Electro Magnetic Field
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Lecture - 29
Magnetic Energy

Last time I had introduced the ideas of inductance and how inductance connects to magnetic stored energy? I am going to go more into detail in it. This time, I am going to introduce some mathematical tools that make such calculations much easier. So, let us see what we came up with last time. We did not fully prove it, because we did not have the mathematical tools to do it.

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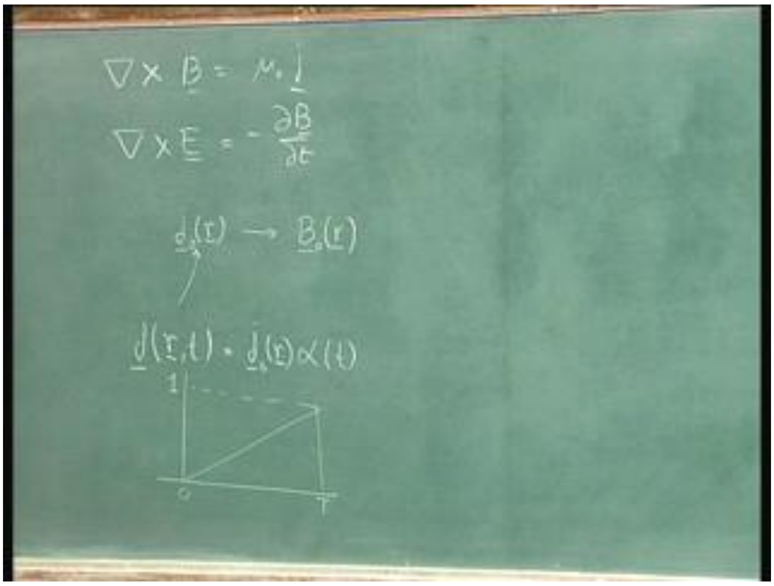

What we said was that you have 2 equations, curl of B equals mu naught j and you have curl of E is equal to minus del b del t. This was the new rule, that earlier, we had curl of E equals 0. Because of Faraday's law, we now know that curl of E is equal to minus del b del t. So, we said supposing, there is a current j, which is distributed through space this current through Ampere's law produces a magnetic field B. But this current must have come from somewhere it does not just appear in one instant. So, let us assume that j of r grew in a uniform way, that is at every point it grew continuously.

So, we defined a new current j of r t, which was the same j of r, I will call it j naught of r and b naught of r. This j naught of r times a function alpha of t and the alpha I choose was starting from t equals 0 to some capital T just linearly, increased to unity. So, when alpha is 1 j is j naught, when alpha is 0 j is 0. So, I am very slowly, turning on this current. But if you look at Ampere's law and I substitute this formula into it what do I get? I get curl of B is equal to alpha of t times mu naught j naught, because that is what I have defined j to be. Now, let me guess, I will guess that my B of r t is some other function f of t b naught of r, substituting in there what do I get?

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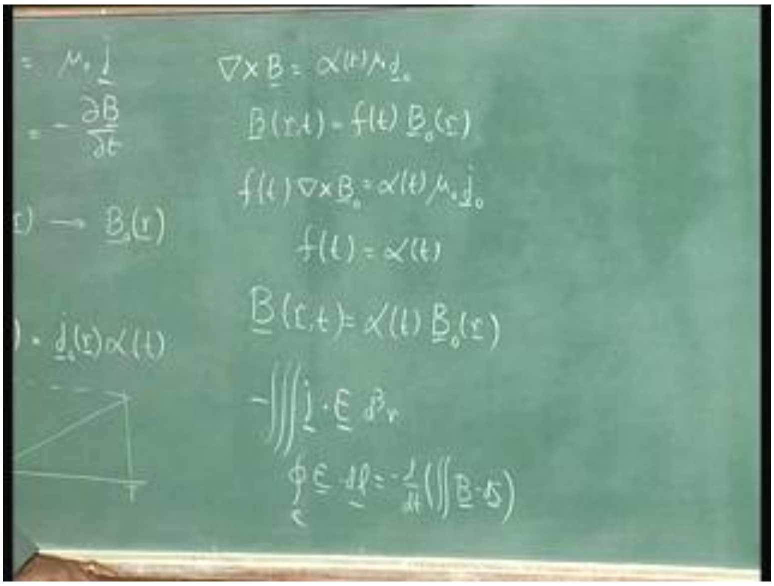

I get f of t curl of B naught is equal to alpha of t mu naught j naught, but I already, know that this is true, curl of B is equal to mu naught j naught. Because that is the original equation, that gave me b naught out of this current distribution j naught. So, I can cancel these out, so I get f of t is equal to alpha of t, which means that my magnetic field b of r t due to this current is also equal to alpha of t b naught of r. Now, why is this important? It is important, because we can now start looking at Faraday's law. Now, if you look at Faraday's law and we say I want to know what is happening to energy input into the field? So, energy in the field is volume integral j dot e d cubed r with a minus sign, J dot e is the energy dissipated.

So, minus j dot E is the energy injected, so j dot e has to be worked out. But I have curl of E is equal to del b del t and the j is equal to curl of B. All these facts I have, so let us see, where we can get using this, what I do is I take the loop integral form of this expression. So, what does that mean, if I take the surface integral on both sides and we apply Stoke's theorem? I get loop integral over some curve c of e dot d l is equal to minus d d t of surface integral b dot d s. So, if I now, want to say what is j dot e I need to understand what happens? If I do this, e and dot on the other side as well. But j is already, connected to E. So, the particular j, I am using is nothing but alpha j naught and the particular e, I have is coming out of here and let us see what that e gives me.

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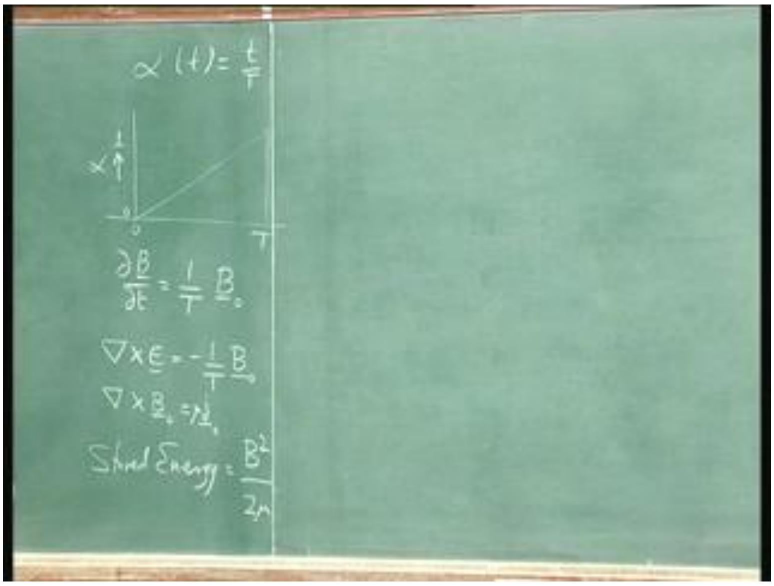

I have chosen that alpha of t is equal to some t over capital T, because it goes from 0 to 1. This is alpha, since this is alpha d d t or del del t is equal to the time derivative of alpha, which is 1 over capital T times b naught. Why, because we already, have that b is alpha of t b naught of r and if I take the time derivative it can act only on alpha. So, I get b naught of t is equal to 1 over t, b of t partial derivative with respect to t is 1 over capital T b naught. So, my Faraday's law says curl of B is equal to minus 1 over capital T b naught and I also have my Ampere's law which says that curl of B is mu naught j naught. It is these 2 terms that are used to get, where I want to get, which is what I had talked about, last time and showed.

I motivated that, you can basically, energy into the field talking about j dot e. The reason is as I inject current as I inject increasing amount of current, my magnetic field increases. As my magnetic field increases, I induced an electric field. That electric field dot j means, I am doing work and therefore, I can take this entire picture and make it into a statement about work done to increase magnetic field. For now, let us look at the result I put down, which is the stored energy was equal to b square over 2 mu. Now, I am going to derive the same thing for an inductor. We have already, done it once, but I think Faraday's law is one place, where revision is the only way of learning things. So, what I am going to do is to recover the circuit theory picture of inductance.

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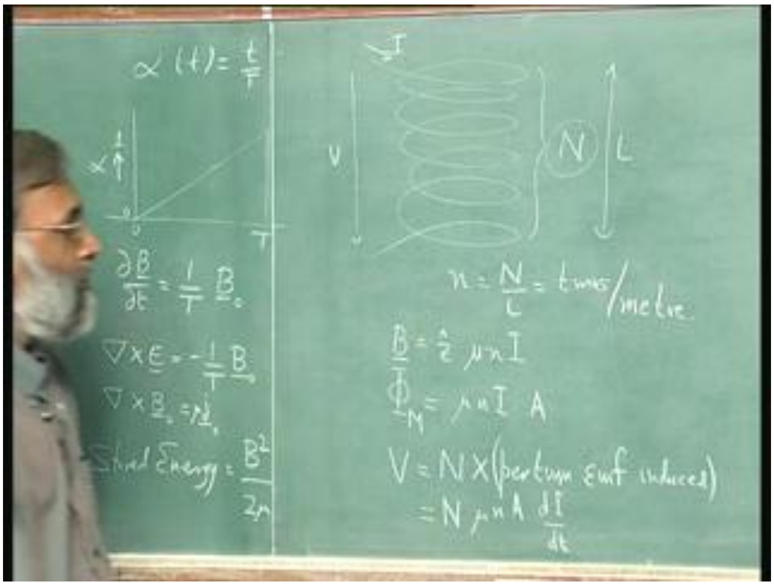

Let us say I have a solenoid, I have n turns and this solenoid has a length capital L. Therefore, my turns per length little n is equal to capital n over capital L, it is equal to turns per meter. Now, if I inject a current, I know that the magnetic field B is in the z direction and it is equal to mu times the number turns per unit length times the current. The flux phi m phi magnetic is equal to the surface integral of this b over a cross sectional area. So, it is equal to mu n I times area, now I know that the amount of voltage that is dropped v is equal to n times the per turn e m f induced. Now, this is not quite correct, there should be a minus sign. So, what I mean is that if there is a magnetic field increase inside the coil there is going to be an induced e m f. This voltage is the voltage required to overcome that induced e m f. That is why there is no minus sign here. So, how much is it? It is equal to n times this piece mu n a times d d t. So, I have taken a minus sign, which is really what, should be there and I have removed it. So, I have got a plus mu n times mu n a d I d t.

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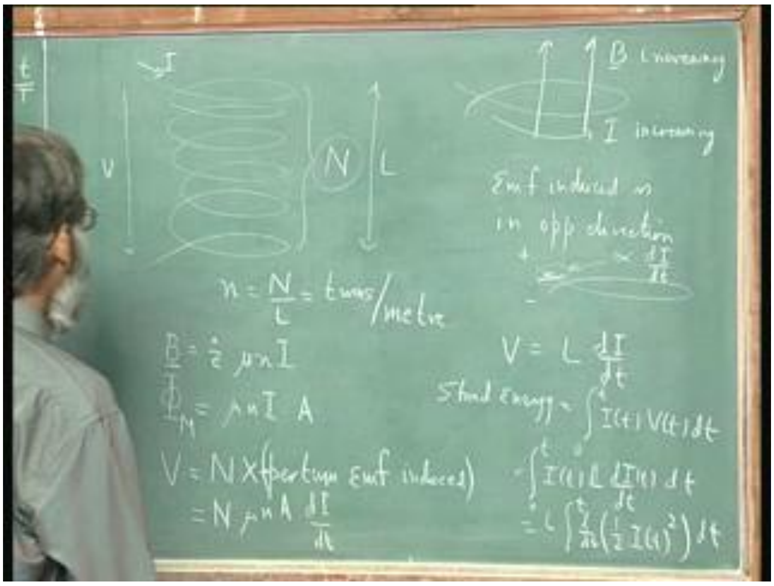

So, let me explain, that sign again, if I have a loop and I have a strengthening magnetic field B is increasing, because the current I is increasing. Now, if the current is increasing and therefore, the b is increasing, what is happening is an e m f is developed. What is the direction of this e m f? The direction of the e m f is, so that a negative b will be produced. So, the e m f is trying to drive a current in the opposite direction. So, the e m f induced is in opposite direction. So, it is like a resistance, it tries to prevent this current from increasing. How does it try to prevent it? It is trying to put a voltage that is trying to drive current in the opposite direction. So, if you still want to impose this increasing I what you have to do is you have to apply an equal voltage in the direction of the current. So, it is as if for every turn for every turn there is a little. So, this is my applied voltage, for every turn, there is a little battery induced battery that is trying to resist my applied battery.

This induced battery is proportional to d I d t, if there is no change in magnetic field there is no induced e m f and magnetic field increases only, if current increases. Therefore, I need to apply a voltage, that is proportional to d I d t or proportional to d phi d t. So, this voltage is therefore, in the plus direction, it is in the plus direction, because I have to overcome a minus voltage. So, really I should have a minus sign here, indicating that v is n times the negative of the, per turn induced e m f. So, overall a positive sign results, but this is nothing more than inductance. So, this is saying just b equals l d I d t, now we know what stored energy is in this system. Because we know, that if you want amount of energy put into the device is equal to integral from 0 to t of I of t v of t d t. It is the amount of current in the direction of the applied voltage integrated over time. But my v of t is already, I of t times l d I of t d t times d t, assuming l is a constant I can pull it out. This gives me this kind of relation, which is integral 0 to t t d t of one half I of t square d t and this of course is a total derivative integrating from 0 to t.

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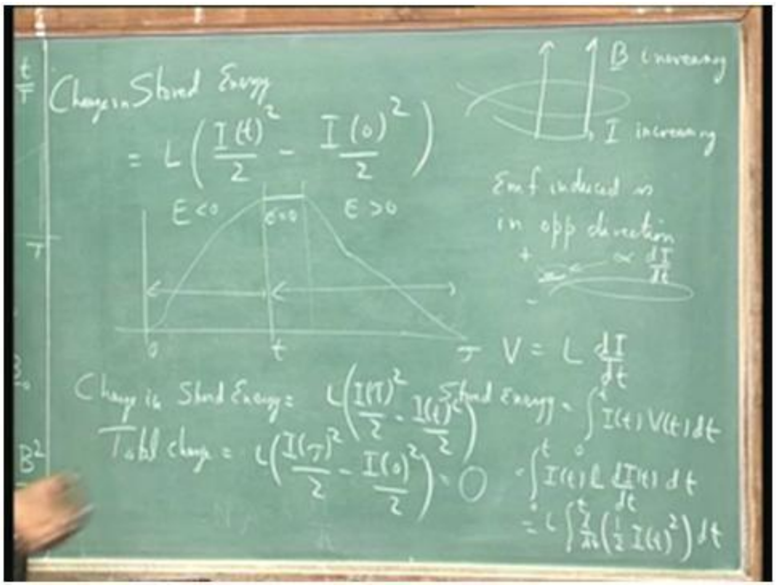

So, that tells me that stored energy, I should say change in is equal to L times I of t square over 1 minus I of 0 square over 1. So, it is nothing more than the inductance formula and of course, it has to be the inductance came from here. But let us, see what sent into it from the point of view of magnetism what went into it is that you had a strengthening magnetic field, because you were trying to drive more current. This strengthening magnetic field due to Faraday's law is trying to apply an e m f in the opposite direction. That is it is trying to drive current in the opposite direction therefore, resisting the increase. This voltage applied in the opposite direction is therefore, having work done on it by the forced current. You are driving a current through the system and the system is resisting.

It is like you are pushing a rock up a mountain and therefore, you are doing work on the system and that work is clearly equal to this amount. Now, the question is where is this energy going? We know one thing. We know that if you now, reduce that current supposing you took this, current and you gave it a reverse direction and brought it back to 0, in this period, we know we have got this. In this period, so this is 0, this is some time t. This is some time tau in the second period, what you have is change in stored energy is equal to l times I of tau square over 2 minus I of t square over 2. I can add these 2 up, because it was a total derivative this, term will cancel out the I of t term will cancel out, here it is plus here it is minus.

So, total change is equal to l times I of tau square over 2 minus I of 0 square over 2. What that means, is I took the current let us say starting from 0 current. So, I of 0 is 0. 1 took it to 10 amps. So, some energy went into this device, I brought it back to 0. That energy came right back, the total change in energy was 0, because I of tau is equal to I of 0, this is not a dissipative device whatever energy, I put in came right back to me. So, energy is getting stored somewhere. This energy cannot be stored in the coils, because it is not there is, so storage in a resistor for example it cannot really, be stored in electric fields. Because supposing I had a situation like this supposing for a while the current was constant. If the current was constant then I know that my magnetic field was constant.

My magnetic field was constant electric field was 0, because curl of E was equal to d b d t. If d b d t is 0 electric field is 0. So, I have electric field let us say negative electric field positive electric field 0. Negative positive is only in the sense that it resists, it resists the increase it resists the decrease. So, here it is trying to slow things down, here it tries to speed things up, but in the middle there is no electric field at all. If there is no electric field energy cannot be stored in the electric field. So, where can it be stored? Well, there is only 1 culprit the culprit is the magnetic field, because it cannot be stored in current since the current by itself, you do not see storage in a resistor. So, it must be some it is certainly related to the current, but it is some special feature of the current and the special feature. We can think of is that the current induces a magnetic field, now how much is that energy, well we know that it is L I square.

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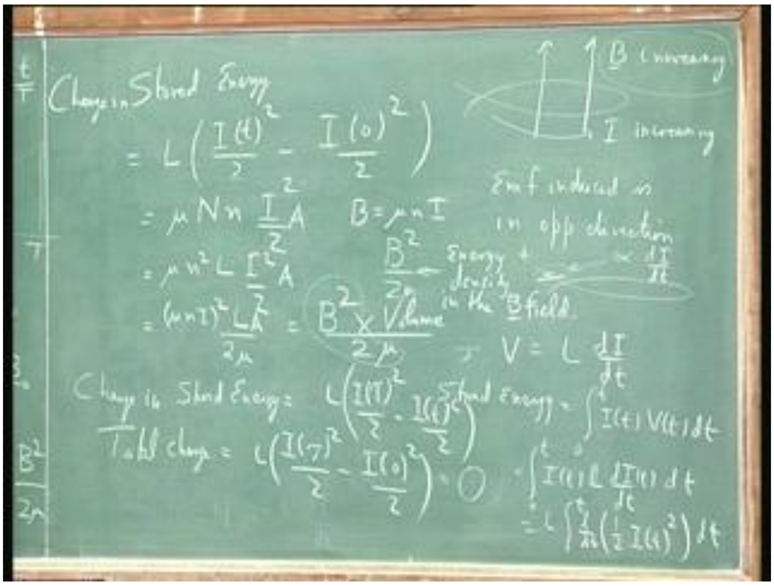

So, we know stored energy is equal to L I square, but now, what is L. You have got l from earlier; it is equal to mu n times n I square over 2, well I can write this as mu n square l I square over 2. Now, the magnetic field is equal to mu n I, so I can see an, I square, I can see a n square, there where I can see only 1 mu. So, I can write this as mu n I whole square times l divided by 2 mu and of course, there is an area also times a. This mu n I square is b. So, it is equal to b square times volume divided by 2 mu and that is where we get back, this same formula that energy in the magnetic field is related to b square over 2 mu. It is the same idea, idea here we will come back to mathematical terms. Idea here is straight forward it is that we have a known device, that we have known magnetic field. In order to increase the magnetic field, we found we had to do work. The work we had to do the total work, we had to do to bring a current up from I of 0 to I of t was one half l times I of t square minus I of 0 square.

We know that this is a conservative kind of change in energy, because if I bring the current back down to 0, I get all the energy back. It is not a resistor in any way. If this were a resistor, if I brought the current back to 0 that has dissipated an equal amount of extra energy. I would not have got any energy back, it is more like a capacitor, whatever, energy went into the device came back, when I gave it out when I brought the current back to 0, but I can write this term down in terms of what I know, about the magnetic field. So, I know, that l is nothing but mu n times capital n times a, so I have written that down, I square over 2, but mu n I is b. So, I can write this, expression in terms of magnetic field and what do I get? I get magnetic field square times volume divided by 2 mu. So, again b square by 2 mu is energy density in the b field. Now, at the end of last class, I introduced the concept of mutual inductance, let me repeat that. So, that you feel more comfortable with it.

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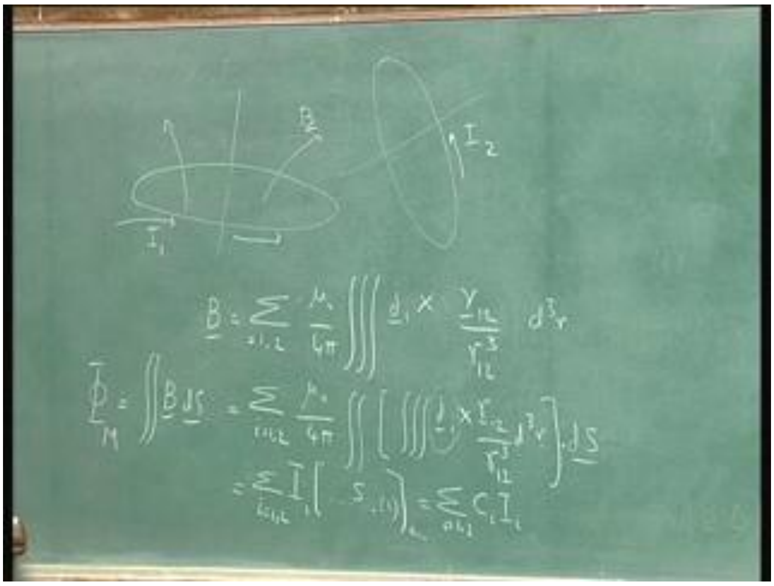

Supposing I have a coil and supposing, I am driving a current I in it, I have another coil and that coil does not have to be along the same axis. And it does not have to point in the same way and let us, say that that coil is carrying a current i. This is I 1 this is I 2, the total magnetic field at every point is due to the magnetic field in I one and due to the magnetic field in I 2 due to I 2. So, the magnetic field b is equal to sum on 1 2 I equals, 1 2 of mu naught over 4 pi volume integral j sub I cross gradient of 1 over r 1 2 with a minus sign d cubed r or if you like j cross r 1 2 over r cubed. So, this is the Biot–Savart law and from the Biot–Savart law, you can see that if I have 2 different currents magnetic field is just the sum of those 2 currents.

So, the magnetic field, here is or rather if I take the surface integral b dot d s. So, this is the total magnetic flux, it is going to be equal to sum I equals 1 2 mu naught over 4 pi integral over the surface of this quantity. So, there is a volume integral to give me the magnetic and there is a surface integral to give me the magnetic flux. So, it is a very complicated integral 5 different dimensions. However, it is linear in j, so if I made j equal to twice its old value, now answer just doubles. It does not matter, how many times

I integrate over space still it is linear in j, since it is linear in j I can supposing I know that, I have fixed currents I 1 and I 2 all the shape of that coil goes into this integral. But I can write this, j in terms of some complicated function. So, I can write it as sum I equals one comma 2 I sub I times complicated stuff which depends on i.

As long as the coil shape does not change this bracket does not change, because it depends only on how the coil goes around in space and how that coil goes around in space. The amplitude of the current comes through i, his is the result for unit current. This is the result for changing that current into its actual value, which means, I can write this, whole thing as a coefficient it is a number. So, I write this as sum I equals 1 2 of some c I. Now, these numbers are not easy to calculate, as you can see I have to do a lot of work lot of integrations to get that number. But I can put j the current as 1 amp calculates this expression. That gives me the constant c then I put the actual current, which is 10 amps through i i and the product of those 2 gives me the correct magnetic flux.

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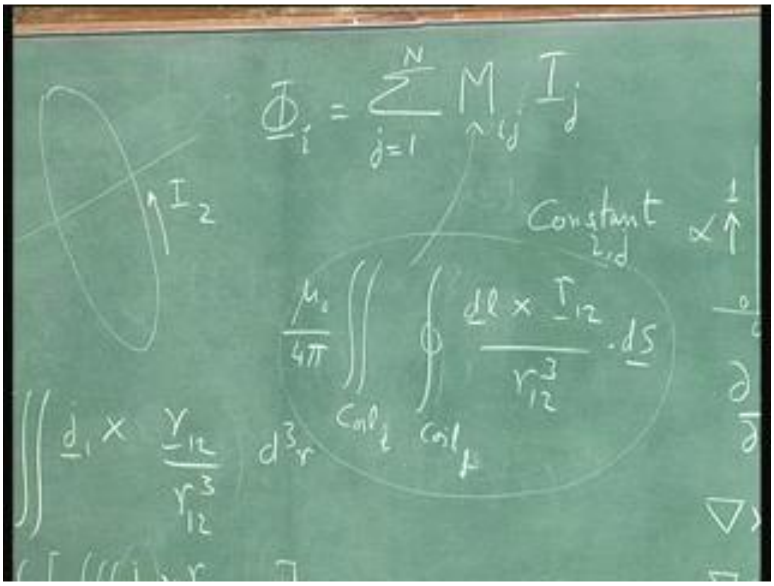

I can generalize this for n coils. So, what I get is phi through coil sub I is equal to sum on j. It goes 1 to capital n some coefficients. Here I will use the symbol that is usually, used M i j times the current in coil j. It is the same idea that constant multiplies the current in the j th coil. So, this m I j, what would it be? It would be mu naught over 4 pi integral over the surface coil 1 or coil I then this current, there is going to be a volume integral.

Now, that volume integral also reduces to a loop integral, because there is no current anywhere else. So, it will be another a loop integral provokes coil 2 of coil j. Then there's unity current, so that j does not matter. But there is a d l cross r 1 2 over r 1 2 cubed dot d s, that is what it is? So, you can see, it depends on current at all, but it depends in a very complicated way on the shape of the coils. This is over the surface of the coil; this is over the loop of the coil. Now, you can make this into 2 loops, that is not difficult. It is just a matter of doing vector algebra. But what for our purpose what is important is this whole thing is a constant, which depends on I and it depends on j. So, that is why, I have written it as m I j.

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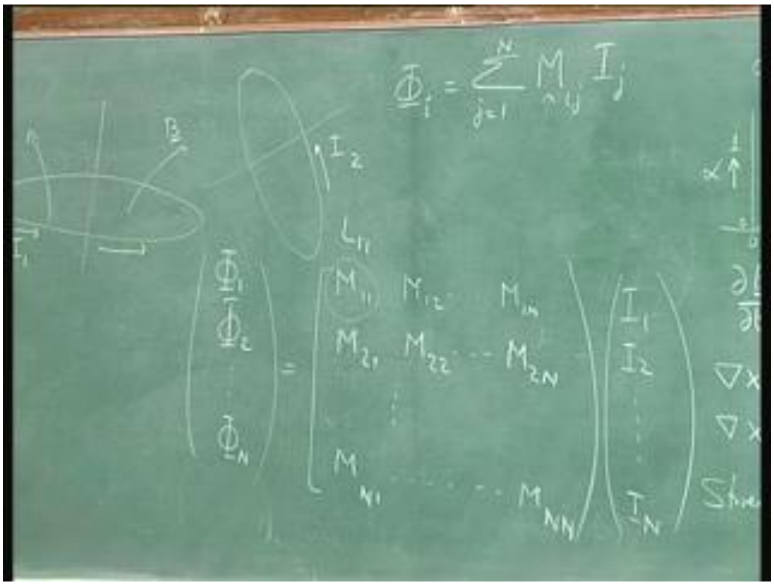

So, it is therefore, a matrix equation and as I wrote down, last time that matrix equation can be written as the flux in coil 1 flux linking coil 2 up to the flux linking coil n is equal a matrix M 1 1 M 1 2 M 1 n. M 2 1 M 2 2 M 2 n up to M n 1 up to M n n, it is a matrix and multiplying this matrix is I 1 I 2 up to I n. So, this is just a linear algebra problem now, if I know these coefficients and usually, the middle one are called l I 1 1 l 2 2 etcetera, because they represent the amount of flux linking a coil due to its own current. So, that is what is called self inductance, these terms are all called mutual inductance, but if I now these numbers then calculating flux is just matrix multiplication with currents. Calculating currents is just inversion, given flux it is just inversion. I can just take the inverse of this matrix multiplied by the fluxes gives me currents. It is also easily, provable that this matrix is always is always invertable. They are physical things; therefore you cannot have a non invertable mutual inductance matrix.

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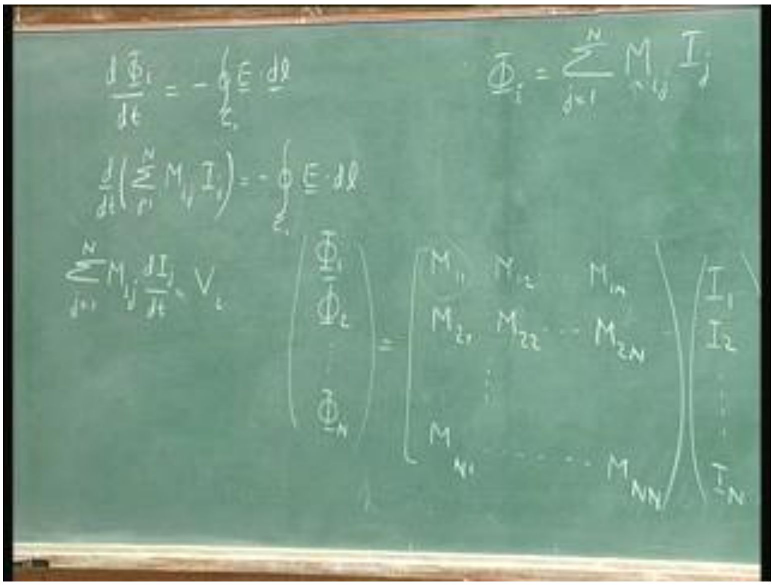

Now, let us, take the time derivative of this, if we take the time derivative of this, then we know that d phi I d t, which is the rate of change of the flux in coil i. That is equal to minus loop integral e dot d l, because of Faraday's law. Curl of E is equal to minus del b del t integrate over the surface, you get loop integral e dot d l is equal to minus d phi d t, but phi I has a representation. So, you can say d d t of sum 1 j equals 1 to capital n M i j I sub j is equal to minus loop integral e dot d l on loop c i. It is on the I th loop here, we have to do this loop integral. Now, let us assume that these coils are stationary. Moving coils we will come to later. So, then this d d t does not capital m, it affects only capital i. So, you get sum j equals 1 to capital n M i j d I j d t is equal to now, what is the loop integral of e dot d l over c i. It is nothing but the induced e m f, so it is equal to v sub I, the minus sign is absorbed, because as you remember this is the resistance, that the loop offers to driving a current.

Therefore, the applied voltage required is minus of this, so the minus sign gets absorbed. Now, this is nothing but a generalization of what we have already done? Because we already have l I i d capital I i d t is equal to v i. That is our self inductance equation, this is a generalization. If I have multiple coils and all the coils varying currents then the total induced e m f is nothing but the sum of all the mutual inductances times. The time rate of change in the other coil, at the molar that gives, you the induced that gives you the applied voltage required to drive your current in your coil. So, now from this we can also work out energy. It is a trivial thing and you can work out, what the stored energy is, I will leave. It as a exercise, for you to try out and once you have struggled with it a little bit may be, I will derive, it in the next lecture or 2.

I want to do one other thing which will set the context for doing the work in the next few lectures. We can see that once, we have curls we are running into trouble with the vector algebra. I can quote you a result, but it is not obvious, when I write curl e cross h you would not know what it is? It is difficult to understand, where terms like this are coming from, how many terms they correspond to... So, we need a little bit of mathematical machinery, which makes all of this rather simple. This machinery now has become very sophisticated and I am going to introduce the simplest parts of it which allow, you not to remember any of the formulas in your book. That is quite important, I mean if you have to keep memorizing all the formulas you do not learn the concepts, whereas if you learn the concepts. And can just work out in a couple of minutes any formula you need the formulas are anyway there in the back of any book. So, you can you come out of the course learning a lot more.

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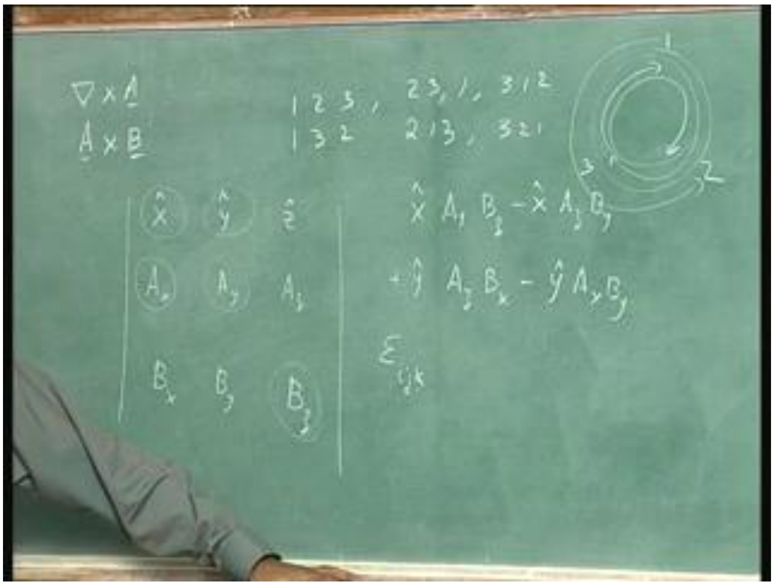

The basic thing we need is to understand how to represent curl or even cross product? As you know, what we do is we write determinant of a matrix x hat y hat z hat a x a y a z b x b y b z. And then we have a standard way of writing down what determinant is, but let us look at what determinant really is. It says take every possible combination of an element from the first row from the second row; from the third row such that, you do not repeat the row. And you do not repeat the column and when you do that make sure that, you always go in the 1 2 3 direction. If you have to go in the 1 3 2 direction, that is you have to go in the other direction then put a minus sign. Let us see if this is true in the way we do it. For example, we say x hat a y b z minus x hat a z b y. So, we took the first row first column, second row second column, third row third column went 1 2 3. So, plus sign, then we went x hat a z b y.

So, we went 1 3 2, so I have put a minus sign, what about the next one. Y hat a z b x minus y hat a x b y, what did we do? We had y hat. So, we had taken up the second column and the first row. Then we did a z b x and a x b z, we cannot do a x a z, because that is not using the third row. We are doubly using the second row; we cannot do a z b z, because that is doubly using the third column. So, they are the only 2 choices y a z b x y a x b z, now if you look at this it is 2 3 1. All these 3 numbers 1 2 3 2 3 1 3 1 2, they are all going in the cyclic direction. So, if I wrote in a circle 1 2 3 1 2 3 goes this way 2 3 1 goes, this way 3 1 2 goes this way. All of them go in the same direction, so they are given the positive sign whereas, 1 3 2 has to go the opposite direction, and therefore I give it a minus sign, similarly, 2 1 3. Similarly, 3 2 1 you can check it for yourself. 2 1 3 for example, goes this way. 3 2 1 goes this way. So, it is just a matter of how you go through the numbers 1 2 and 3 and that is what gives you the signs. Now, there is a symbol which represents all of this.

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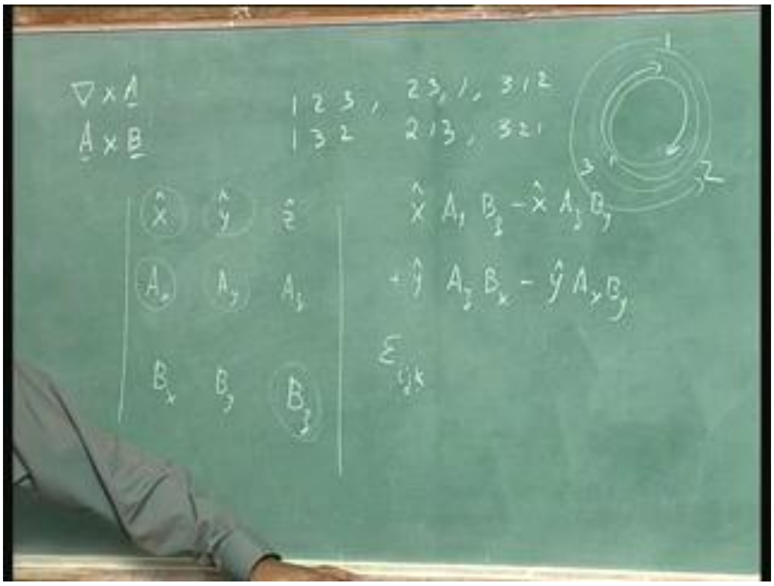

Specifically, it is called epsilon I j k, it is called the unit anti symmetric tensor. It is a big name for something very simple, it is equal to 0 if any 2 indices are repeated. So, if we have 1 2 2 or 2 3 3 then it is 0, they have to be different indices for it to be non 0. It is equal to 1 if cyclic; it is equaled to minus 1 if anti cyclic. Now, if you had this epsilon i j k then clearly the determinant of A cross B is nothing but sum on j k I is of I unit vector times this epsilon i j k a j b k. Because if you look at it, you are using e sub I a sub j B sub k and depending on whether you clockwise or you go counter clockwise. You put a plus sign or a minus sign and that is taken care of by the epsilon I j k. Now, this sum actually, goes through a lot of cases, we do not care about. For example, I equal j equals k equals 1. We do not want that, but that is where epsilon i j k is equal to 0. So, this is a short hand way of defining cross product. It does not look short, but it is, we also suppress, this sum. It is assumed that if I appears twice it is summed over j appears twice it is summed over and if k appears twice it is summed over.

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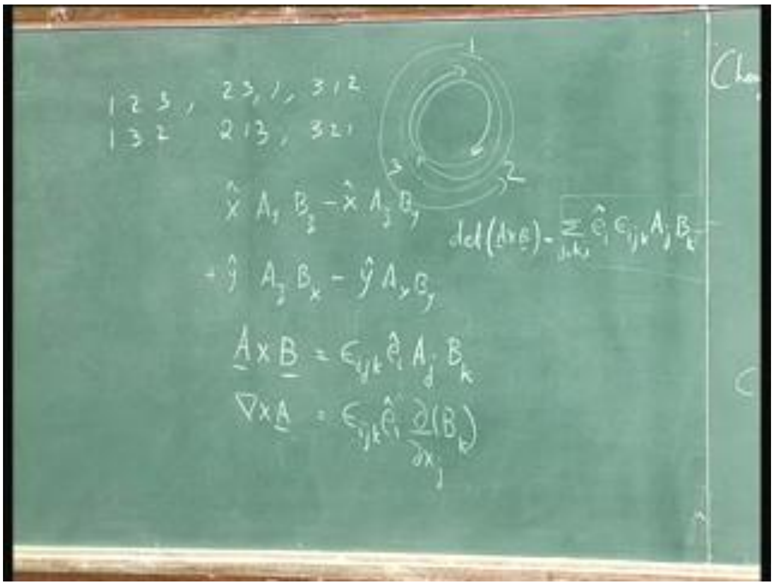

So, we write a cross b is equal to epsilon i j k unit vector along I j th component of a k th component of b. What happens with curl? Curl of B is the same thing, It is epsilon i j k unit vector along I derivative along j of the kth vector, because it is after all just the matrix. So far, so good, but now, there is something new, this epsilon i j k has a lot of symmetry properties. Because of its symmetry properties, you can simply a lot of expressions. But before I do that, let me also introduce some other things.

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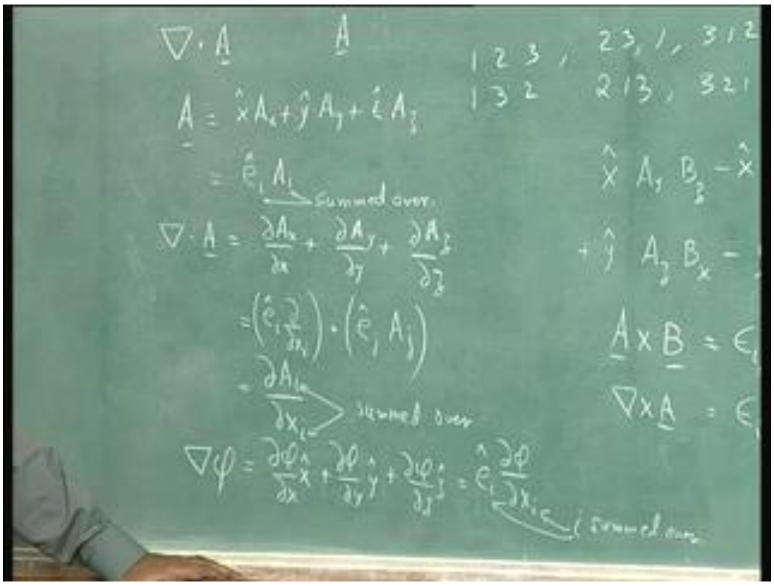

First of all how do I represent divergence? And how do I represent a vector itself. Well a vector A is equal to x hat A x plus y hat A y plus z hat A z. So, I represent in the same notation by saying, e sub I a sub I meaning e one along the first direction times the component along the first direction. Second unit vector times component in the second direction plus, third unit vector times component in the third direction. So, this is my vector and what is divergence? Divergence of a is equal to del a x del x plus del a y del y plus del a z del z. Well it is equal to if you want to write it out it is equal to e I del del x i dot producted with e j A j. That is, this is the gradient operator, this is the vector and if you take the gradient operator and dot product, it with the vector it gives you this result. So, what is it equal to, it is equal to del a I del x I, because e I dot e j is 0 unless I is equal to j. So, again I appear twice, so summed over, similarly here I appear twice summed over. What about gradient of a scalar?

Well, gradient of phi, It is equal to del phi del x along x plus del phi del y along y plus del phi del z along z. I used this form, which is e sub I times del phi del x i. This is my vector component in the I th direction, so this is gradient again I i. So, I is summed over, this is usually called Einstein notation. It is, because when Einstein introduced relativity mathematics became so complicated. We had to invent shortcuts, but he did much more complicated things and we are going to do very simple things. But nonetheless, this notation is very useful, it is very useful, because it is very compact and it turns out there is a lot of things you can do. And you can see lot of simplifications, which you cannot see if you are going to write out everything in great detail.

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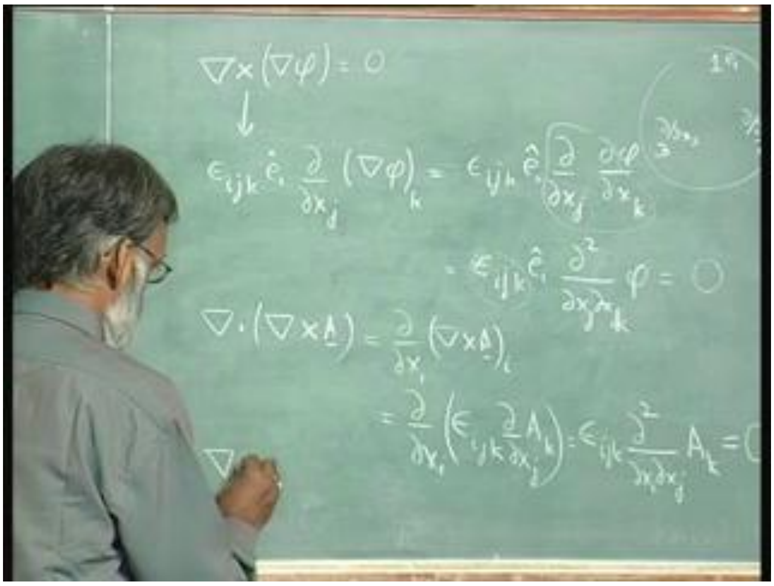

Now, let us see if we can derive some of our older results, we said that curl of gradient phi is 0. We did that by showing that you have 2 identical rows and therefore, second partial derivatives are continuous. It went to 0, but can we do it in using our new notation, well what has this become. This becomes, epsilon i j k e sub I del del x j of gradient phi in the k th derivation. That is coming out of the definition of curl, definition of curl is epsilon i j k unit vector along I del del x j vector along k. So, unit vector along I del del x j vector along k, but this vector along k is actually, del phi del x k, because that is what gradient is. So, now you have this expression, you have these 2 derivatives. So, you can write it as e sub I well you can write the epsilon i j k out, E sub I del square del x i del x j acting on phi.

Now, if i and j are equal epsilon i j k is 0. If i and j are not equal we know, that sorry j and k. We know, that when j and k are not equal, you can have i j k or I k j. That is you can go through your 1 2 3 cycle. Let us say this is e 1 and this is del del x 2, this is del del x 3, you can either go through it this way or you can go through. It this way and when you go through, it this way you get a plus 1, when you go through, it this way you get a minus 1. And therefore, this term combined with this epsilon i j k automatically, gives you a 0. The anti symmetric properties of epsilon immediately, tell you curl of gradient of phi is 0. What about divergence of curl, well what is divergence? Divergence is del del x i of curl of a component along i.

So, put that down, del del x i times epsilon the I th component is epsilon i j k del del x j a sub k. Why, because if you go back here, this is the definition of curl, now, if I want the Ith component, it is whatever is multiplying e i. So, I remove e I and the rest of it is the Ith component, which is what I have written. Epsilon i j k del del x j a k. So, I can write it out again. Epsilon i j k del square del x i del x j of a k, almost the same thing, but not quite. I have my first 2 indices rather than my last 2 indices, but epsilon does not care; Epsilon does not like any of its 2 indices to be the same. So, i and j cannot be equal and if you interchange any 2 indices sign changes. So, i j k and j I k give you opposite signs. Both of them are present, and therefore whatever you get out of one term is cancelled by what you get of the other term.

So, again it is equal to 0, so we have proved that curl of gradient of phi is 0. Divergence of curl is 0. Now, what we will prove? Next time is what we develop; next time is a technique for doing things like divergence of a cross b or curl of curl of E. These are going to be very, very important when we get to the next stage in electromagnetic. It is even important for proving this very simple fact about magnetic energy. And it is only, because we have not developed these techniques. That I have been you know just talking instead of deriving, using this simple structure we can. In fact, make all the vector calculus calculations trivial and if you know it. You do not need to memorize any of the formulas at the end of the book. Your book has all sort of formulas for getting the divergence the curl then vector identities none of them are required. We have already, figured out how to get many of these results. The remaining results will come out of using this epsilon i j k concept. So, I will complete that next time.