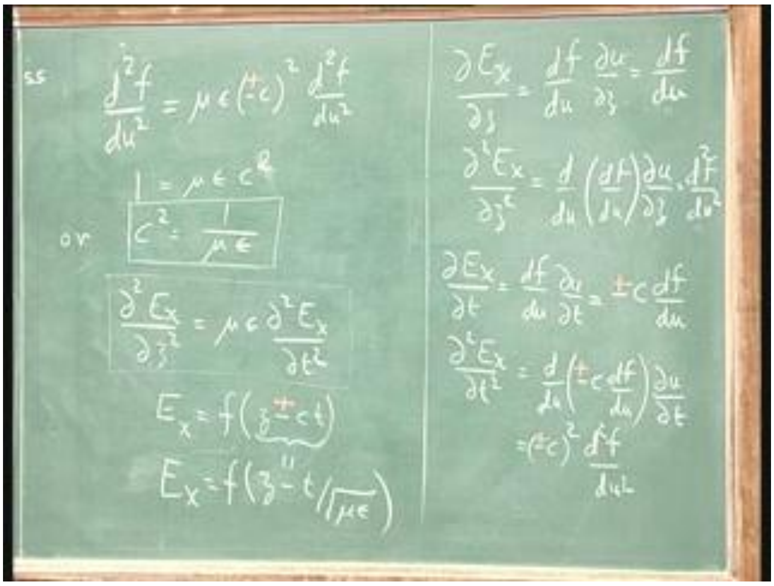

So, if I look here I get plus C which gives me the same answer the same C, which means that both this solutions an arbitrary function of z minus t over square root of mu epsilon. And an arbitrary function of z plus t over mu epsilon of both solutions of this equation, which means, that this guess that we made let look for the solution of this type is a good guess is that guess of given answer. So, let me summaries, what you got up to right now we started where the generalized Amperes law and we derive the wave equation.

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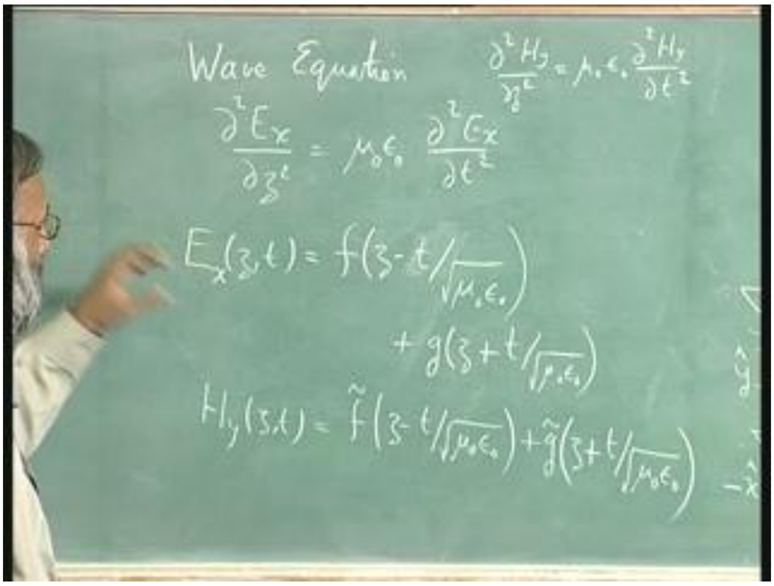

Actually, we derived it in 1 dimension and the wave equation looks like del squared Ex del z squared is equal to mu epsilon del squared Ex del t squared. And for this problem, I assume vacuum having done this, I assumed that the dependence of z and t, where connecting. And I found that any solution of this following type, Ex as the function of zt is equal to some f of z minus ct or I should say, ct I should say t divided by square root of mu naught epsilon naught plus. Any other function g function of z plus t over square root of mu not epsilon not is the solution of this equation now, I could have started with this same equations. And I could have gone the reverse way instead of eliminating H through this equation, I could have eliminated E through this equation in which case, I leave that to you for a exercise. You will find that, this wave equation becomes, del squared Hy del z squared is equal to mu naught epsilon naught del square Hy del t square.

It is a same equation, whether you work with E or you work with H you get solve the same equation. And you would find Hy of zt is equal to some, I put a fiddle on top of it minus t over square root of mu naught epsilon naught plus some other g z plus t over square root of mu not epsilon naught. So, both have the same structure, Ex depends on some function of z minus t over mu epsilon square root plus an arbitrary other function of x plus t over mu naught epsilon naught. Whereas, H as the same form, but possible different functions, now the different function are not really, the different functions, because they are connected out. We have Paradise law and Amperes law, which means where I if I take the z derivative of tx it connects over the time derivative of Hy. Let us try that out, we have to apply this equation or we like this equation. So, I apply that equation, what will I get? I will have to take the z derivative of Ex. So, I will get I will have to define my symbols, let me call us z minus t over mu naught epsilon naught as u z plus t over mu naught epsilon naught as v.

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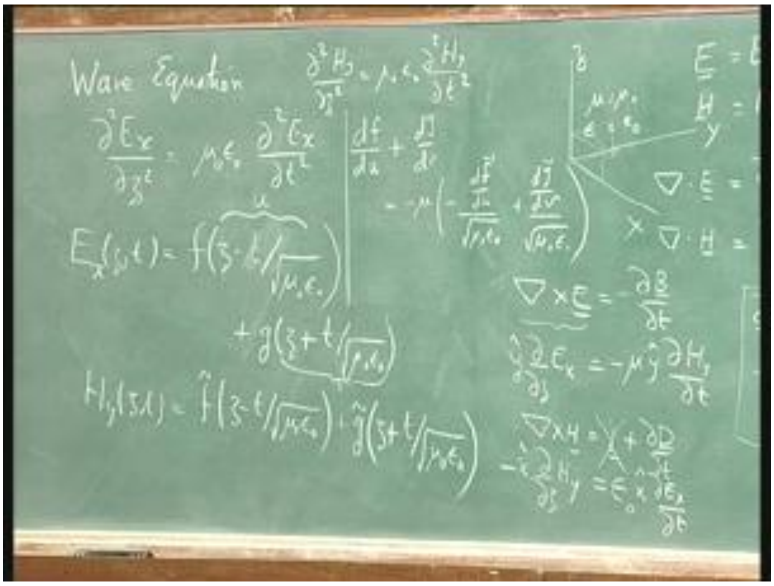

So, I will get df du times 1 du dz is 1 plus dg dv derivative with respect to z is again, 1 that is the left hand side dx del z is equal to minus mu. Now, the time derivative of this which is minus df twiddle by du divided by square root of mu naught epsilon naught plus dg fiddle dud v divided by square root of mu not epsilon naught. Similarly, if you use this equation, you will get another pair of equations, so clearly df f and g are related to f delta and g 2. We will come back later to exactly, what the relation is, but solving any 1 of this 2 amongst the solving both. So, now, look at what kind of solution you are forming? For this, I am going to assume this form for electric field.

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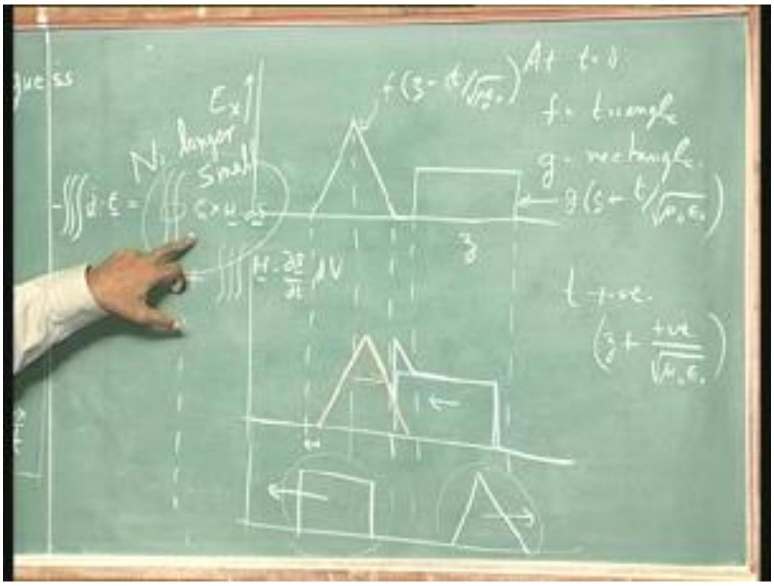

I am going to assume that this is Ex versus z at t is equal to 0,, so t equal 0, I am going to assume that f is a triangle and g is a rectangle. So, if I draw my f and g yet say, my f is a triangle, which is f z minus t over square root mu epsilon naught. And it and my rectangle is this is g of z plus t over square root mu not epsilon naught. So, at t is equal to 0, I have some electric field now, what is this electric field going to do as time changes supposing t is now, equal to a larger value. What is going to happen is this positive value divided by a constants, means that z has to be larger. In order to equal the same value of u, that is u is equal to z minus t over square root mu naught epsilon naught when t is equal to 0 u is z t is equal to 1u is z minus 1 over square root of mu naught epsilon naught.

So, I need a larger z in order to get this, same u what will happen is that for the larger for positive time right. Draw the same graph underneath, I am drawing dotted line to show where the triangle was and where the rectangle was, my triangle is going to move, it is going to be starting and ending later. So, you can see, there is a shift, the shift is due to the fact due to time positive time I require a larger value of z to reach the same value of z minus t over mu naught epsilon naught for positive time. This triangle is moving in this direction what happens? If I look at the rectangle, now at positive time z plus something positive divided by square root of mu naught epsilon naught is the argument of g. So, to reach the same value of this quantity, I need smaller z, because this is already present.

Therefore, my rectangle is going to start and end at smaller z the rectangle is moving left. Now, the final electric field, I have is the sum of the tool is that, what is going to happen? I have no electric field up to this point, I have electric field due to this triangle, this electric field drops again, due to the triangle. Then suddenly, this rectangle, this is my net electric fields a electric field is due to the sum of both triangle and rectangle as the rectangle moves in the left, triangle moves right, going to meet each other they are going to pass right through each other. And you look at some later time this; same picture will now, become a triangle moving right and a rectangle moving left. So, it starts out with the rectangle moving out right, triangle on the left the 2 of them merge.

And then the rectangle keep moving to the left and the triangle keep moving to the right. So, a very strange equation, when you think about it is a equation that preserves, object. And just let us them keep going whatever, the shape I have taken deliberately 2 shape which adjust arbitrary 1 triangle. And one rectangle, those shapes are preserved thus another thing was noticing, it is that after long enough time these thing are going to reach infinity. You are going to reach; however, for a surface, they are going to reach it and the amount of energy that reaches the surface amount of signals reaches the surface is equal to the initial signal. So, this means, that of previous calculations, when we calculated magnetic field energy we did a volume integral j dot E. And then we did integral in time, we are looking at this is the power instantaneous power dissipated.

Therefore, instantaneous power introduced I am say; that this thing is equal to there is a surface integral E cross H dot ds. And then there was a plus the volume integral of H dot del p del t dv and we ignored this piece the E cross H piece. And we kept this that was correct, because coulomb law and bias Evert law told as E cross H is very small for now, you look at this kind of situation. Electric field is along x magnetic field is along y E cross H is along z. And this object is moving in Z which means, if I put a surface here then this object reaches here, E cross H is going to be non zero. And is going to be large is not going to be small therefore, this term is no longer small. In fact, this is the term now, become is very important and it is called the poyrting flues. We will come back to it sense it is a central concept in wave theory, but the lesson to be learned that several lessons.

One is that we have a general solution possible, this is the linear equation there, where 2 kinds of solutions, I could see z minus ct and z plus ct and I can use any linear combination of them. Because the equation was linear and I choose it some arbitrary function f of 1 solution plus the arbitrary function of solution g of the other solution when I graph the solution, I find f and g just move they do not change shape. Because f is only a function of this combination whatever, value of t there is just a shift. So, this full p s look like some z naught at any given time, it is a z naught, but the shape is the same. This is a shifted shape at different time in the case of the minus sign term. The shift is to the right in the case of the plus sign the shift is to the right, so these are moving electric and magnetic objects and because they are moving electrical and magnetic object.

They are called waves will come back to more familiar waves, very shortly, but these are also waves is nothing un wave like about it. Now, the other thing that should be notice down here is we solve this problem and we got this factor. So, we can ask, what is the significant to this factor? Well, for that go back right here. And say supposing, I that t equals 1 at t equals 0 this object z equal u naught at t is equal 1 where is the object well, you can take it for either of this equation. I am going to take negative sign case, we have that z minus 1 over square root mu naught epsilon naught is equal to u naught, I want to solve where the u not is at t is equal to 0, it was z is equal to 0 naught t is equal to 1.

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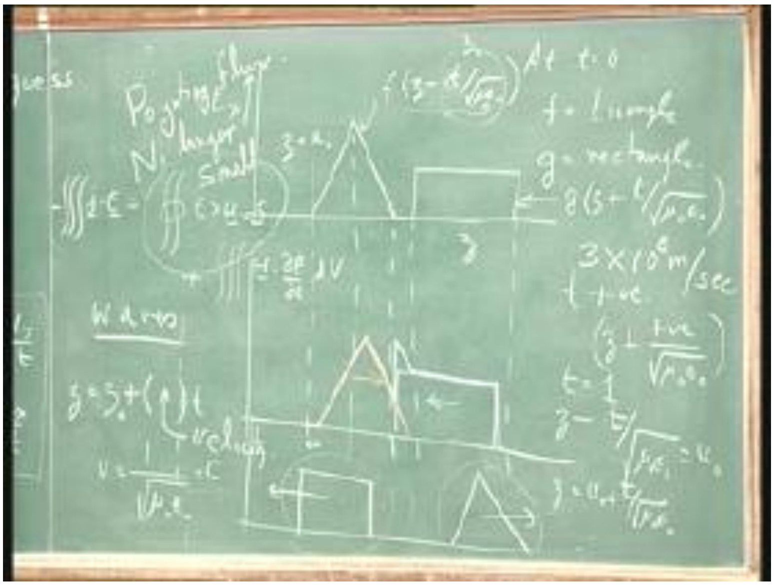

It is z minus 1 over square root of mu not epsilon naught equal to u naught. So, what value of z is that z is equal to u naught plus 1 over square root of mu naught epsilon naught more generally. If I left tb anything, this 1 is t and z is therefore, equal to u naught plus t over this now, you have studied kinematics. So, you know that if you get any solutions, that says z is equal to z not plus some constant times t the constant is velocity. What is the velocity in this case, this velocity v is equal to 1 over square root of mu naught epsilon naught, when you put in the numbers it is naught is no accident. Because it is design, that way you get this is equal to the speed of light in vacuum that is its equal to 3 into 10 to the 8 meter by second.

So, you just coming out of the equation the equation have built in them, that any disturbance, that you have can be solve for. And you will get one part of the disturbance moving forward with the speed of the light. One part of the disturbance moving backward with 0 planes or if you like one moving right the other moving left, this velocity is nothing. But speed applied ones; Maxwell’s saw this, well I will not know, exactly the history whether it is well known to Maxwell’s. But once we say, this feature here we can realize something, they realize that the entire theory of light the entire theory of updates is just part of Maxwell's equations.

Because this is really, talking about the propagation of radiation propagation of light so tremendous kind of simplification. Because you do not want to have 2 separate theories One theory to explain light, one theory to explain electricity and magnetism same theory explains, transformers same theory explain inductors, machines, generators. It also explains, cell phone wireless medium wave transmission, machine satellite transmission. It also explains radiation from the sun, it also explains all kinds of variations and it is a, it is a enormous simplification. That resulted in physics, once this additional term to Amperes law was at as I said only one thing, I want to do, now which is supposing instead of talking about arbitrary functions f and g supposing.

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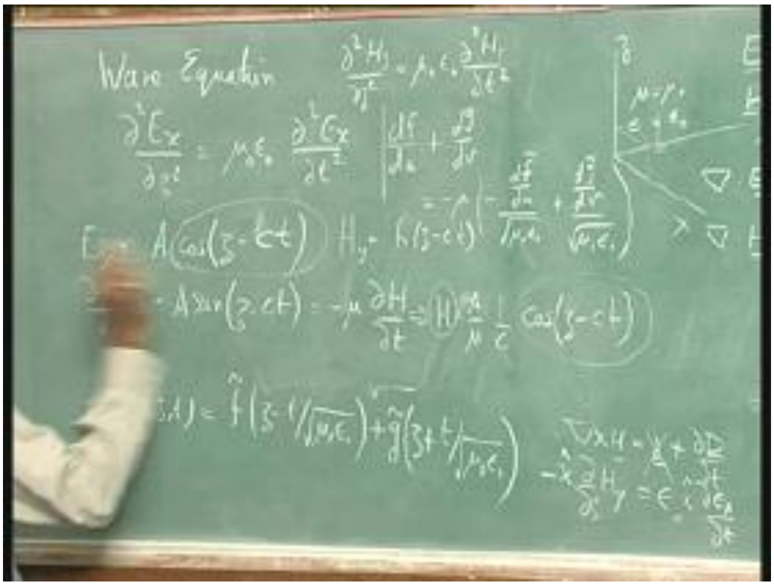

I had assume Ex is equal to some A cos of Z minus ct call it as ct, if I assume this form, then what do I get? Well, I can apply this equation. And what I get is del E del z is equal to minus A sin z minus ct and this is supposed to be equal to minus mu del H del t. Now, I am going to assume Hy is also a function of z minus ct, this is the function of z minus ct plus, that one will also d, because we know disturbance of moving as objects. So, that the electric field must carry the magnetic field with it magnetic field must carry the electric field with it. But what is the shape of Hy? Well, that is easy I just integrate with respect to time may be integrate with respect to time, what you get? It implies, H is equal to A over mu times the integral of sin, but this sign of u naught sign of t. You have to get a 1 over c and then cos of z minus ct this probable a minus sigh here. Because of z minus sign no, because when I do the integration the minus sign goes away. So, you get that the magnetic field also look like cos of z minus ct. And the electric look like cos of z minus ct and this just allow us that, tremendous simplification and it allows as to, do much simpler analysis of waves. That is why for the rest of this course, we will be look at what are monochromatic wave waves, which look like sin and cosine I have continue that next lecture.