ElectroMagnetic Field

Dr. Harishankar Ramachandran
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture - 28
Magnetic Energy

Good morning. Last time we had looked at Faraday's law, and I had made the point that Faraday's law comes out of Coulomb's equation. I am going to look more deeply into this and develop the connection between Coulomb's law and Faraday's law and see that there is nothing at all strange about what we are saying in the behavior of electric and magnetic fields when there is time variation. So, let me remind you what we did last time.

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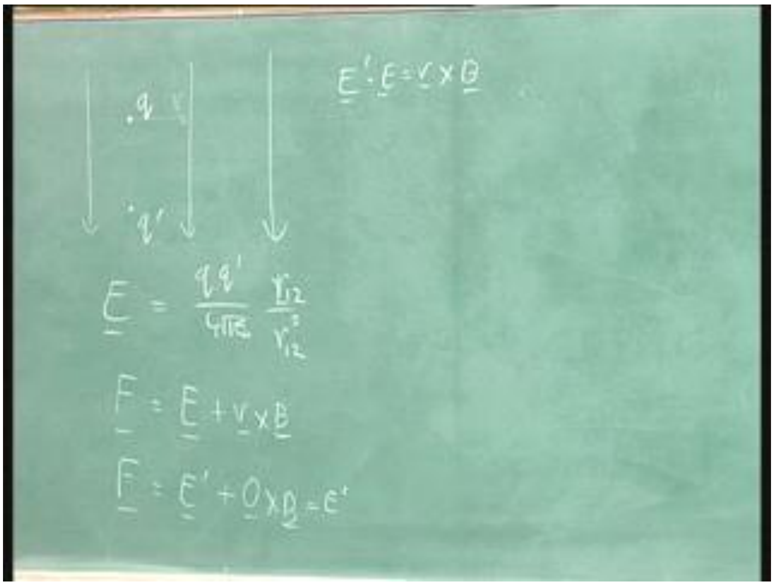

I said that supposing we have two charges q and q prime and supposing we have a uniform magnetic field. I talked about how the force between these charges which is Coulomb's law gives me an electric field, which says q q prime over4 pi epsilon naught 1 over r 12 cubed the inverse square law, and you also had a force equation which said f is equal to e plus v cross b.

Now this e is valid when things are stationary. Supposing, this charge is moving with a velocity v in the frame, where this charge is moving with a velocity v, I have that the electric field is e plus v cross b. The electric field is e plus v cross b, because this charge is stationary. So, the electric field everywhere due to this stationary charge is known. V cross b is known. So, I have an answer. But if I moved with this charge q, then the same force f is equal to some other electric field plus 0 cross b, because the velocity is 0. And, therefore, it is equal to e prime.

And since it is the same force whether I move with the charge or I remain stationary, I concluded that e prime minus e is equal to v cross b. That is there is an additional electric field which is equal to v cross b. Now this is just a requirement from Coulomb’s law and the fact that we cannot have measurements telling us different things depending on whether we are sitting down or we are walking. Measurements must say the same thing. It does not matter what the scientist is doing. Now I am going to take a slightly more complicated problem and try and work out the same thing and try and understand a little bit more about what coulomb’s law is telling us.

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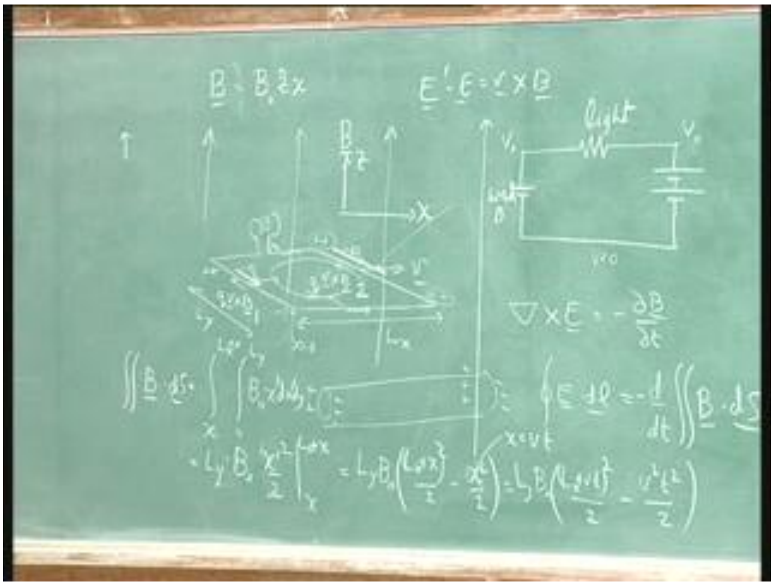

I start with a loop. Let us say it has a bulb, this loop as a whole is moving with a velocity v, and well, let us say there is a well magnetic field b. Now this magnetic field is uniform; well, you can see that this part of the loop v cross b is going to give me a force in that direction for positive charge q. This part is also moving with the same velocity. So, again I am going to get a force the same force q v cross b. This part of the circuit, the velocity is along the circuit along the wire. So, v cross b is going to be in this direction. Similarly, velocities along the wire v cross b is going to be 90 degrees to the wire.

Now you take consider the electrons in this circuit. The electrons have negative charge. So, along this wire, the electrons are feeling a force opposite to this q v cross b which is in this direction for electrons. So, the electrons are pushed from here to here. Out here the electrons are not pushed along the wire; they are pushed sideways. See the wire were like this, the electrons are all trying to go to the other side of the wire. So, there is minus sign and there is plus sign.

Now that is not productive, because if you got a wire like this if this side becomes slightly negative and this side becomes slightly positive, there is no current that is going to flow as a result. The same thing is going to happen on this wire, because the force is at right angles to the direction of the wire. On this leg of the circuit, the force and the electrons is again in this direction. It is minus e, because the charge of the electron is minus e. So, once again electrons pile up here. There is positive charge here.

So, I am going to have positive charge, negative charge, negative charge, positive charge. So, if I have to model this, I will write this circuit as a battery another battery connected with this is my light. This is q v cross b. This is q v cross b. Now, of course, I have written q v cross b, but what is this q? We have to ask what kind of battery is set up. We have to ask how much force is there over the whole wire. So, this q will end up being the number density of electrons times the charge of the electrons times the length of this wire.

And of course, the electrons are pushed this way; the ions or the remaining atoms are left behind, and that is why you have positive charge here and negative charge here. It does not really matter what the value of this q is. What matter is this electric field and this electric field are exactly equal which means if we imagine v equals 0 here, it is some v naught here. It is also v naught here and therefore, no current will flow. This lamp will not light.

Now let us think of it in terms of Faraday’s law. Faraday’s law says curl of e equals minus del b del t or loop integral e dot d l is equal to d d t with a minus sign surface integral b dot d s. Now what is b dot d s? It is just this magnetic field b minus d d t of some b naught times area. B is uniform; it is not changing in time. Area is constant because this circuit is not deforming. So, it is equal to zero. So, what it means is if you integrate right round the circuit, it is telling you the net battery if you go right round is equal to 0 because of which these two batteries essentially cancel out. And you are left with a light, but no power source to drive a current through it. So, it is saying the same thing. This picture and this picture are consistent, okay.

Now let us change this problem. I have this same circuit, but I have decided to make my magnetic field variable in strength. Magnetic field is weak getting strong, getting stronger, getting stronger, getting even stronger. Now what is going to happen? If I look at it from this point of view, I have weak b strong b. That is in this side, magnetic field is weak. On this side, magnetic field is strong. The force q v cross b is still there, but here I have to say q v cross b 1 and q v cross b 2, because the magnetic fields are not the same.

Now this v cross b the velocities are the same. Magnetic field is stronger which means that the force experienced by the electrons is strong here but weak here. Now we can draw our circuit diagram differently. We can say that we have a small battery here, but we have a strong battery. If we have a strong battery and a weak battery; obviously, part of the e m f is going to get cancelled out, but part will be left and because of the part that is left current is going to flow. The current, of course, is in the opposite direction of the flow of electrons.

So, the current flows this way and the lamp will light. Can we understand it from this point of view? It is the same thing because if you look at the surface integral of b dot d s, but we have to first define what we mean by b. So, let us say b is b naught along z direction, but it is a function of z. This is the z direction, and this is the x direction. So, it is b naught x in the direction z. So, for larger x it is stronger. For smaller x it is weaker.

Now we now want to calculate this surface integral; let us assume that this has some length l x. This has some length l y. So, the surface integral b dot d s is equal to integral 0 to l x integral 0 to l y b naught x d x d y. And so, it becomes l y times b naught times x d x from 0 to l x which is l x squared over 2. Now this is if the loop is starting at x equals 0. But as the loop moves, it is no longer going to be at x equals 0. It is going to be at some other x in which case this integral becomes integral from x to l x plus x, and let me put some primes to indicate these are dummy variables.

Then the integral is b naught. The integral in y is still l y, but the integral in x becomes x prime square over 2 between x and l x plus x. So, it becomes l y b naught times l x plus x square over 2 minus x square over 2. So, this is the surface integral of b dot d s, and it is now a function of x. Now this x is really a function of time. X is nothing but v times t, because this loop is moving with a velocity v in the x direction. So, through this, surface integral b dot d s is a function of time. I can replace x by v t, and I get l y b naught times l x plus v t squared over 2 minus v square t square over 2.

Now if I take the time derivative, I will get a term from here, and when I get that term, I am going to get exactly the same answer, but this answer does not tell me exactly where the induced electric field appeared. It tells me that if I go round this whole circuit, this is the total amount of flux that is changing, and, therefore, this is the total induced e m f. I will leave to you the calculation here. It is trivial. You can now take the time derivative. The two will go away and you will get l x plus v t on one side and you get v t on the other and so, you get a constant e m f. That constant e m f will drive a constant current, and, therefore, the light will turn on.

Now there is yet another thing that we can do. We have found that if this loop moves at constant velocity, we are going to have a v cross b force and we can understand that v cross b force from Faraday's law; they are the same story, but supposing in this picture we moved with the circuit. Suddenly, the circuit is not moving because we are moving with it. So, it seems like it is still. But now what is happening?

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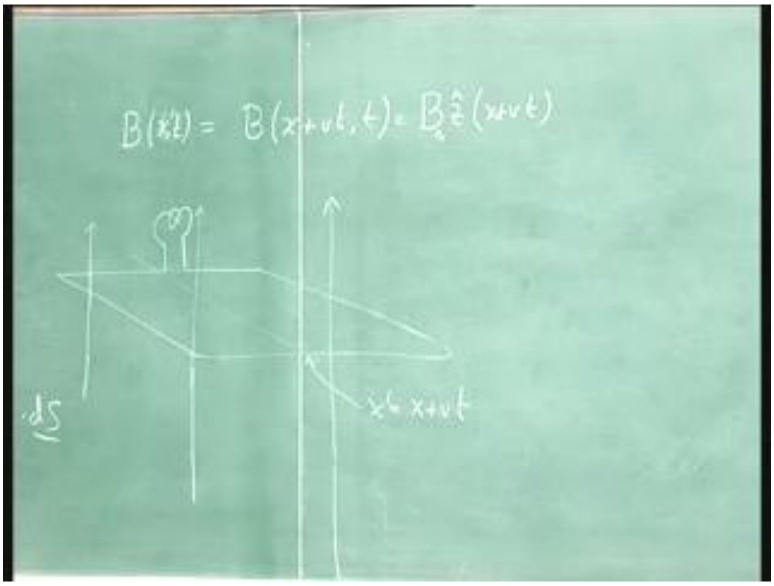

What is happening is my circuit is here. At t equals 0, I have a magnetic field like this. But as time progresses, this magnetic field is getting stronger. So, instead of being movement of circuit through a non-uniform magnetic field, it is now a stationary circuit, but b is changing in time. How is b changing in time? B of x t in this new coordinate system where I am moving with the circuit is actually equal to, in order to make it better I will just put x prime t is actually equal to the original b at x plus v t comma t because I know that this circuit is moving in time.

So, what I think is change of b in time is nothing but measuring b at different positions in space. Let me repeat that. I have a magnetic field that is not a function of time, but it is varying is space. The circuit is moving in space, and as it moves, it sees different strength of b. But supposing I moved with the circuit, now I do not know the circuit is moving, because as far as I am concerned, it is staying still. But what am I seeing? I am seeing an ever stronger magnetic field in time.

How is this? Because if I fix any particular x on the circuit and I measure the b there; that b is nothing but in the original picture, it is this x prime would actually be x plus v t, because in time, I am moving with velocity v. So, it will be the original magnetic field at x plus v t comma t and the original magnetic field do not depend on t at all. So, it is equal to b naught along z into x plus v t. It is only a function of the first coordinate. It is a function of time through the velocity. But in this circuit which is staying still because it is staying still there is no v cross b force. Even in this circuit, I must see my light bulb light up, and it must light up in the same way. Well how can I do that?

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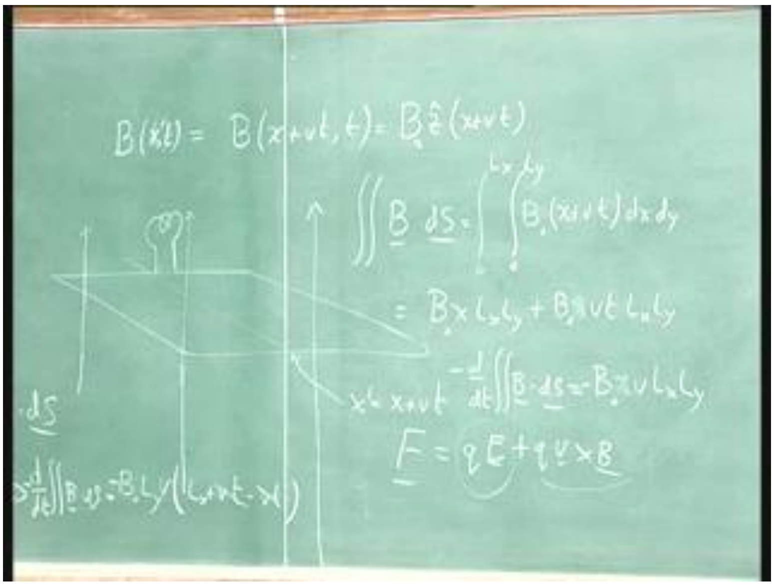

Let's calculate what surface integral b dot d s says. It is saying integral 0 to l x integral 0 to l y b naught times x plus v t d x d y. Now I do not really care about the constant part, because I am going to take the time derivative of this. But I will get one term b naught x times l x l y. It does not matter. It is just a constant plus b naught x v t times l x l y. Now I take the time derivative, what do I get? Minus d d t of surface integral b dot d s is equal to minus b naught x v times l x l y. Sorry, there should be no x here V naught v t; so, b naught v l x l y.

Now if we go back to this picture and take the derivative, what do we get? We get this implies d d t minus sign of surface integral b dot d s is equal to what? Well, I get the b naught l y. Nothing can go wrong there. I will get a minus sign from here and then I will have to take the derivative. Derivative of each term is twice this factor times the time derivative of that factor. Time derivative will give me a v times l x plus v t minus v t. The v t is canceled out, and it is after all the same answer. It has to be the same answer.

If it is not the same answer, the same current will not flow. If the same current did not flow, it would mean that whether I was sitting in a car or I was sitting on a stool, the behavior of the circuit would change and that cannot be. The circuit does not care what the observer is doing. Circuit does what it does. So, now you can see that there is some internal consistency about this whole picture. You start with Coulomb's law and let motion happen. The moment motion happens, you have this fundamental equation, and because of this fundamental force equation, you conclude that the v cross b force must be matched by an electrical force.

So, when there is motion, it is v cross b. When there is no motion, it is called electrical force. There are not two forces. There is only one force. They just trade off. So, then when you look at a circuit that is moving in a non-uniform b field, you can immediately see that the difference in v cross b's can be understood as change in flux. That was quite easy. But now you are seeing something else. In a non-uniform b field, if you move to the speed of the circuit which means v goes to 0 that same change in flux is now showing itself directly as electric e m f. So, we are seeing three different things.

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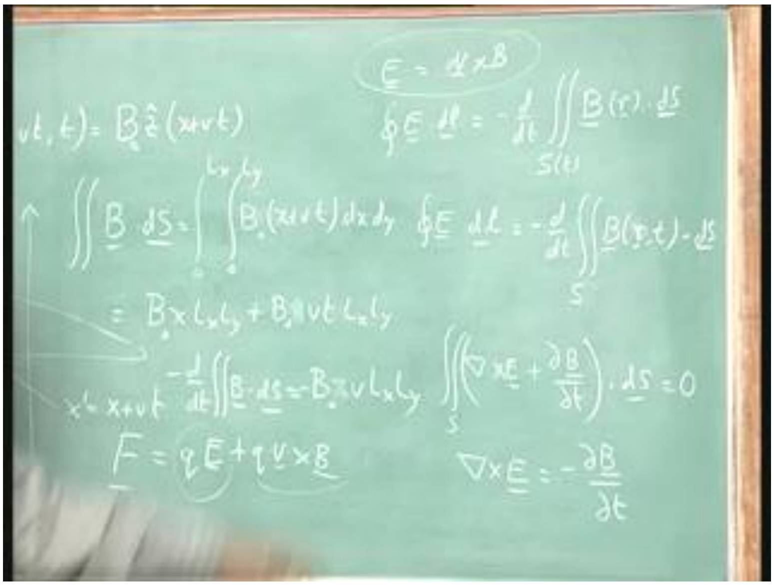

We are seeing the e equals v cross b. That is the direct motional e m f. Then we are seeing loop integral e dot d l is equal to minus d d t of surface integral of a moving surface b which is only a function of position dot d s. That was the moving circuit in a non-uniform field. That gave me an answer, and we found that this was also equal to minus d d t of a constant surface, but now b was function of r as well as t dot d s. So, both these results, the result where b is a function of t and therefore, you get an e m f or b is not a function of t, but the circuit is moving and therefore, you get e m f.

Both of them come from e equals v cross b. It is all motional e m f. In fact, all of Faraday's law is motional e m f. It is a very important thing to understand because it simplifies our understanding of Faraday's law completely. Otherwise, you end up learning three different pictures of Faraday's law. I am thinking of them as three separate formulas. It is not three separate formulas you have to learn; it is just one formula, and that one formula is whichever one you want to call it generally dependent on time. And as I did last time, all you have to do now is take the d d t inside.

Let us assume that the surface is not changing. Let us take a constant surface. Take the d d t inside. Apply Stokes' theorem and you get surface integral curl of e plus del b del t dot d s equals 0, and from that, we get one of Maxwell's important equations called Faraday's law. Actually Faraday's law ought to be this equation or this, and this is Maxwell's way of writing the same equation. They are equivalent and I will show later that they are, in fact, you can derive this equation from here. It is by now obvious I hope that starting with motional e m f, we can derive these rules, and starting from here, this is obvious.

There is one very important topic that Faraday's law allows us to work out. Up to now, we have not talked about energy in the field. We have talked about energy in the electric field, but we never talked about energy in the magnetic field. The reason is actually very obvious.

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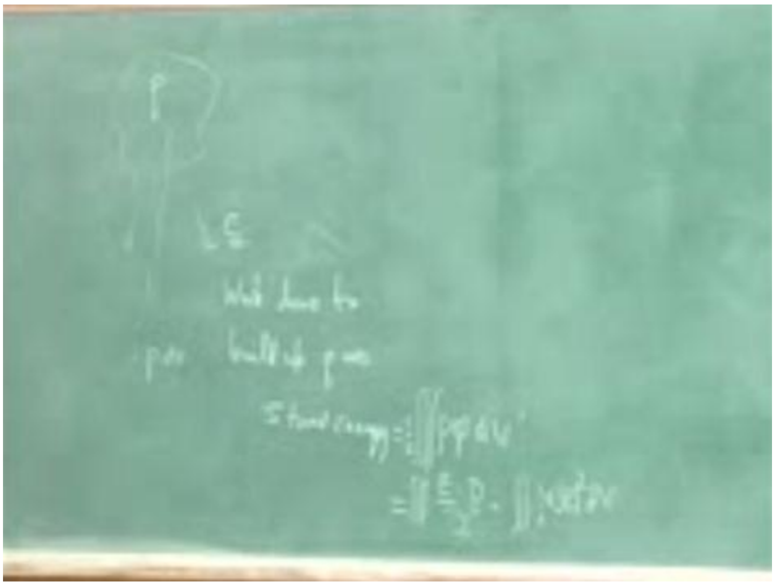

If I take any charge distribution row and I take a charge little bit of charge row d v and I add it to this row. I know that because of this charge density there is an electric field, and I also know that when I push this charge from infinity to wherever it belongs, I have to do work. So, I can work out work done to build up row which gave me a result which was that total energy or stored energy was equal to volume integral one half row times phi d v. And then using Poisson's divergence theorem and the gradient relation between phi and e, you can show that this is equal to volume integral e dot d divided by 2.

This comes from doing integration by parts because divergence of d is row and gradient of phi is e. So, if you work on it, this is what you will get or more commonly you will see volume integral one half epsilon naught e square d v, and if it is a medium, you will get epsilon in it. So, we have done this derivation for electrostatics and the primary way we did it was to work out how much work we had to do to build up the charge. However, it is not clear how we worked out how much work we did to build up the current. There was not charge to be built up in the case of magnetic field.

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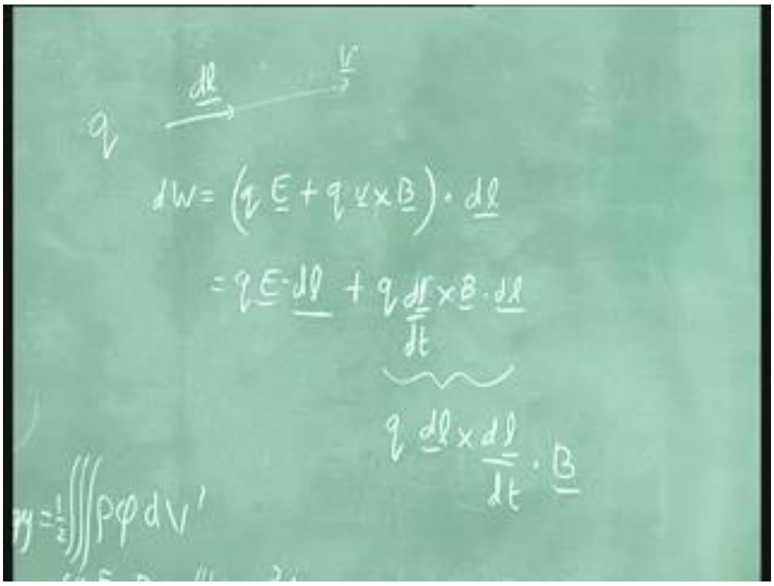

And even worse there was a problem that if you moved a charge q from one point to another if you moved it at distance d l, then you know perfectly well that the amount of work done d w is going to be equal to q e plus q v cross b dot; this is the force dot d l. So, we did electrical work q e dot d l. So, if the electric field was aligned with the direction we moved, we had to do work or we had work done on us, but what about this?

It became plus q v cross b dot d l. But what was v? This charge is moving in space and the rate at which it is moving is v. In other words, it is d l d t. That is what v is and this is the problem. You can change this triple product and write it as you can always rotate. So, you get d l cross d l d t dot b. The cross product of d l and d l d t is 0. By that I mean if the charge moved in a particular direction; obviously, its velocity is also in that direction. Charge cannot move in one direction and have a velocity in another direction.

So, when you do v cross b dot d l, you are not doing any work. So, if you assemble charges, you do not have to do any work to assemble them as far as the magnetic field is concerned. Only the electric field does work on a charge, and we have already taken into account the work done by the electric field that came into our definition of capacitance. So, stored energy became one half c v square.

Now the question is what do we do about magnetic field. Is there stored energy in the magnetic field and if there is, where does it come from? Well, the answer of course, is there is stored energy in the magnetic field. There has to be and what is more. It is all coming from Faraday's law.

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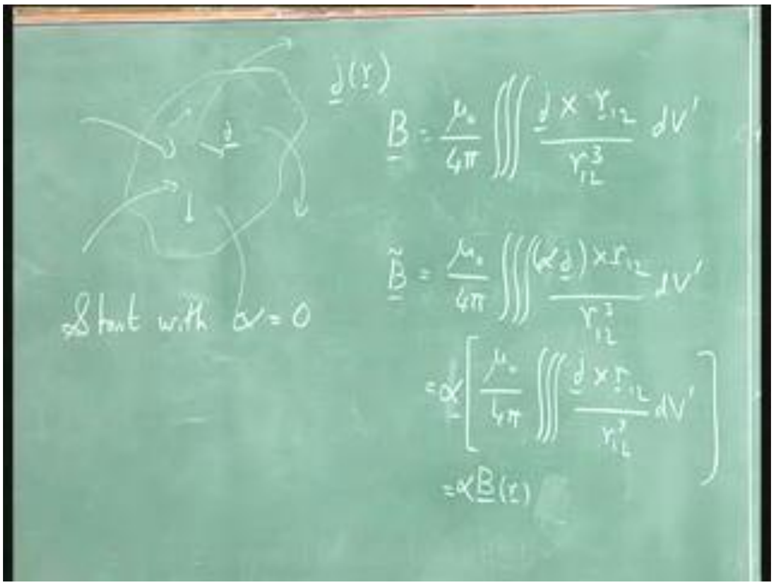

To take a look at that, let us see how it will work out. Let us say that I have some area where there are currents. So, I have current in space j of r t. I am going to remove the t dependence J of r. I know that given currents, I can calculate the magnetic field. This is the Biot-Savart law. So, I have currents, because of the currents I have the magnetic field. Now the question is how do I calculate how much work it took to bring those charges to where they were. The first thing I see is supposing I have this current distribution.

Supposing, everywhere I change that current distribution from what it was to 50 percent of what it was, then I would get a new b field. Let us call it b tilde. It is going to be mu naught over 4 pi volume integral 50 percent of j cross r 12 divided by r 12 cube d v prime. But this 50 percent this factor of half is a constant. I can pull it out. What is it? It becomes half of mu naught over 4 pi volume integral j cross r 12 over r 12 cubed d v prime.

Now what I have got in the square bracket is the original expression which means this is equal to the original b as a function of r divided by 2. So, if I took the currents that generated a magnetic field and reduced every current by the same amount brought it to half its value, I get back the same magnetic field except ay every point it is half the strength and half is only a number. I can take this whole thing and say I will multiply it by some number alpha and then I will just get alpha multiplying out. I will get alpha times b.

So, if the magnetic field had a particular shape, this shape will remain. All that will happen is it will become weaker or stronger by the factor by which I make the currents weaker or stronger. What I will do is I will take this idea and I am going to build up my currents from zero. So, I will say start with alpha equals zero. There is no current. There is no field. I will assume it took no energy.

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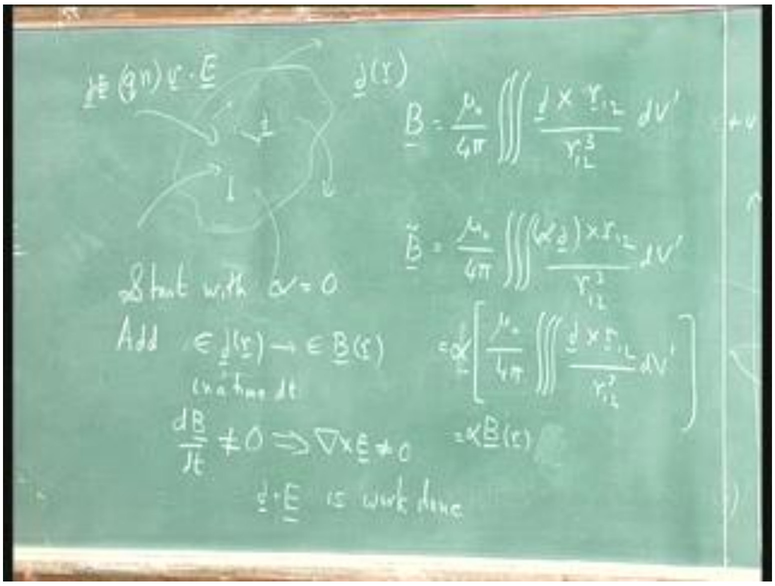

I add a very small amount. I call it epsilon j of r bringing it from infinity. Now when I did this when I brought epsilon from infinity, the magnetic field went to epsilon b of r. Now I must do this in time. So, I took in a time d t, okay. Now the thing is if I took a time d t to bring this charge this current in, in that time d t this magnetic field developed which means there is a d b d t. It is not 0, and I already know if d b d t is not 0, it implies curl of e is not equal to 0.

So, I must be in order to bring this current in, I have to create an electric field. Now what does this electric field do? At the point I am bringing in this first bit of current, the electric field does not do anything, because there is no current to interact with it. But if there were a current, then what would happen is this electric field would, in fact, have to work on the currents because I know that if I have any current j and I have an electric field e, then j dot e is work done.

That is because j represents j is what? It is some charge times the number of charges times v. So, what is j dot e? J dot e is q n times v dot e. But e dot v being nonzero means that charge is moving in the same direction as electric field which means work is done. Either work is done on the charge or work is done by the charge, but it is no longer zero. So, this is how the electric field comes into the picture now. As I build up this magnetic field, there is a d b d t which means there is an electric field. This electric field in turn does work on the currents that are already there and in turn builds up the amount of energy in the field. Let us see if we can make it a little more precise.

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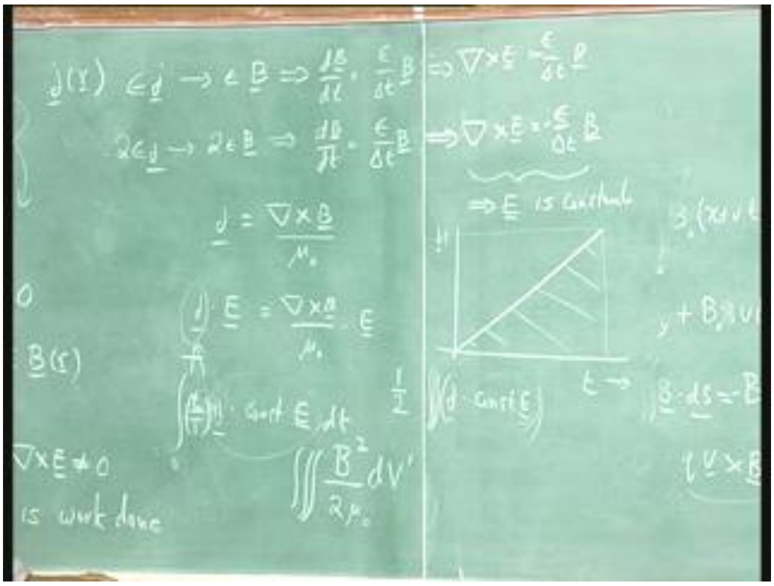

The first time I introduced epsilon j, I got epsilon b in a time delta t which implies d b d t which is equal to epsilon divided by some delta t of b. So, this time that was taken to bring in epsilon amount of b which implied in turn curl of e was equal to epsilon over delta t b. Now I bring in the next epsilon of j. So, there is twice epsilon b, but the amount by which b has changed is the same. So, d b d t there is a minus sign here is equal to still epsilon over delta t b. And so it implies curl of e is still minus epsilon over delta t b and so on and so forth.

I just keep doing this. So, what do I get? At every stage, curl of e is equal to epsilon over t b, but this curl of e implies an electric field that is constant. This current j is equal to curl of b over mu because of ampere's law. So, I now want j dot e which ends up being curl of b over mu naught dot e and some mathematics is involved. But what is important to notice, j is growing. J is some alpha j dot constant e and I am integrating this in time. Alpha is a function if time. Once again, I say once again because I followed the same argument for capacitance.

What happens is this alpha, the amount of current that is present; this may be the total amount of current present. I increase this current linearly in time. So, this alpha of t is nothing but t over capital T. It is the fraction of current that is present at any time and if I do the integral from 0 to t, I just get this the rest of it comes out. So, I get j dot some constant e integrated over all space multiplied by 0 to t of t over t d t, and that integral gives me basically a factor of half, because when I do an integral, I am basically only taking the area of the triangle as compared to the area of the rectangle.

So, this factor of half comes. The normalization constant of 1 over t will come. That 1 over t is basically related to the strength of the e field. The slower I turn on j, the smaller the d b d t and therefore, weaker the e. When you do all the math, what do you get? You get that the stored energy is b square over 2 mu naught volume integral d v. It is actually quite easy to do it using vector calculus. I am trying to avoid the calculus, because I do not see the point in doing elaborate formulae, because that does not convince anybody.

The thinking I want you to have is the following. I have to build up this current distribution bit by bit, and as I introduce the current steadily, what is happening is I develop a d b d t. I have to. I cannot add current without increasing the magnetic field, and as I increase the magnetic field, I have a d b d t. If I increase the current steadily that is if I use a linear increase of current, my d b d t is constant which means I have a constant electric field, because you can see curl of e is equal to a constant.

This constant electric field dot the currents that are present which is this current gives me the amount of work that is being done to add that little bit of current, because to add that little bit of current, I induced this uniform electric field and that uniform electric field did work on the existing current. When you integrate from beginning to the current time, what you end up with is an integral t over t integral 0 to capital T j dot e. And that integral t over t d t gives me the factor of half, and this capital T is nothing but a normalization constant. And so, I get half times the volume integral of j dot induced electric field.

Now you have to understand; this field did not come from Coulomb's law at all except in the sense that all of our electromagnetics comes from the Coulomb's law. It is not due to charges. It is due to time rate of change of magnetic field. This constant electric field dot j represents certain energy, and when you do the calculations, you get that energy is b squared over mu naught and a factor of two comes out, because of this area of a triangle being half the area of a rectangle.

When we do pointing theorem, the expressions will be clear. Clear that is as in the sense they come out of vector calculus, but I think the expressions are not terribly important. What is important is to understand why it is so, and it is so because to build up any current distribution, you have to build up the magnetic field, and to build up the magnetic field, you have to have induced voltage. The induced e m f does work on the very current that you are trying to increase, and, therefore, you end up doing work which is reflected in stored magnetic energy. This is the basic principle of inductance.

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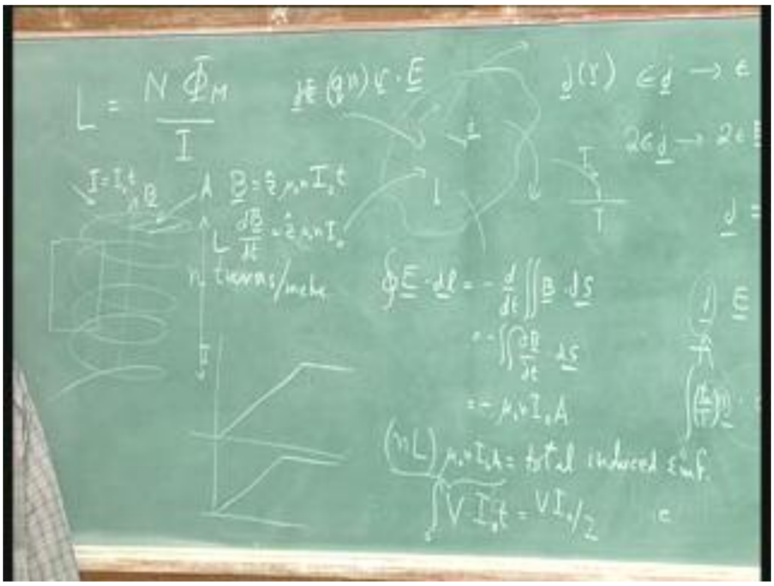

Earlier we had defined that inductance was equal to n phi magnetic by i, but that was not really a useful definition. It was just ad hoc completely arbitrary. What is more useful is to ask how much stored energy do I have in a system consisting of number of coils. So, supposing I have a solenoid. The solenoid has n turns per meter has a length l has an area cross section a, and let us say there is a current i. Due to this currents, a magnetic field is produced. And we know that using Stokes' theorem that the magnetic field induced is mu naught n i. B is along the z direction times mu naught n i.

Now if this b is now going to be built-up slowly by increasing this current and if this current is some I naught t. So, between 0 and ones I get I am steadily increasing I till I reach I naught. Then what is going to happen is my b is also mu naught n I naught t. So, both current is going to increase and magnetic field is also going to increase at which point it is kept constant. Now this magnetic field has a d b d t. So, d b d t is equal to z hat mu naught n I naught. I know now that there is an induced e m f.

So, loop integral e dot d l for one turn is equal to minus d d t of surface integral b dot d s which is since the loop is not moving minus surface integral del b del t dot d s. I already know my d b d t. That is mu naught n I n I naught. So, this becomes minus mu naught n I naught multiplied by the area. This is the loop integral, but there are n loops per meter. So, I actually have a net loop number of turns which is n times l times mu naught n I naught a is equal to total induced e m f.

Now this induced e m f has a minus sign. It is actually opposing the magnetic field that we are trying to build up, and therefore, we have to do work. It is like I am driving a current i, but this e electric field is trying to resist me. So, I have to drive a current through this battery. And because I have to drive a current through this battery, I have to do work on this battery. So, how much work do I do? The work I do is this induced e m f; this is a voltage v times the current which is I naught t. This is the power integrated from 0 to 1 which is v I naught divided by 2, and if you look through all the terms that are here that is how we get our inductance.

So, you can see that ultimately it is the same story. In order to build up stored energy in an inductor, I have to increase the current from zero. When I increase the current from 0, I am having to have an induced e m f. That induced e m f has to be overcome by my currents, and, therefore, I am injecting energy into this device. Where is that energy going? It is going into the fields, and specifically, it is growing into the magnetic field and the expression for it is b square over 2 mu naught. I will formulize this expression next time, and we will work out on some examples with inductors.