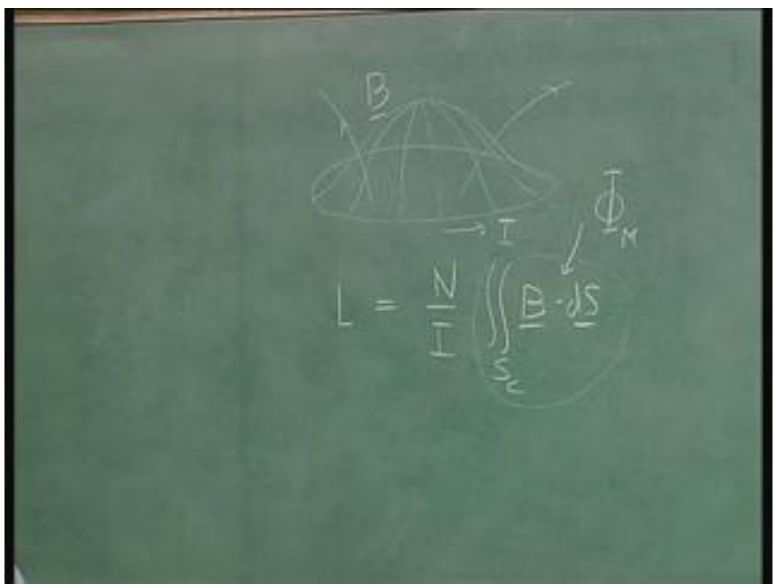
Electromagnetic Fields
Prof. Dr. Harishankar Ramachandran
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 26

Mutual Inductance

Good Morning. Last time I had talked about inductance and I would like to complete that discussion today and then revisit the crossed electric and magnetic field problem and with that I would probably come to the end of static magnetic fields. This will be the last lecture in magnetostatics, okay? Let us see what we defined inductance as being.

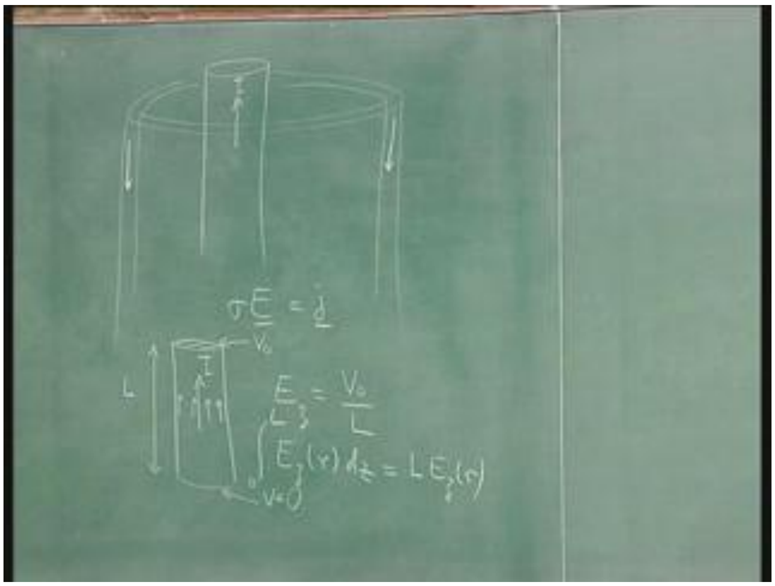
(Refer Slide Time: 01:35)



We defined that if you have a coil and the coil has a current I, there will be magnetic field threading that coil. You define any arbitrary surface that connects to that loop and then you define that conductance l is equal to one over i the surface integral b dot d s. If there are n turns at this coil, we put a factor of n. So, the idea is that if you have a surface integral b dot d s that links a loop then the particular surface you choose does not matter.

All that matters is the loop to which that surface connects and this particular integral is called magnetic flux which I will denote as phi n. So, the number of turns times the magnetic flux divided by current. Now we did calculate inductance for several simple cases. Now what I am going to do is to take one of those cases and look at it in a little more detail.

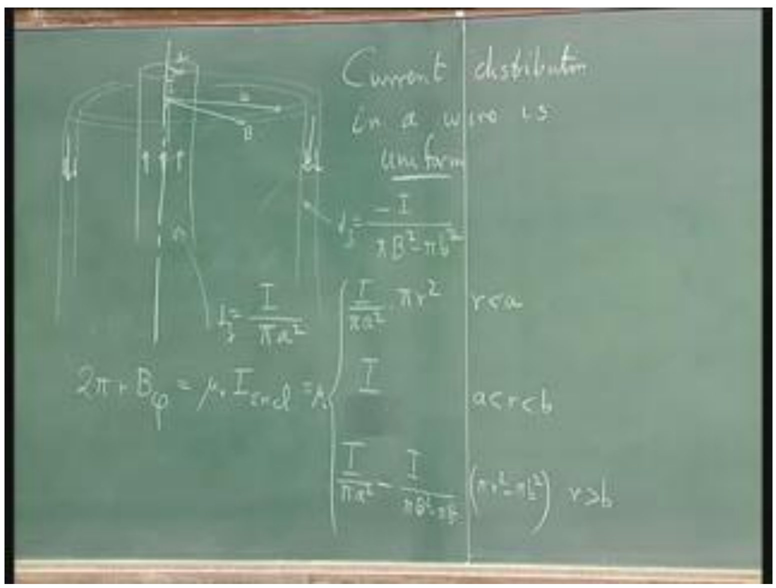
(Refer Slide Time: 03:24)



I am going to take the case where I has an inner conductor carrying a current i and outer conductor carrying the return current. So it is also got a thickness and the current is returning on that outer conductor, the magnetic field develops in the region. Between now the first thing that I would like to ask is, how is this current distributed within this wire. For that let us go back to just looking at a single wire. If I drive a current I, in a wire the wire has a conductivity or a resistivity which means that the current cannot flow without there being a voltage drop because sigma e is equal to j. So, if I take this, then it tells me that if i integrate this over a distance d l, I will get a voltage drop. So, what does that mean? Supposing if i put this top lead at a voltage v naught and I put this bottom lead at voltage zero and this height is some length l.

So, my electric field is going to e is along the z direction, e z is equal to v naught over l. Now this e z is basically going to be uniform. So, in other words it is the same e z here it is the same e z here and it is the same e z here. Because, if it were different e z's at different radial positions then over the same length, I will develop different voltage drops. That is if e z were a function of r then I could do an integral zero to l d z. That will give me the voltage drop but that integral would correspond to l times e z because e z does not depend on z. It depends on r only. But this is not allowed because the entire top of the wire is at voltage v naught, entire bottom of the wire is at voltage 0. So, e z cannot depend on r. It is a constant. If e z is not a function of r, neither is j.

(Refer Slide Time: 06:24)



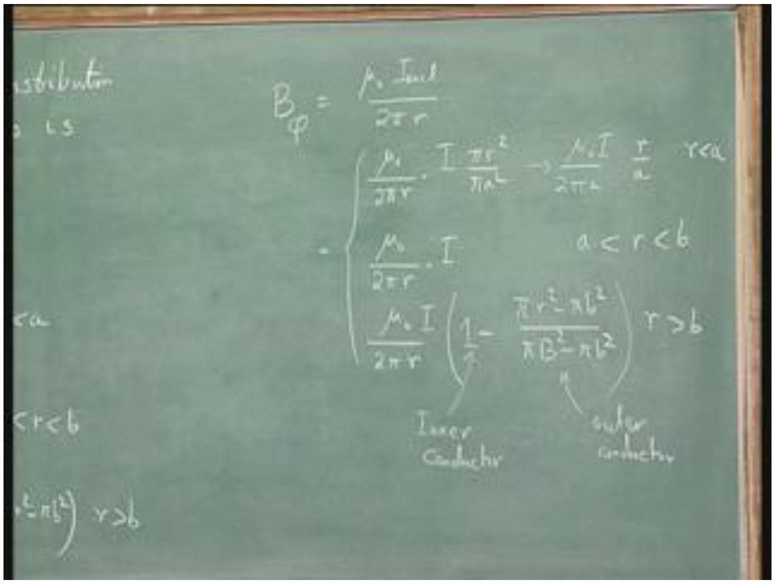
So the current distribution in a wire is uniform. The thinking just comes from the fact that if I apply a voltage the voltage drop is uniform and therefore for uniform conductivity the current density is also uniform. Now I am going to generalize the problem, I did last time because I am going to say that there is some magnetic field even inside the wire. So, when I calculate inductance and I say flux linkages. I am not only going to consider this flux, I am also going to consider the flux inside and then I am going to ask what is the total flux that is linked by this current. I am going to be slipshod because I want the idea to come through rather than be exact.

Now what I know is this current is a uniform current and the return current will again be uniform. We just worked that out. So now let us look at what the electric field what the magnetic field is going to be? In the wire the current j is going to be j z. It is equal to i divided by the area which is pi a square where a, is this radius. Now for the outside there is a radius b and there will be a outer radius, I am going to call it capital b. So, the return current is going to be minus i because this is in the opposite direction divided by the area of the outer conductor which is pi capital b square minus pi small b square. So, I have upward current in the inner conductor downward current in the outer conductor.

Now I can apply stoke's theorem. So, I get 2 pi r b phi, phi is the polar angle is equal to mu naught current enclosed. But, this current enclosed is going to be different things in three different areas. For r less than a, the current enclosed is going to be mu naught times this j i over pi a square times the area of enclosed by r which is pi r square, this is for r less than a. For the region between a and b the current enclosed is i and the for the region greater than b but less than capital b there is positive current enclosed here and negative current enclosed here.

So, it becomes i over pi a square minus i over this current pi capital b square minus pi small b square. This is the current density times the area which is pi r square minus pi b square. This is the current density. This is the area cross section over which the current density flows. This is for r greater than b. So, I have three different current densities and they therefore give me three different kinds of magnetic field. If I solve this, what do I get?

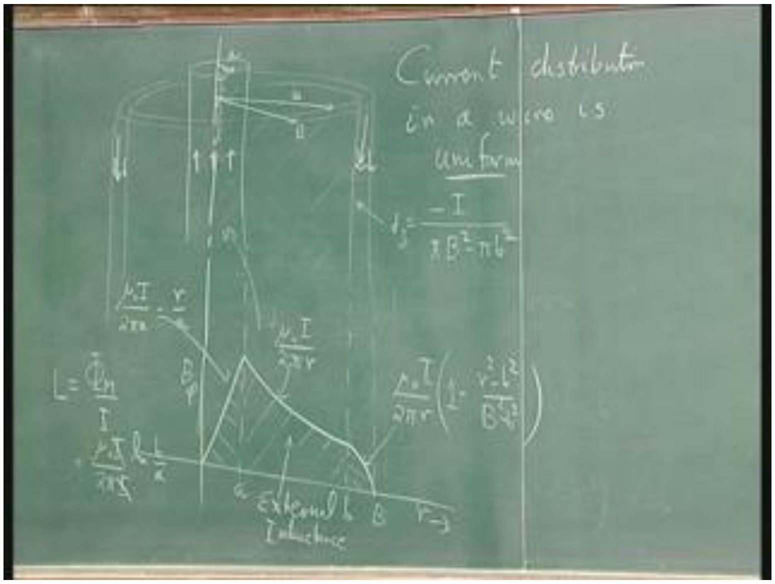
(Refer Slide Time: 11:20)



I get the magnetic field b phi is equal to mu naught, I enclosed divided by 2 pi r which is equal to mu naught over 2 pi a times i. I am going to take this I am going to take this and this and combine it. So let me write it out properly. Then may be it will be clearer, mu naught over 2 pi r times i, times pi r square over pi a square. This is the current enclosed and this is the mu naught over 2 pi r term. What I will do is, I will cancel out one r. So this becomes mu naught over 2 pi a times i times r over a. So the magnetic field actually grows from 0 to a. It is actually an increasing function of r.

Now, in the region a less than r less than b it becomes mu naught over 2 pi r times I, the total current i because r square becomes a square and the two terms cancel out. This is a less than r less than b and in the region beyond b it becomes mu naught over 2 pi r times i, times one minus pi r square minus pi b square divided by pi capital b square minus pi b square. This is the current in inner conductor and this is the current in the outer conductor, clearly when r is equal to b this cancels out, this becomes minus 1. So, one cancels with minus 1. There is no magnetic field. So if I had to plot this field let me go back here and plot it in this graph.

(Refer Slide Time: 14:09)

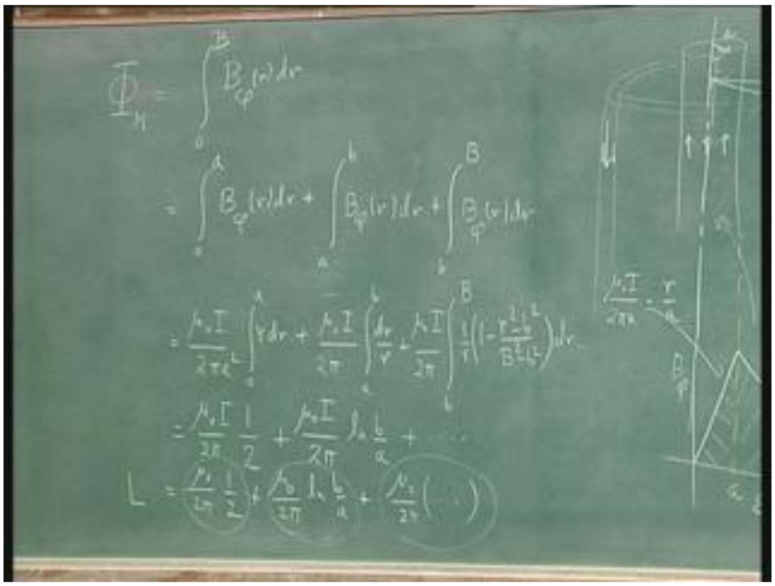


So, this is r equals zero. This is r equals capital b small b a. So, the magnetic field b phi initially increases. It increases linearly up to coming out of the wire. So, this is the portion which is mu naught i over 2 pi a times r over a. So it is increasing linearly. Once it is outside the inner conductor, but still in the gap then the current decreases. This is mu naught i divided by 2 pi r. So now it is one over r the dependence. Finally the magnetic field drops to zero and it drops to zero parabolically and if I am not sure I can draw it properly. I think it is like this. This is mu naught i over 2 pi r times 1 minus r square minus b square over capital b square minus small b square. This detailed curve I have not drawn. So I hope I have got the curvature, right?

So, this is the kind of curve I have and this is the magnetic field I have seen already. It is the magnetic field outside the conductors in the gap between them and last time we used this to define inductance. We said that the inductance l was equal to the magnetic flux divided by current. Magnetic flux is equal to the integral of this along r. So mu naught i over 2 pi log b over a divided by i. So the i cancels and I got mu naught mu naught over 2 pi log b over a. If I take a distance l then the factor of l also comes. This is what we had. Now this flux which is outside conductors this is called external inductance.

Well I suppose the flux is called external flux and the inductance due to this flux is called external inductance and that is what we calculated last time. However you can see that there is flux in here also and flux, here also and this magnetic field is also created by this current, so if you asked how much is the total flux then obviously you must add in this portion as well. So let us do that.

(Refer Slide Time: 17:49)



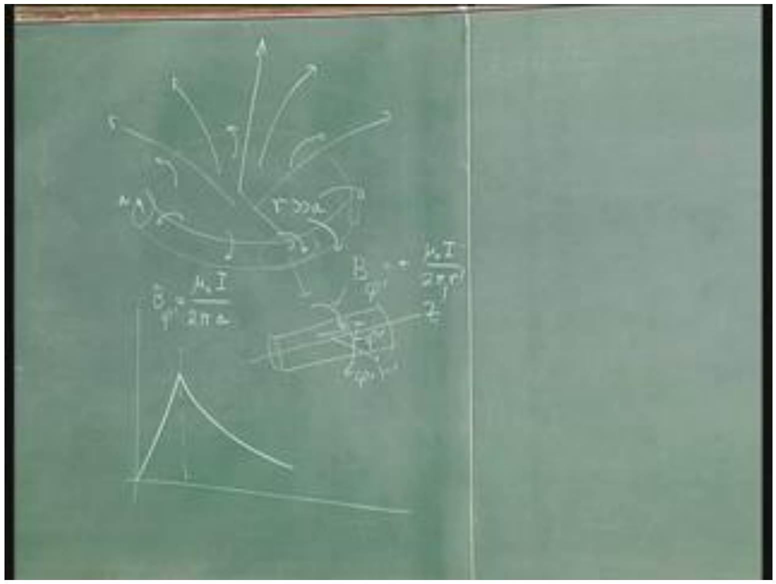
So, I will say that the flux phi m is equal to integral zero to capital b of b phi of r d r. I am taking per unit length of l in the z direction. So, it breaks into three parts. There is a part going from 0 to a b phi of r d r plus a part which goes a to b b phi of r d r plus a part going from b to capital b b phi of r d r. So, we write this, out what do you get? This b phi of r is mu naught i over 2 pi r over a. So, two pi a square integral 0 to a r d r. That comes from this factor mu naught i over 2 pi r over a square. So, the a square is here and r is in the integral plus mu naught i over 2 pi integral a to b of d r over r. That is, this piece one over r d r and the third terms which is plus mu naught over 2 pi integral b to capital b of one over r times r square r square minus small b square over capital b square minus small b square d r.

So, there are three terms. Each of them is multiplied by mu naught i over 2 pi but they have different dependences on a and b. If I write it out again, it is mu naught i over 2 pi. This becomes r square over 2 between 0 and a. So, it is a square over 2, the a square cancels with a square. So, you just get a half. This term gives me we have already worked this out last time. Integral of one over r is log r between the limits a and b. So that is l n b over a and this piece where it gives me a more complicated expression and I am not going to really write it out because it is not important. It is another term similar to this term. Now when I divide by current, I am going to get the inductance.

So if I divide by current I, get mu naught over 2 pi times 1 over 2 plus mu naught over 2 pi log b over a plus again a mu naught over 2 pi times a complicated mess of terms due to this integral. Now this term is the term we have already seen. This is our external inductance. This term and this term are the terms that we have not seen so far and they are the terms that correspond to internal inductance. Now why is internal inductance important. Well when you talk about magnetic field so far we have only talked about magnetic field as something created by current.

But, when we talk about faraday's law starting from the next lecture we will realize that magnetic field has stored energy and this internal inductance is quite important when you talk about stored energy and at low frequencies. This self inductance can become an important factor when you are trying to calculate the net inductance of any system. So, we have talked about two kinds of inductances; external inductance or the normal kind of inductance and internal inductance which is the inductance due to the fields within the wire or within the conductors themselves. And the important thing to remember with internal inductance is that it is always present and in fact it is a good thing that it is present.

(Refer Slide Time: 23:20)

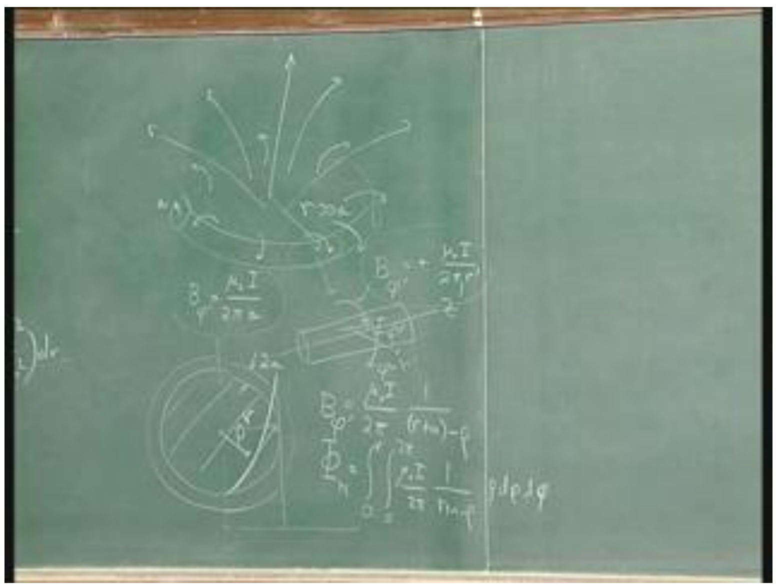


Because supposing you think of a coil a loop. Now, a loop actually has consists of a wire with finite radius some radius a. If you go very close to this loop and we assume the loop has a radius r which is much greater than a, if you go very close to this loop you know that you can approximate this portion alone as a straight segment. It is only true if r is much greater than a. But if it is then you can say locally it looks like a straight line and if it looks like a straight line I know how to calculate magnetic field. The magnetic field along this direction is this is z along that direction. The magnetic field is assuming i is in this direction, the magnetic field is going round and round that wire. So that magnetic field b in this new co-ordinate system.

Let me give it a name z prime r prime phi prime. In that coordinate system the magnetic field b phi prime is going to be negative minus mu naught i over 2 pi r prime. So, what does it mean? It means that I think my phi is wrong. Phi should be this way. So it is plus. It means that very close to this wire the magnetic field is actually going round. Further, inside at the centre for example the magnetic field is going straight up and if you go intermediate points it flaring out. Now, how big does this magnetic field become? Well, it becomes as the highest value it gets is when r prime is equal to the radius of the wire. We saw that already.

When we did the plot of the magnetic field, the magnetic field increased till it reached the edge of the wire and then it decreased. So, this is the maximum value that the magnetic field reached. So, the magnetic field at the surface of the wire b phi prime is going to be equal to mu naught the current in the wire divided by 2 pi a. As a, becomes smaller and smaller the maximum magnetic field becomes larger and larger. Now let us just take this as a given formula and let us try and work out what it would say for the inductance of a loop.

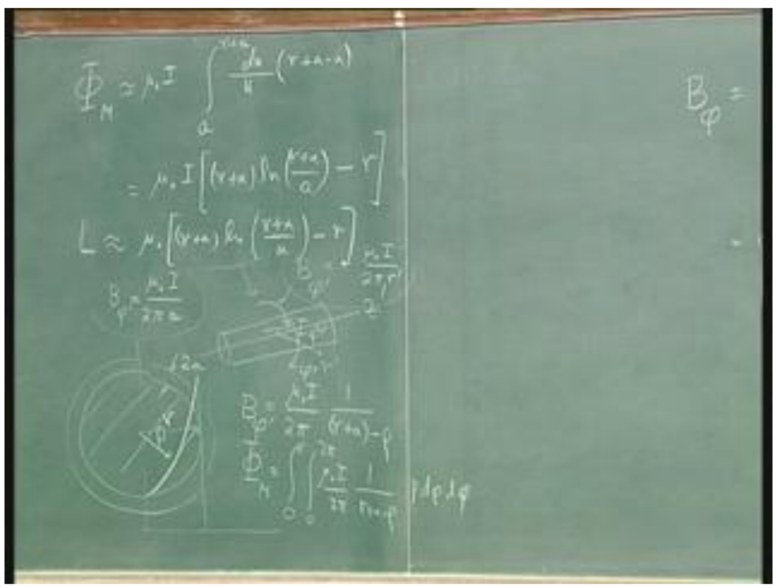
(Refer Slide Time: 26:27)



I am looking at it from above. So my loop is a circle. This is the thickness which is 2 a, and let us pretend that this formula is valid. It is not really valid except very close to the wire but away from the wire, it is actually an under estimate. Because of the bending of the wires this field will be higher than what I would get by using this formula, so using this formula I get that b phi is equal to mu naught i over 2 pi. This radius is r and the magnetic field is actually one over the reverse direction. So one over r plus a minus row where row is my general coordinate. So, if I have plotted b, I am saying that the magnetic field is maximum there and then comes down as one over r. If you believe this, it is actually magnitude wise quite correct, it is wrong by a factor of 2 or 3.

We can now work out the flux the magnetic flux would therefore be equal to the b phi due to this b phi prime due to this magnetic field. But phi prime is nothing but z because if you look at this direction phi prime is around the wire which means it is upwards. So, it is equal to integral over this surface. So I have to integrate this b which is really b z over this interior area. So, it is integral 0 to r integral 0 to 2 pi of this quantity mu naught i over 2 pi over r plus a minus row row d row d phi. It does not depend on row at all. So I can pull out a 2 pi. I will do the integral here itself.

(Refer Slide Time: 29:04)

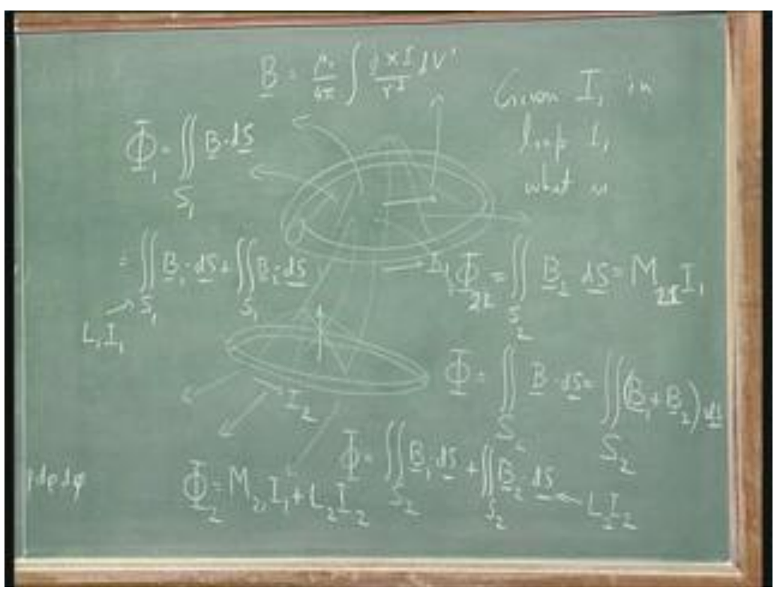


So I get phi m is approximately equal to mu naught i. The 2 pi cancels out and I have an integral from 0 to r. Now since the denominator is a complicated function, I will change my units to u equals r plus a minus row. Then d u equals minus d row. So, I will just flip my integrals. So I will get a, to r plus a d u over u times row. What is row? Row is r plus a minus u. So you can work this out, mu naught i times, this is one over u. So it is log u. So r plus a times log r plus a over a minus the u cancels out minus r. So, this is the flux that is linking outside the wire. So, this is the external flux.

Now you can that there is a one over a here which means the smaller the radius of the wire is the greater the external flux. If the wire actually went to zero radius the enclosed flux becomes infinity which means that the inductance l. I will have to divide it by i. So, mu naught times r plus a log r plus a over a minus r. So, you can see that the inductance would go to infinity as the gauge of the wire became finer and finer. Of course I will have to drive the same current through it, that would be difficult because the thinner the wire is the greater its resistivity. But if I had a super conducting wire of very negligible radius the inductance would be very high. So, in that sense I really do not want zero radius wire.

I would rather have that thickness and live with the internal inductance because the internal inductance is all set and done in rather small quantity. If you look back at this formula, this is ln b over a. That is it is just half and it is smaller number than this. So, this term does dominate but this term is present. So, in a certain sense the issue of internal and external inductance has more to do with the gauge of the wire and our decision of how much of the flux should be outside the wire and how much should be inside. So that is the concept of inductance. There is another concept and it is a concept that is quite important.

(Refer Slide Time: 32:47)



Supposing I have a wire and I have another wire and I have a current i 1 in the first loop and I have a current say i 2 in the second loop. These may actually point in different directions. These are just general loops. Now this current loop will create its own magnetic field. Some of these magnetic fields is actually going to come and go right through this loop as well. Other parts of the magnetic field will not go through this loop. So, you can see that one can actually imagine talking about how much of the magnetic field intersects this loop. In other words we can ask given i 1 in loop one what is surface integral s 2. That is draw a surface here, draw a surface on this loop of b dot d s and I will say b one dot d s because it is the b due to current i 1. This magnetic field is proportional to this current and therefore the amount of flux.

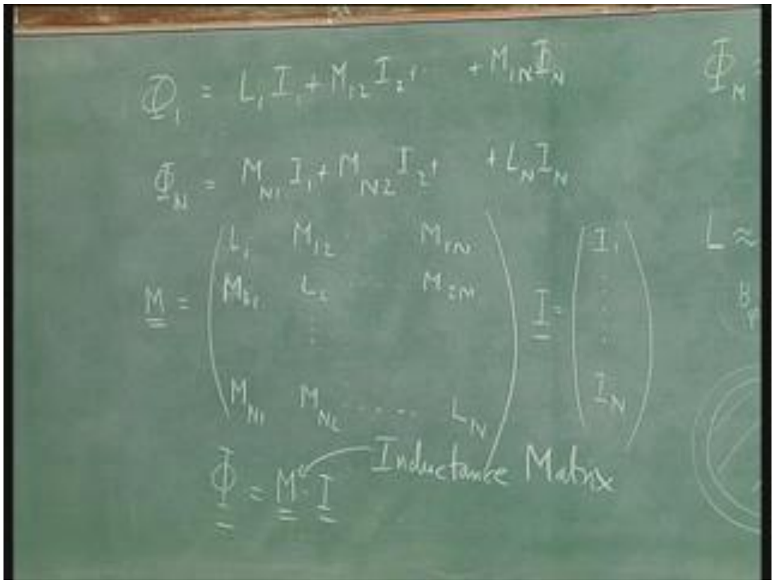
This is the flux due to current one in current in loop two. So I will call it phi 1 2. This phi 1 2 is proportional to i 1. It is not proportional to i 2, even if I have a current here the flux that the portion of the magnetic field due to this current is different. In fact if I wrote the total magnetic field phi it will be equal to surface integral over s 2 b dot d s. But this magnetic field is due to this current and this current and if I remember my biot savart law, I know that b is equal to mu naught over 4 phi volume integral j cross r over r cubed which means that if I have two separate currents they give me 2 separate fields which I just added up vectorially. So I can write this as surface integral s 2 b 1 plus b 2 vector addition dot d s b 1 dot d s is this term, b 2 dot d s is nothing but the self-inductance, self-flux term.

So I can write this phi as surface integral over s 2 b one dot d s plus surface integral s 2 b 2 dot d s. Now we already know what to do with this term. This is nothing but l i because if you have a current in a loop and it generates a magnetic field the inductance of that loop is nothing but this flux divided by the current in that loop i 2. So, this flux is l times i 2 and because it is coil 2 i will put it as l 2. This flux however is a new flux. It is this phi 1 2 and it is not due to i 2 at all it is due to i 1. It is due to the current in this loop rather than the current in this loop. So, it is given a new symbol. This phi 1 2 is called m 1 2 i 1.

Actually I suspect it is m, I should say phi 2 1. I will call this m 2 1, it does not matter because they are equal to the same number. But the nomenclature is, I think this phi 2 1 i 1. So, I can write this equation now as an equation in terms of i 1 and in terms of i 2. The equation will look like the magnetic flux is equal to m 2 1 i 1 plus l 2 i 2 and this is the flux threading this loop. So I will call it phi 2. Now what is the situation with loop one. Well, I can do the same thing. I can say the flux in loop one is equal to a surface integral over the surface of one. So I define a surface there s 1 b dot d s. But b again can be broken into two parts, the j due to i 1 and the j due to i 2.

So, it will become surface integral s 1 b 1 dot d s plus surface integral s 1 b 2 dot d s. This surface integral of the over surface one due to the magnetic field generated by current one we have already looked at it. This is nothing but l one i 1. This term is the magnetic field that is threading loops one due to the current in coil two. So, this is given, this is just like phi 2 1 except it is called phi 1 2 and it is also proportional to this current. Therefore I write it as m 1 2 i 2. So corresponding to this equation I have another equation. This equation looks like phi one is equal to l one i 1 plus m 1 2 i 2. Now you can see that this is matrix equation. It is, I take all these currents i 1 i 2 may be there are more coils i 3 i 4 and out of them I generate a number of fluxes. So, I can write a matrix equation out and the matrix equation; will look like this.

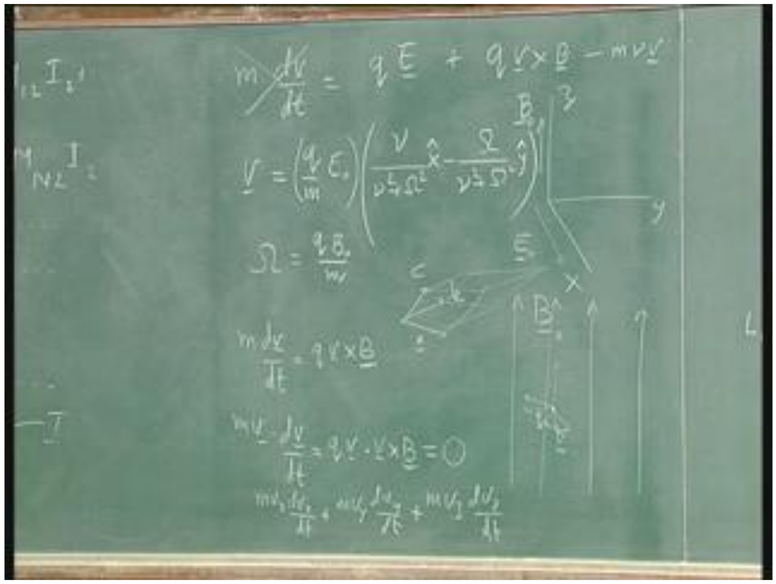
(Refer Slide Time: 40:47)



Phi 1 let me write out the equation first. Phi 1 is equal to l 1 i 1 plus m 1 2 i 2 plus plus m 1 n i n, similarly phi 2 etcetera. Phi n is equal to m n 1 i 1 plus m n 2 i 2 plus l n i n. I will just generalize that same result to m coils carrying n currents. So, this is a matrix equation, I will define a matrix m. To call a matrix, I will usually put two lines underneath. It is nothing universal, but it is my notation l 1 m 1 2 up to m 1 n l 2 sorry m 2 1 l 2 m 2 n so on, up to m n 1 m n 2 up to l n. So, this is a matrix and I have a current vector I, which is i 1 up to i n. Then my flux vector phi is equal to this matrix multiplied by the current vector i. This m is called the inductance matrix. The diagonal elements you notice are called l 1, l 2, l 3. Sometimes you call it as l 1 1, l 2 2, etcetera. These are called self-inductance terms because l 1 represents the amount of flux threading coil one due to current in current in coil one.

So, that is a self-inductance term but the off diagonal elements m i j corresponds to the flux that links coil i due to current in coil j. So, those are called mutual inductances and that’s where this symbol m came from. We will understand much more about the inductance matrix once we have studied faraday’s law. Because really without faraday’s law we cannot go any further. The crucial point about inductance is it represents stored energy and to understand it represents stored energy we have to do faraday’s law. So, I am going to finish this lecture by going to the crossed electric and magnetic fields problem and look a little bit more in detail at what is happening there. Let me remind you we have done a problem for a charged particle.

(Refer Slide Time: 44:32)



Problem was, I solve the force equation m d v d t is equal to the forces on the charge which are the electric force q e the magnetic force q v cross b and the friction force which is minus m nu v. So, these three forces are the forces we can typically expect on a charge. You could also expect gravitational forces if the object was very large and there are systems where gravitational force is important. For example if we are doing industrial manipulation of dust for example many kinds of abrasion powders are made by chemical processors where electric fields are important. Once the dust is formed out of the gas it is actually quite heavy. It is a micron sized dust particle. That dust particle not only responds to the electric and magnetic fields. It starts responding to the gravitational field, so then I would have a m g there.

But for this problem let us assume that we do not have any such systems. So, we have only electric field magnetic field and friction. So now, we take this and last time I took you through the steps to solve this problem. I assume that I have x y z. I assume that I have a magnetic field. I going to call it b naught, I have an electric field. Let us call it e naught. But as a worked out last time you do not have a velocity only in the direction of the electric field but because of the magnetic field you have velocity in the y direction as well. So, when you work out what the answer is what you find is the velocity under which this left hand term is missing that is under steady state conditions d v d t is equal to 0. That steady state solution is proportional to q over m, e, naught, this is the force with which the electric field is pushing the particle, but there are two components.

One component in the direction of the electric field is nu over nu square omega square in the x direction and there is a component in the y direction which is minus capital omega over nu square plus omega square in the y direction. Now what exactly is this capital omega. Capital omega I defined as q b naught over m. Now if you look at a magnetic field, let us say it is a uniform magnetic field strength is b naught and let us assume that I have a charge q with some velocity v and let us assume v is in the x and y direction, b is in the z direction.

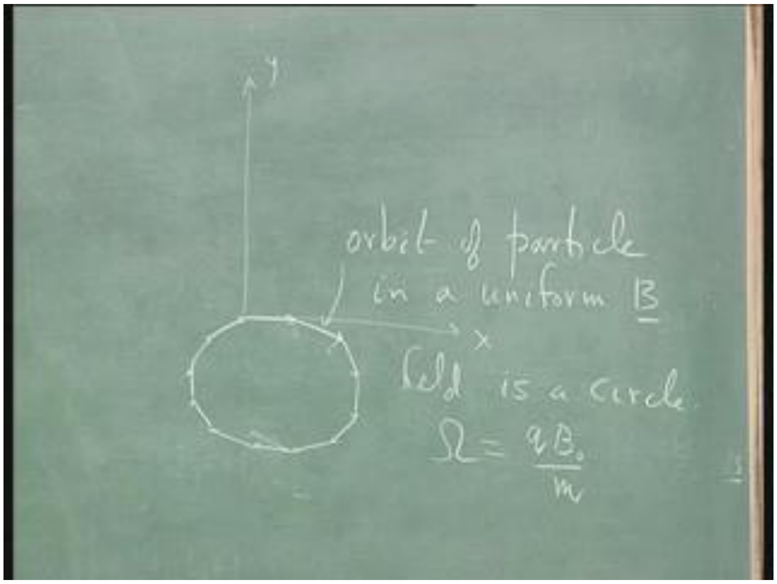
So, this is the z direction, v is in the x and y direction. Now what will happen? Due to this velocity and the b v cross b force is going to be present. So, you can keep it in your mind these two vectors and you find that there is going to a force this way. Now because of this force there is going to be a d v d t. Let us ignore the e. Let us ignore the friction. Just take these two terms m d v d t and q v cross b. So, what happens is because of this magnetic field the velocity experiences acceleration in this direction. So, the velocity changes direction. It starts pointing this way. But when the velocity points this way v cross b points that way 90 degrees again. So the velocity changes direction again and so on and so forth. So, what happens is that the velocity vector continuously starts rotating. Does it change in magnitude?

Well you can prove that it does not because take that equation m d v d t equals q v cross b. Let me dot product both sides with v. So, I get m v dot d v d t is equal to q v dot v cross b. Now you know that any triple vector like this v dot a dot b cross c the value is the volume of the parallelepiped. So, if you have a vector a vector b and a vector c you have to form the parallelepiped and it is the volume of this parallelepiped that corresponds to v dot v cross b. But two of the vectors are the same vector. So, this is a parallelepiped in which two of the arms are pointing in the same direction and a little bit of thought will tell you that means the volume is 0.

Another way of thinking about is v cross b is perpendicular to v and perpendicular to b. So we dot it with v dot the result with v we are going to get 0. So, this is equal to zero, but what is the left hand side? Left hand side is m v x d v x d t plus m v y d v y d t plus m v z d v z d t. You know that you can always take these into the derivative. So, what you get is this is equal to d d t of one half m v x square plus one half m v y square plus one half m v z square. The total kinetic energy is equal to the left hand side the rate of change of the total kinetic energy. It is equal to 0 which means one half m v square is equal to constant.

The magnetic field cannot do work on a particle. That is because the force is always at right angles to the velocity. Since the force is always at right angles to the velocity it can change the direction of the velocity. It cannot make it grow, it cannot make it shrink. So, a particle that is in a magnetic field keeps changing its velocity direction without ever increasing in speed or decreasing in speed. So, the only thing it does is it keeps changing direction. So, what will happen to the orbit? It is easy to derive but instead of deriving I want to draw pictures.

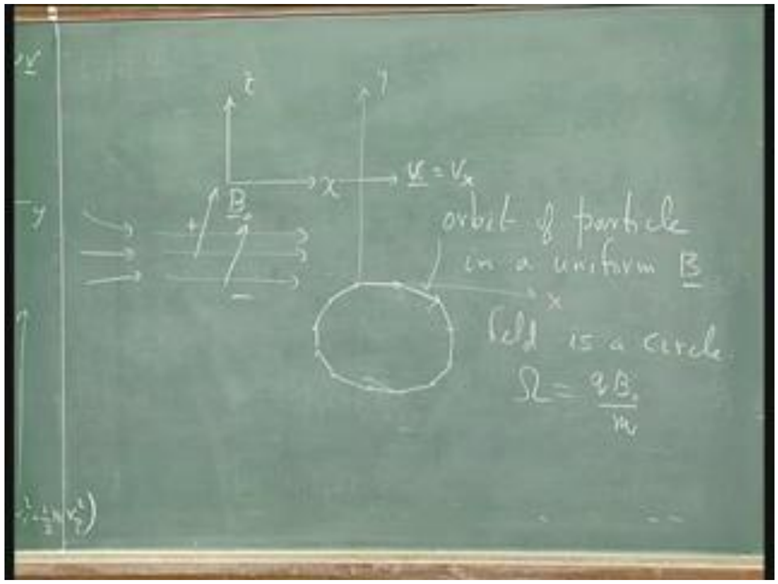
(Refer Slide Time: 53:09)



Looking from above, let us say this is y this x and let us say the particle started with a velocity along x. The magnetic field is out of the board. So, v cross b so the force is downwards. The velocity bends. Now the force is this way. Velocity bends some more. It goes right round and round in a circle. So, the orbit of particle in a uniform b field is a circle. The rate at which it goes round and round is nothing but this value omega which is q b naught over m; so, this omega is talking about the number of times particle will go round and round in a circle per second.

Now, if you back and look at this formula what it is saying is if the magnetic field is very weak, then you have the collision term dominating. Then you get q over m e over nu which is nothing but our conduction current. If the magnetic field is large then this term goes away. Instead you have q over m e over capital omega which is at a 90 degree angle and this is what is called the e cross b drift. This is the e cross b drift which I have already been talking about last time. Now this same equation can be viewed in different ways. You can view it as a conduction current that is limited by magnetic force. You can view it as a e cross b drift which is having a leakage current due to collision frequency. One other way you can look at it is the following.

(Refer Slide Time: 55:50)



You have a magnetic field in the z direction and let us say that you have flow in the x direction. That is your velocity is equal to v x. Now if your velocity is equal to v x then your problem cannot be same as this because here I have velocity both in x and in y. That is all right. I can always rotate my coordinates so that I point along velocity in which case the electric field will not point along x alone. There will be a slight bit of electric field in the y direction. Now what does that give me? Supposing I have a strong magnetic field and I force a charged fluid in the x direction. What these equations actually tell me is this. Term is small this term is large.

So I am forcing fluid along a direction, automatically a electric field will be developed in a 90 degree direction. So, if I have some kind of pump some which is forcing fluid through a region where there is magnetic field and if this fluid is a charged fluid then automatically, an electric field will develop and this electric field can extract the energy of the flow and convert it to electrical energy. This is nothing but m h d power generation. It is the same equation. It is just how you tilt your head and look at it. Okay, with this I am stopping the topic of static magnetic fields and next lecture onwards I will continue with faraday's law.