Electro Magnetic Field
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Lecture - 31
Generalised Ampere's Law

Good morning. This lecture we changed gears and we leave behind both electrostatics and magneto statics. And we are going to discuss completely new phenomena it is the phenomenon that Maxwell introduced in the second half of the 19 century. And it is a extremely important idea he joined together different fields of physics and made them into single subject.

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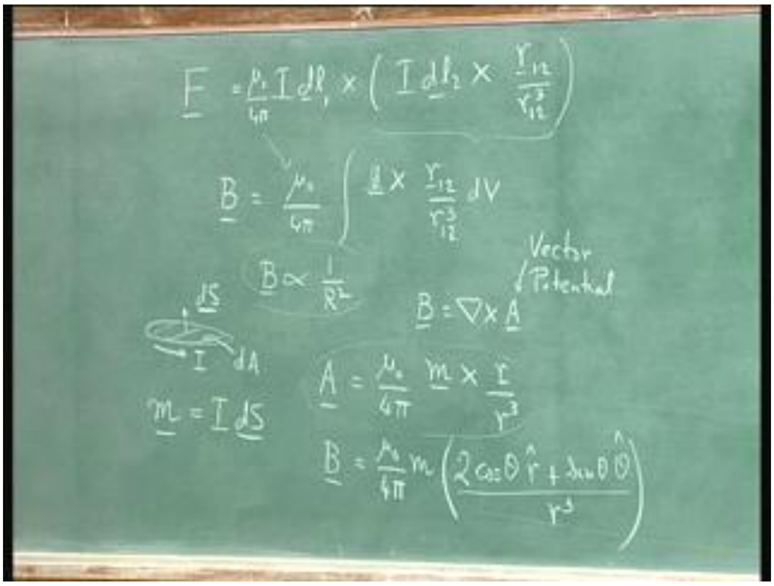

To bring that topic into focus I am going to review a little bit first of all you remember back. We said that magnetic field was got from a force equation observed remind you which said that is equal to I dl 1 cross I dl 2 cross r 1 2 over r 1 2 cube. And this is mu naught over 4 pi and this bracket was identified just like we used we identified the corresponding bracket equation in electrostatics. We identified this with the magnetic field and so we defined something called magnetic field which is mu naught over 4 pi constant came in integral over all space. This I dl times d A became j cross r 1 2 over r 1 2 cube d V just to remind you. The j d V is really I dl which is j the vector under d l I am putting associating with j.

So, d l times d A, because current is current density times area and this d l d A by d V, so that is why j d V came from. And, so I defined a magnetic field integrated over all space which was the current cross current density, cross r over r cube. So, magnetic field is proportional to 1 over r square as you go far away from your currents magnetic field drops as 1 over r square it also has an inverse square law. Some few lectures later we looked at the problem of field due to a loop the loop has is carrying a current I and it has a area d A. And, because of the right hand rule the area is given a direction I will call it d S. So, if I have a loop of current and if it points in the direction d S I defined a quantity called the magnetic moment m which is I d S.

And in terms of this m you could find out what the magnetic field was first of all we had also found out that B is equal to curl of something. That something is called the vector potential in terms of this vector potential I could work out B if I knew A I just take the curl. And I get B and for this small loop if you work out what the vector potential is you find that is equal to mu naught over 4 pi m cross r over r cube. This is surprising or it is surprising if you do not think about it, because here magnetic field is going like one over r square here vector potential itself is going like one over r square. If you take this vector potential and find the magnetic field due to it you get B is equal to mu naught 4 pi times. The magnitude of m times twice cos theta along r plus sin theta along theta divided by r cube now just put things into context let us look at what electrostatics has to say.

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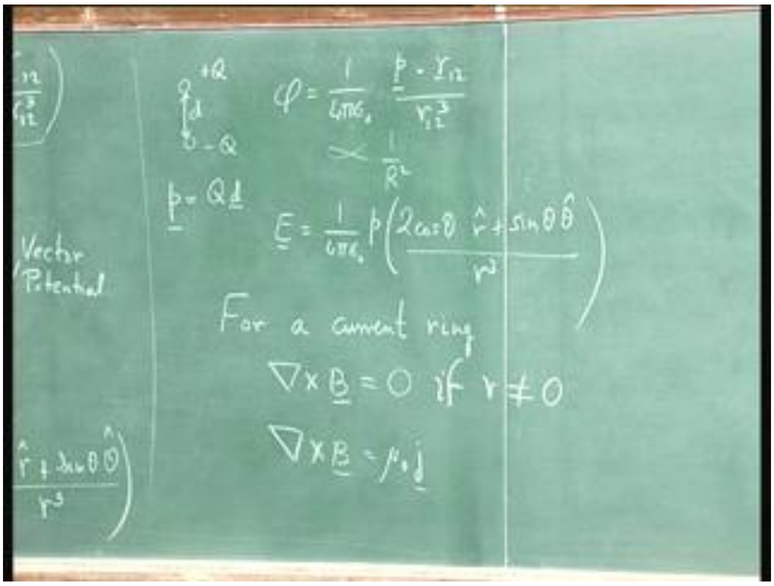

If you have a charge plus Q and a charge minus Q which are a distance d apart I can define something called the electric dipole p which is equal to Q times d. That is the charge times the distance between them the corresponding analogy is I have defined a magnetic moment which is the current times. The area both of these are vectors because they have a direction now, in terms of this p if you work out what the potential is the potential is p dot. That is a multiplying factor 1 over 4 pi epsilon naught p dot r 1 2 divided by r 1 2 cube. So, the potential unlike for a charge the potential due to a dipole is proportional to 1 over square of the distance.

And if you work out the electric field E is equal to 1 over 4 pi epsilon naught times the magnitude of p times twice cos theta in the r direction and plus sin theta in the theta direction divided by r cube. Now, I am sure you can see the symmetry between these 2 directions B with some multiplying factor times. The magnetic moment times a vector function E is another normalization constant times the dipole moment times the same vector factor. So, both E and B have the same form when you consider that electric field is due to a dipole. And magnetic field is due to a ring which is why magnetic rings tiny rings of current are considered to be magnetic dipoles.

Now, using this concept of a magnetic dipole I further argue this is only an argument. The actual proof is given in your text book that you can construct any general current density by just stacking all these little rings. And, so if you know the properties of ne ring we sought of know the property of magnetic field in general. And I showed for you that for a current ring curl of B is 0. But actually curl of B is 0 if r is not equal to 0 if r is equal to 0 curl of B is not equal to 0.

And we looked at that problem and we finally, concluded that curl of B in general is equal to mu naught j. Now, you could have done the same argument you would have taken the divergence of E you would have found the divergence as 0 except at r equal 0 at r equal to 0. We know how to calculate the divergence, and so we could have got a divergence statement for electric field as well. Now, the importance of all this is the following supposing I go back to the original equation this is where everything came from even this calculation came from this equation.

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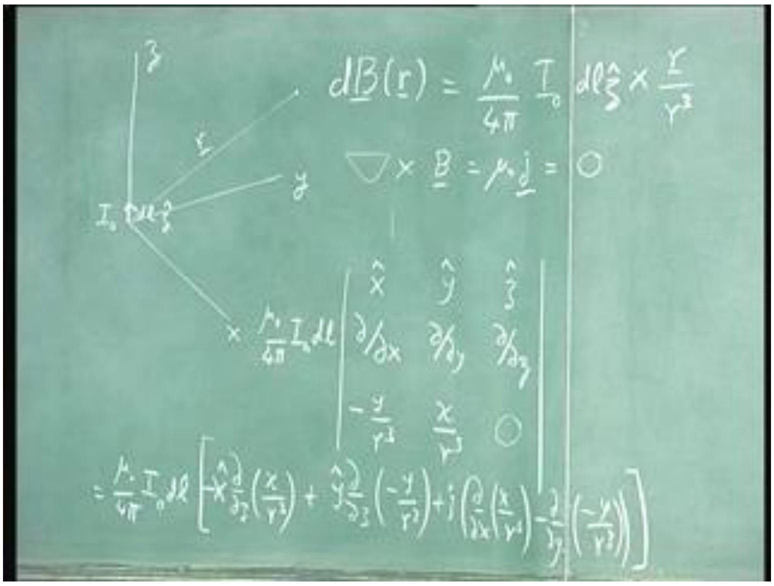

And let me consider let us say that we have x y z and that I take a small piece of current I naught for a distance d l along z Bio Savart law tells us that such a piece of current creates a magnetic field far away this is r. And the magnetic field at this point B of r I should say d B of r is equal to mu naught over 4 pi times. This current element I naught d l along z cross r divided r cube this is just the fundamental starting equation. We had now, what happens if I have only this piece of current and I take it is curl. And I know that curl of H is equal to j let me use curl of B is equal to mu naught j.

So, since my only current is here curl of B should be 0 everywhere, so let me try taking the curl of this quantity. So, the curl of B is going to be unit vector along x unit vector along y unit vector along z derivative along x derivative along y derivative along z the x component of this vector. Then you pull the constant parts out mu naught over 4 pi I naught d l the x component of this vector is z cross minus y. So, it is going to be minus y over r cubed the y component is going to be z cross x component. So, it is going to be x over r cube and the z component is going to be 0 z cross z, component is going to be 0.

So, that is what the curl is going to be, so I can evaluate let me right it out it is going to be equal to mu naught over 4 pi I naught d l times the x component is x hat del del y of 0 minus del del z of this quantity. So, minus x hat del del z of x over r cube the y hat component plus y hat is going to be del del z of this quantity minus del del x of 0. So, it is going to be del del z of minus y over r cube and then the z component is going to be these 2 terms which is del del x of x over r cube minus del del y of minus y over r cube. So, it is a vector and if this is to be 0 then each of these components must be 0 except at x equals y equals z equals 0. That is the origin, because that is the only place we have current.

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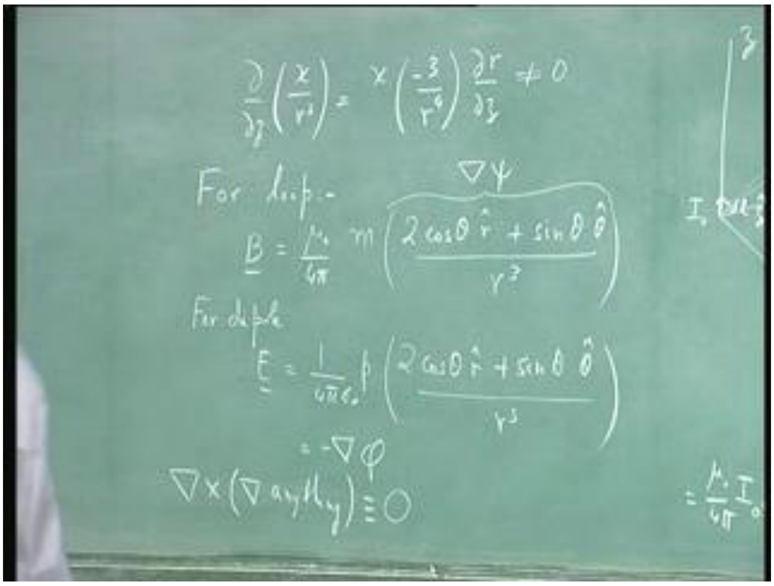

But if I take the z derivative of this term for instance what is it equal to del del z of x over r cube is equal to del x del z is 0, so I x is a constant that is this partial differentiation is concerned. So, it becomes minus 3 over r to the power of 4 I take the derivative with respect to r therefore, minus 3 over r to the power of 4 del r del z. But del r del z is not clearly 0, if I move in the z direction r is going to change. Therefore, this quantity is certainly not equal to 0 it can be 0 if x is 0, but it does not have to be whereas, here we are claiming that is 0 no matter what. So, this is not equal to 0 neither is this for the same reason and you can work this out and you find that this is not 0 either. So, we have a problem the problem we have is that we started with bio savart law we applied an equation that we know should give us 0 and answer is definitely not 0.

So, we have a problem here this either this definition is wrong or this equation is wrong one of the 2 must be the case. Now, in the case of the magnetic loop we did not have this problem when we wrote down the magnetic field for a loop. We found that magnetic field B was mu naught over 4 pi magntitude of the magnetic moment times twice cos theta r hat plus sin theta theta hat divided r cube. Now, this does not say that we do not have a curl; however, we know something else we know that for a dipole. The electric field is equal to 1 over 4 pi epsilon naught times the same quantity twice cos theta r hat plus sin theta theta hat divided by r cube. And we know something else we know that; however, we got this it came out of the gradient operator acting on potential. So, if we take the curl of B apart from a few constants it is like taking the curl of E I am taking the curl of E means taking the curl of a gradient.

And we know that curl of gradient of anything is identically 0 and so for the loop we can sure that this piece is the gradient of something. And, because it is the gradient of something the curl of B is 0 which means amperes law is satisfied. Whereas, for Bio Savart law amperes law is not satisfied something is going on we have a law with that. We have satisfied ourselves about it predicts everything that we know about in magnetic. And yet it is failing this basic law the moment we use the basic building block of magnetic field as I said either this is wrong this is wrong. Now, we will come to the how to chase this law when we start looking at it slightly differently. But this is the current message namely if I start with Bio Savart law I can work out an expression for curl of B. And that expression is definitely not 0, this is no way it can be 0 everywhere and yet we have a law saying it should be 0.

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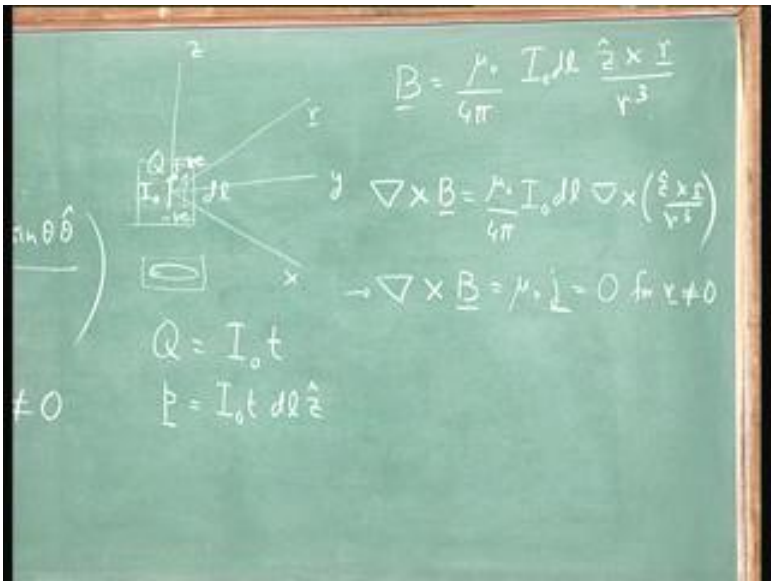

If you have a current I naught that is steadily going from minus d l over to to plus d l over to in the z direction it is like I have a discharging capacitor current is steadily flowing. So, negative charge is building up here and positive charge is building up there how much positive charge if I call this Q Q is equal to I naught times t. Because I naught Coulombs per second are flowing into this point, so the charge must be I naught times t assuming that the charge is 0 at t equals 0. So, I have a dipole my dipole p is equal to the charge I naught t times the distance between the charges which is d l in the z direction there is 1 way to make sense of this situation.

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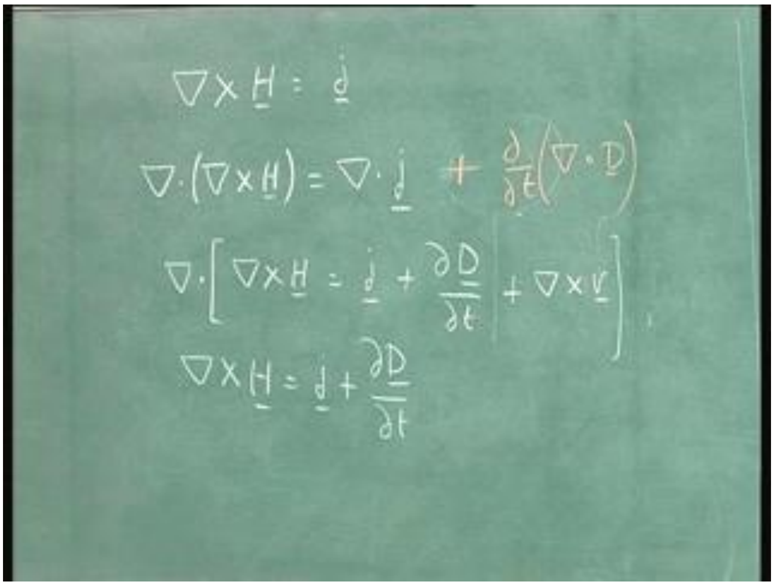

Supposing you take curl of H is equal to j now, I can take the divergence of this equation this is nothing, but amperes law. So, when I take divergence of this equation that is what it gives me, but divergence of a curl as you saw last class. And in fact, several classes you have seen this is equal to 0, but divergence of j we know is equal to minus del del t of the charge density. The reason is if you have a region in which this current flowing out then it must be that the charge contained is reducing. So, divergence j is definitely equal to rate at minus the time rate of change of charge density, so in other words this equation is only true for statics.

So, only true when charge density is constant and this particular case is a case where charge density is not constant it is continuously building up, because of this time derivative of this time dependence. So, what would you require to make this equation correct? You would require an, another term you would really want plus let me use an, other color chalk I want a term. That goes like plus del ro del t if I had a plus del ro del t then 0 is equal to minus del ro del t plus del ro del t and then that would be 0 equals 0. So, I would be happy the question is how do I get this plus del ro del t well I have another equation I know that Gauss's law has told me divergence D is equal to ro.

So, instead of saying plus del ro del t I can say plus del del t of divergence of D why does this matter? It matters, because I have taken divergence of this whole equation. So, I can pull this divergence out. And I can write this equation as divergence of and I am going to do this in a way that is not correct. But it makes it clear curl of H equals j plus del D del t of course, I should not do this I should. In fact, take them all to one side. And then correct terms I am just showing that I can pull this divergence out the first term looks like curl of H second is j and third is del D del t. Now, of course, this is not quite correct, because you know that I can always add any curl and this equation would still hold.

So, I could not only add del D del t, but I could add plus curl of anything upto this I can have because when I take the divergence the curl will vanish. And I do not know what this thing is, but I do know that I do need del D del t here, because I need to cancel out the divergence term that comes out of taking the divergence of j. So, this where we had Maxwell reached this point he said curl of H is equal to j plus del D del t. And of course, this is not unique you can always add a curl to this equation and at that point he did not choose to add anything he said let us suppose he did not add anything let us see what we get.

So, this additional term we had was required just to conserve charge and it is also motivated here. Because we can see that just when curl of B fails to be 0 your electric field is time varying everywhere see you can surely expect to see that. The electric field starts talking to the magnetic field the working out of the details becomes very messy see anyway. It is not necessary also, but the idea is that the moment you have a dipole a time dependent dipole forming your d E d t at the same time curl of B is not 0. So, this is the equation Maxwell came up with and with that he came up with 4 equations altogether which he said by complete in themselves.

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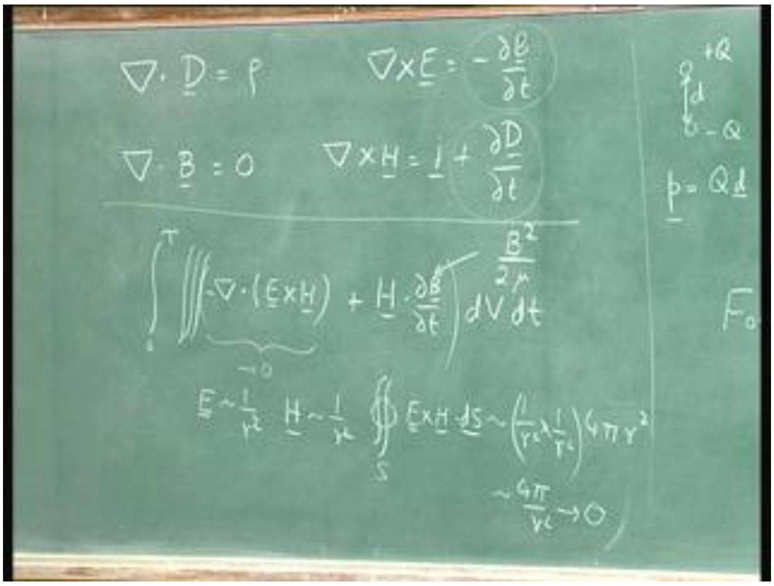

These equations were divergence of D is equal to ro curl of E is equal to minus del B del t divergence of B is equal to 0 curl of H is equal to j plus del D del t. So, you can see what has changed since we left electro statics first we were forced to use this term in order to explain Coulomb's law for moving systems. And secondly, we were forced to add this term, because you need to have charge conservation both of these have been forced on us I mean. It is not as if they were required by observation they just forced on us by the fact that our equations are not suitable for time dependent systems. Now, this system of equations is what are called Maxwells equations are very powerful. And we will spend the next few lectures looking at just what they are saying, but I want to go back to something I said last time in order to emphasise why these equations are important.

If you remember when I was trying to talk about magnetic energy I had this integral 0 to capital T volume integral over all space of and I had a term which was divergence of E cross H. And there was an, another term which was H dot del B del t now, signs I do not remember which was which way d V d t. And in discussing this issue of magnetic energy I dropped this term I said this goes to 0 the argument by which I went to 0 here was I said electric field. If charges are far away goes like 1 over r squared magnetic field if currents are far away goes like 1 over r squared. So, if I take this and I apply Gauss's law to it I would get surface integral over a surface E cross H dot d S which is something like

1 over r squared. And 1 over r squared integrated over a surface which is 4 pi r squared that is the area of a surface.

So, it will go like 4 pi over r squared which goes to 0 as I make the surface larger and larger the importance of this term would become smaller. And smaller and this why we dropped this term this is why we were able to get an expression for magnetic energy which said B squared over 2 mu. But there is something wrong with this statement as it stands it is correct, but once you are adding this extra term del D del t. It turns out E does not go as 1 over r squared H does not go as 1 over r squared and. In fact, the surface integral of E cross H dot d S does not go to 0. And this is the remarkable thing that this equation has done just adding 1 term has completely changed. The character of all the equations and the importance again of this statement is that this one term once you added it was able to combine electricity magnetism and optics.

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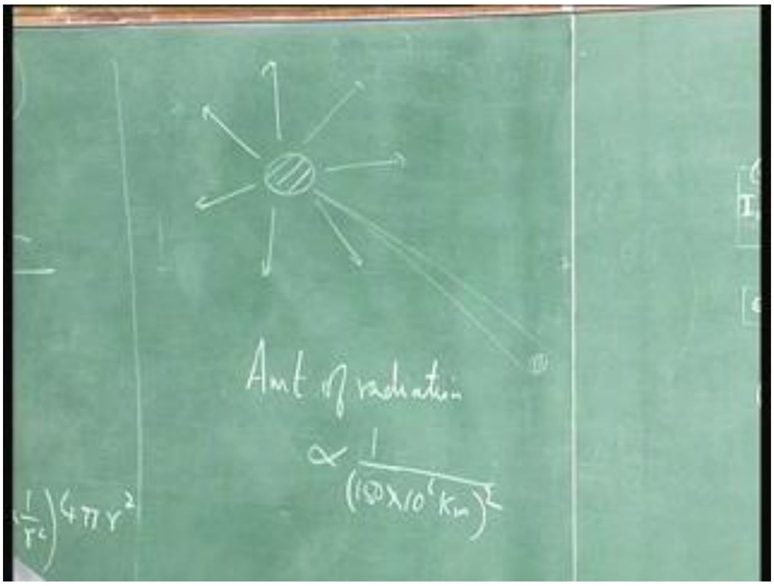

Now, think about what optics is saying optics is saying that you have for example, a star our sun is somewhere 100 and 50 million kilometers from us. And the sun we have told is shining in all directions and we are 100 and 50 million kilometers away. We get whatever little bit of radiation happens to fall on the earth and we know that the amount of light. That is falling on the earth amount of radiation is proportional to 1 over 150 into 10 to the power 6 kilometer square of the sun sun's total radiation. We know this is true because we know this is part of optics we know in optics the light weakens as 1 over r square radiation weakens as 1 over r squared.

So, the question immediately comes up is what can light possibly do with electricity and magnetism in electricity. And magnetism the energy stored in the field it goes like E squared or B squared will go as 1 over r to the power of 4 whereas, here are saying amount of radiation goes as 1 over r squared. So, there is something very different between electromagnetism of optics or the theory of optics. And the theory of electricity and magnetism which is why until Maxwell came nobody connected the 2 up to them. They were 2 separate fields Newton studied light Eigen studied light and Faraday Maxwell studied electricity and magnetism. But when he added this one term suddenly Maxwell obtained some new effects.

And he found the theory of radiation was nothing more than a special case of electricity and magnetism. This is a very profound thing it suddenly simplified an entire field of physics. optics just became a special case. And because of that we were able to generalize optics we were able to say may be there is an optics. That works at larger wavelengths may be the wavelength of light does not have to be a millionth of a meter may be wavelength of light could be 1 meter. If the wavelength of the light were 1 meter what would it is frequency be and that is where radio waves came from the theory of radio waves came, because Maxwell recognized that. The theory of optics was nothing more than a special application of electricity and magnetism the moment you realize that you can start changing the variables.

So, it is alright we are eyes can see optics may be there are other rays also and when you solve these equations you find. In fact, there are other rays. And when you look at what those other rays are you find infrared rays you find ultraviolet rays you find x rays and of course, you find micro waves radio waves and other longer length waves. So, this particular generalization that Maxwell did he just added 1 term, but it is far more than adding 1 term people have been adding term to these equations. The thing he did was he unified the whole set of different observations and gave a single theory that explained all of them.

And that is why today we do not talk about this as Faraday’s law and this as amperes law and different laws. They are just Maxwell’s equations, because he was the 1 who made sense out of it all the rest of them were giving individual laws he said they are not individual laws it explains something which is interconnected. We will come back to this, because this is explaining the inverse square law of radiation has to be the final goal of any electricity and magnetism course on you know that you can believe that electricity and magnetism can explain all of that.

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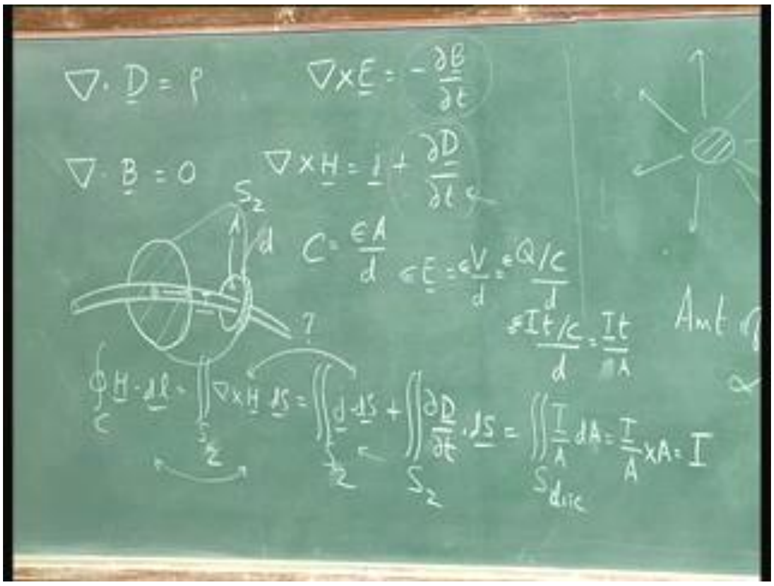

There was one more way of looking at amperes law and that was to say supposing I have a wire and the wire is carrying a current. Now, I can draw a loop around the wire and I can do loop integral H dot d l if I do loop integral H dot d l I know that. This is equal to surface integral over that connecting that loop curl of H dot d S which is equal to surface integral J dot d S this is equal to I enclosed. We have done this a dozen times now, therefore, loop integral of H dot d l ought to be giving me the same answer. But you can make a thought experiment you can say supposing I break this wire at 1 point. And I put a disc 2 discs which is separated by a dielectric very thin dielectric the discs have an area A and the dielectric has a thickness delta or d.

So, I know that the capacitance of this disc is going to be epsilon A over d where epsilon is the dielectric constant however the dielectric is an insulator. There is no current flowing in the dielectric, so now I have this loop drawing it dark just to emphasize. Now, I can connect this loop by a surface that intersects the wire that is how I got I enclosed, but I could take the same loop I do not move the loop. But I could connect it by a surface that did not cut the wire at all that slides through the dielectric what that does mean it means that my discs loop integral is still equal to surface integral let me call this surface S 2.

So, call it as S 2 curl of H d S that is still 2 which is equal to S 2 of j dot d S that is what amperes law says is equal to I enclosed. But, there is no current enclosed because on that surface there is no current flowing at all there is no current flowing here. Because it is air there is no current flowing where the surface goes through the dielectric, because dielectric is a non conductor this is equal to 0. So, I have got 2 separate answers in 1 case I have got the answer loop integral H dot d l is equal to I enclosed which is I in the other case I have got loop integral H dot d l is equal to 0.

But it is 1 or the other I have a current flowing the current is not 0. So, loop integral H dot d l cannot simultaneously be 0. And I it must be 1 or the other or there must be something wrong with the equation, but this is Stokes law nothing can be wrong here. It is just a vector identity this is definition if I integrate the surface of over a surface j dot d S it is the current going through the surface. So, there is no problem, so the only place where there can be a problem is here is true that curl of H is equal to j. We already know the answer it is the third different way of looking at it curl of H is not equal to j supposing I had a del D del t dot d S well in the place where intersects.

The wire electric field is very small, because we know that copper is a good conductor, so there is hardly any electric field this term is 0 whereas, this term is large. Now, when I go through the capacitor my electric field E is equal to the voltage divided by d in the direction from the plus plate to the minus plate. And the voltage V is equal to Q over C divided by d because charge content is heavy and what is Q Q is nothing but I times t divided by C divided by d. Now, this is the electric field let me put in the expression for C, so what do I get? I get the electric field is equal to I t divided by epsilon A.

If I want to work with D I have to use epsilon E and the epsilon goes away, so it is I t over A now I want the derivative in time of D. So, this becomes in this when it is intersecting the dielectric there is no current, so this surface term goes away surface integral S 2 of I over A dot d S of d A. Now, this expression I over A is only correct over the region of the disc away from the disc there is no field, so I can take 0. So, it is S disc, but the area of that surface is A, so it is equal to I over A times A which is equal to I. So, after all I get back amperes law I got amperes law by using a simple surface by the fact that it intersected the wire. And I got j dot d S over that surface was equal to I and if took a surface that went through.

The dielectric j dot d S is equal to 0, but this del D del t term gives me the same number gives me I. Now, this is why I did not do those complicated vector calculations because I think I can convince you of the answer more easily directly. So, for these 3 reasons firstly, if I take the basic building block of Bio Savart I find that curl of B is not 0 in the way amperes law requires it. And I recognize in Bio Savart that I am having a time evolving dipole electric dipole, so there is a d E d t present. And when d E d t is present my curl of B was not 0 secondly, we looked at charge conservation when we looked at charge conservation.

We managed to get that we need a term that looks like del D del t and thirdly when we tried to do curl of H and compute it is value by 2 different means. We find that in 1 case we get I due to the conduction current and in other case you get the current I due to what is called the displacement current this del D del t is called. It is only a name ultimately it is required, because the time changing electric field induces magnetic field even as the time changing magnetic field induces electric field. So, both of these are happening you have an electric field caused by time rate of change of magnetic field you have magnetic field caused by time rate of change of electric field.

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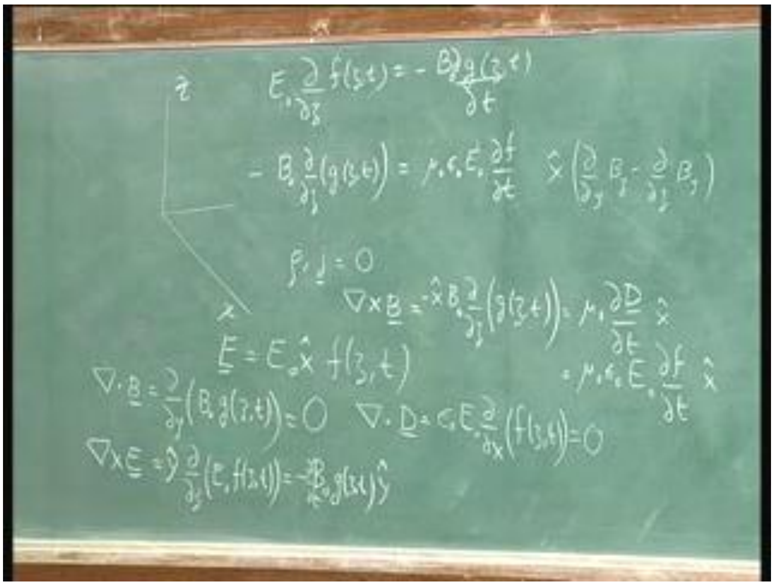

So, now I want to end the lecture today by putting all these pieces together and showing you that they give you entirely new. So, what I am going to do? I am going to assume that I have an electric field in the x direction and a magnetic field in the y direction. And this electric field E is in the direction and I am going to assume it as some dependence on z and t. So, the electric field does not change in the direction x, but it can change in the direction z I am going to assume the magnetic field B as some B naught in the y direction and some other function g of z t. And then I am going to try and apply the various Maxwell's laws and see what comes of having such case.

So, when I take divergence of B the direction of B is y, so it is equal to del del y B naught times some function g of z t which is equal to 0, because g is a function of z. And I am taking the derivative with respect to y if I take the divergence of D which is epsilon E I get epsilon naught E naught times del del x of f of z t which is again 0. Because f is not a function of x f is only a function of z and t I will further assume that there is no ro there is no j. Now, if I want curl of E I clearly need something that can differentiate E and a only spatial direction in which E is varying as z. So, it must be del del z of the electric field along x E naught f of z t, so if I have something whose derivative is z acting on the x component must be in the y direction.

And this we are told is equal to minus the magnetic field luckily the magnetic field is in the y direction B naught g of z t. So, that gives me an equation because both are in the same direction, so I can write down this equation. The equation I can says del del z E naught outside of f of z t is equal to minus B naught g of z t sorry this should be del del t of, because it is minus the time rate of change del g del t. Now, if I apply the same thing to B I need curl of B is equal to well B is in the y direction and it depends on z. So, I must take the z derivative del del z of B naught comes outside g of z t, now if I have the z derivative of something along y, it must be along the x direction.

So, it is in the minus x direction because you look at the standard term for curl along x it will be x hat del del y of B z minus del del z of B y. So, I do not have a B z I have only a B y, so there is a minus sign, so that minus sign is here this curl of B is equal to mu naught j. But, there is no j and is also equal to mu naught del D del t. D is nothing, but epsilon E, so it is mu naught epsilon naught times del del E of electric field E naught del f del t. So, let us write that equation as well it says that again both are in the x direction. So, I have an equation now it says minus B naught del del z of g of z t is equal to mu naught epsilon naught E naught del f del t, so I have 2 equations with which I can try and solve the system.

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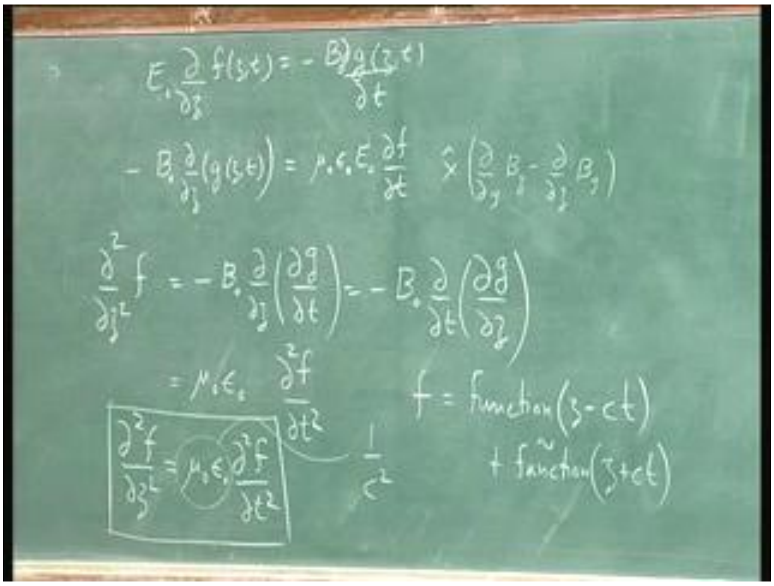

But this involves g and this equation for g involves f, so I need to eliminate g if I do not eliminate g I will not get an equation for 1 variable I require del g del z. And I have del g del t, so I will take another derivative with respect to z, so I will take E naught del square del z square of f is equal to minus B naught del del z of del g del t. Now, as you know second partial derivatives can interchange, so I can write this as minus B naught del del t of del g del z. And now I can use this, because I have B naught del g del z right here, so it becomes mu naught epsilon naught E naught del square f del t square this E naught is common.

So, I can remove it from the equation and I get the final equation which is del square f del z square is equal to mu naught epsilon naught del square f del t square. This equation is the equation Maxwell derived and this is the equation that suddenly changed everything. Because mu naught epsilon naught when you work out the numbers is nothing but 1 over the square of the speed of light in vacuum. Now, it is not an accident in fact, I would say mu naught and epsilon naught are deliberately obscure constants. If you used a different system of units this equation would naturally have had 1 over c square in it mu naught.

And epsilon naught were derived more for convenience of views and not for convenience of thinking. But what is more important is if you look at this equation I will derive it next time you will find that. There are solutions f is equal to some function z minus c t plus some other function not the same function I will put a Quidde of z plus c t. So, when you look at this you find this system support solutions which move without decaying they can go to the end of the universe without obeying. The inverse square law of Coulombs law I can have a source that generates this disturbance I will go far away without losing it is energy. We will do the rest of the derivation next time when we complete the wave equation.