Electromagnetic Fields
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Lecture – 25

Inductance

Good Morning. Last time we derived a couple of expressions for the magnetic field in different geometries. Today what I am going to is to connect up these calculations to a very important concept namely inductance. We cannot really formulize this concept of inductance till we reach faraday's law. But I am going to introduce the definition and I will make it make sense once we have done stored magnetic energy. So, what have we done?

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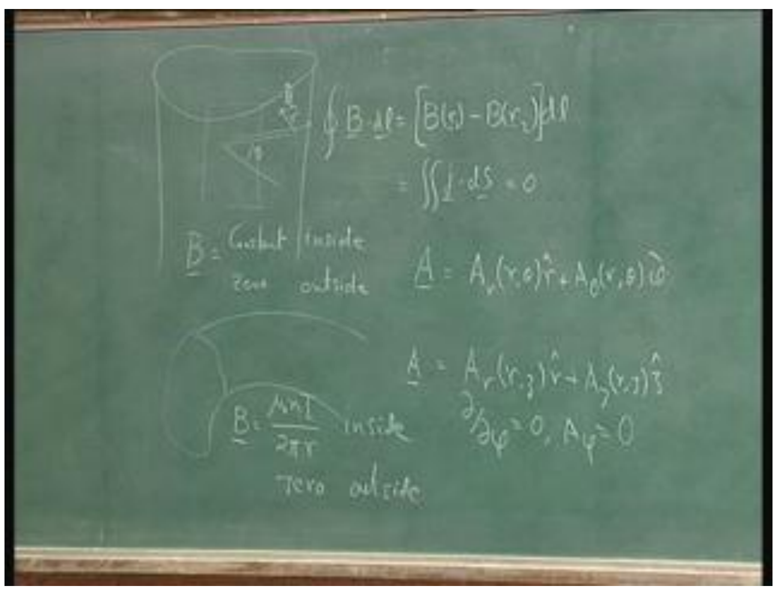

We looked at an irregular solenoid and we concluded that the magnetic field is constant inside and 0 outside. We did that by proving that the magnetic field is in the z direction and then if it is in the z direction, you can always draw Stoke's loops and since there is only magnetic field here and here you have loop integral b dot d l is equal to b of r 1 minus b of r 2 times d l. But, there is no current entering. It is supposed to be equal to surface integral j dot d s which is 0.

So, that must mean that the magnetic field is constant along this direction. Similarly we worked out the case for an irregular torus and we proved that the magnetic field is equal to mu naught n i over 2 pi r inside and 0 outside and the way we proved it was quite important. We proved it by talking about the vector potential a, and showing that if the vector potential a, has certain symmetries. In this case vector potential a, is not a function of z and because the currents are all in r theta.

We said that a is equal to a r of r theta along the r direction plus a theta of r theta along the theta direction where r and theta are this is the r direction, this is the theta direction polar coordinates. If you assume just this much the rest follows. Similarly, here I had to work a little bit to show that the vector potential a, was in the r z plane. So it was a r as a function of r and z along the r direction plus a z a function of r and z in the z direction del del phi was equal to 0 and a phi was equal to 0.

Using just that information I was able to prove this. These kinds of proofs are quite important. I mean after all the solenoid and the toroidal solenoid are perhaps the most important magnetic structures we have in electrical engineering. So, if we can derive expressions for them, then there must be some use for this vector potential, okay? So now we have got some expressions. Let us look at what those expressions are saying.

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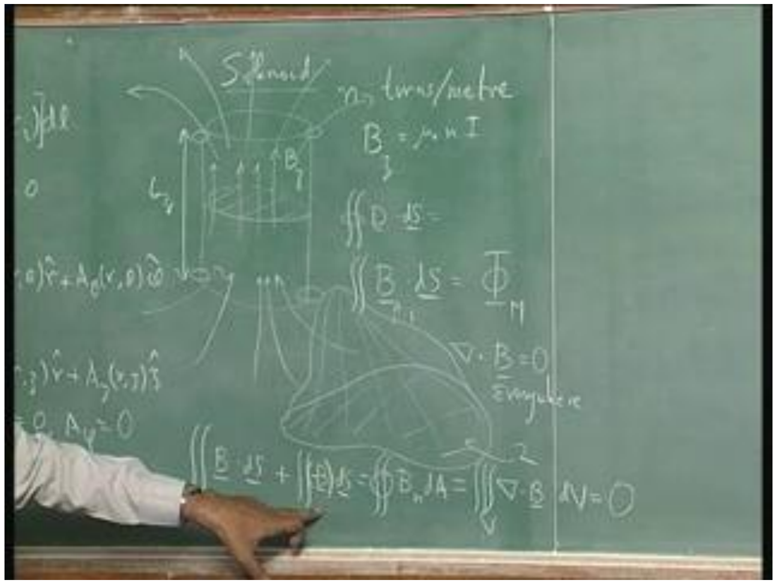

In the case of solenoid I am going to take a circular solenoid, there is no need now to go for irregular and other things. I have n turns per meter. I already know that b is along the z direction and it is equal to mu naught n I where I is the current per turn, n is the number of turns per meter. So, b z is in the z direction and in fact b z is constant within the solenoid. Now this is true for an infinite solenoid. For a finite solenoid there are always going to be n defects. So, if you had a solenoid that was only at length l z, what will happen is in the middle of the solenoid you will have uniform b but it will start diverging as you reach the end.

So, what happens is actually a little bit of flux leaks out through the side coils towards the end. So, it is not true that b is totally along z and it is not true that the amount of b that is enclosed is constant. So it is an approximation for finite length solenoid but it is exact for an infinite length solenoid. Now I want to introduce a new concept which is I have this solenoid and the solenoid has a cross section. In this case a circular cross section. So, just as for displacement vector, I defined the amount of flux leaving a surface of course in that case I used close surfaces and I talked about this as the electric flux leaving a surface.

Similarly, I can talk about a magnetic flux leaving this surface. So, I can talk about surface integral b dot d s. It turns out to be an extremely important concept and it is called the magnetic flux usually denoted like this. In fact very often you would not even see the m below it just phi. So, this magnetic flux is the integral of b over a surface which surface. Well, it does not matter which surface because, supposing I have some loop and I connect it by a surface. I could also connect it by some other surface. So, one surface is along the plane the other surface is pulled out.

So, I would get different answers perhaps but I would not because if I call this top surface say one and I call the bottom surface two, then I can do something. I can say the surface integral of one b dot d s minus the surface integral over surface two b dot d s. Now what is minus surface integral mean? It really means plus minus b dot d s. I can always take the minus inside which means here b is pointing upwards. There, I am taking the opposite sign. I am asking b to point downwards. In that case what does this mean? This is nothing but integral over the entire surface b normal d a because on this surface I am doing b dot d s.

On the bottom surface, I am doing minus b dot d s. So, I am taking the total amount of b. That is trying to leave this volume so that is surface integral over the closed surface b d a. But we know Gauss' law. The divergence theorem tells us this is nothing but volume integral over the enclosed volume some v divergence of b d v. This is just coming from the divergence theorem.

Any vector field dot d a over a closed surface is equal to an integral over an enclosed volume divergence of the vector d v. But now I have a well-known result. I know that divergence of b is equal to zero everywhere. It is a fundamental statement and it came out of the fact that b is an integral of j cross r over r cubed. So, divergence b is zero which means this is zero. So, it says integral over surface one minus integral over surface 2 is 0 for any pair of surfaces 1 and 2. But what does that mean?

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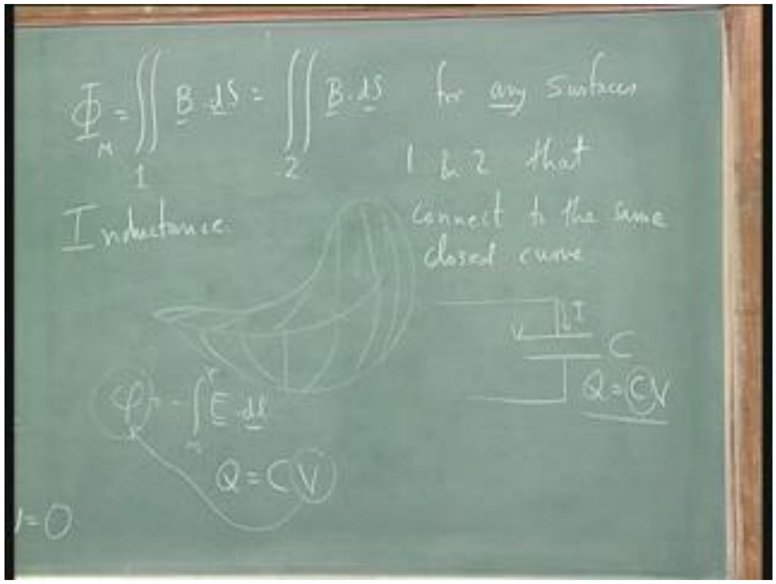

It means integral over 1 b dot d s is equal to integral over 2 b dot d s for any surfaces 1 and 2 that connect to the same closed curve. That is the only condition. That 1 and 2 must somehow connect to the same curve. Now note that this curve does not even have to be on a plane. I have a plane that is like a cucumber, right? I can draw a surface that looks like this and I can draw another surface that looks like this and the top surface and bottom surface will both give me the same answer which means, what this means is that talking about magnetic flux is meaningful and I think that is an important point.

When we talked about electric field we defined a quantity phi which was minus integral from infinity to r e dot d l and we did not know whether this things was meaningful or not. We could only say it was meaningful because it did not depend on how we got from infinity to r. Once we knew that then we had a powerful new concept called the scalar potential or the electro static potential. Now, similarly here b is some vector field. It is a complicated field. But you find that if you integrate b on any surface that connects to a closed curve, no matter what that surface is you get the same answer which means that, this concept of phi magnetic is a good concept.

It does not depend on how I drew my surface, any surface will do. So, it is very analogous to talking about electro static potential. You should see these two as analogous. As you see these two as analogous then you can ask potential is stored in capacitors, right? I mean potential you have relation q equals c v and voltage is nothing but potential. So, there is a characteristic of potential that is coming into circuit devices.

Similarly, the moment you define something that does not depend on the detailed shape of electromagnetic objects you can expect that will enter a line diagram. It is quite important here. So, pay attention. Supposing I draw a circuit and I wrote this and I said there is a capacitance c. If the behavior, of this object the relationship between current entering and the voltage was a very complex function of the shape of this object. If it depended where the current entered, if it depended on which part of the capacitor was where, then you would not be able to write a simple relation like q equals c v.

It will become much more complicated, it would depend on the distribution of charge. It would depend on other things. Once I did that, it will no longer be useful in circuit theory. But we know ahead of time that capacitor is a very well defined object. It is defined by a scalar equation. So, all the vector details are hidden in c.

And that comes out of the fact that actually you have certain linear relationships that define what is happening inside c and you have a scalar quantity, the potential that does not care about the path by which it is calculated. All these things go into making this possible. Similarly now we have a new scalar quantity. It is called magnetic flux. Now it is a scalar quantity that is a function of an entire loop. But, it does not depend on the surface that connects the loop and we can confidently expect that because it is such a simple idea here. There is another circuit device that is connected up with this scalar quantity and of course that is the inductance.

Inductance is the connection between magnetic flux and the current through the coil just as in capacitors capacitance is the connection between the potential and the charge that is stored. So the stored charge and the used the current that is flowing through the coils are the quantities that directly define the fields. The charge is what generates the field here. Current is what generates the field here. Associated with the charge, there is a concept called electrostatic potential. Associated with the current there is a quantity now which we have found which is magnetic flux.

So, there is a big symmetry here and it is very important to appreciate it because every time you see a symmetry between electrostatics and magnetostatics. We should use it to simplify magnetostatics. We should not get confused and frightened by the fact that magnetostatics seems to use much bigger formulae. It is almost for point to point that is same as electrostatics. You just have to appreciate the fact that it is a same thing and then things become much simpler. I am going to now introduce what is the definition of inductance. But as I said this definition would not make sense till we get to faraday's law.

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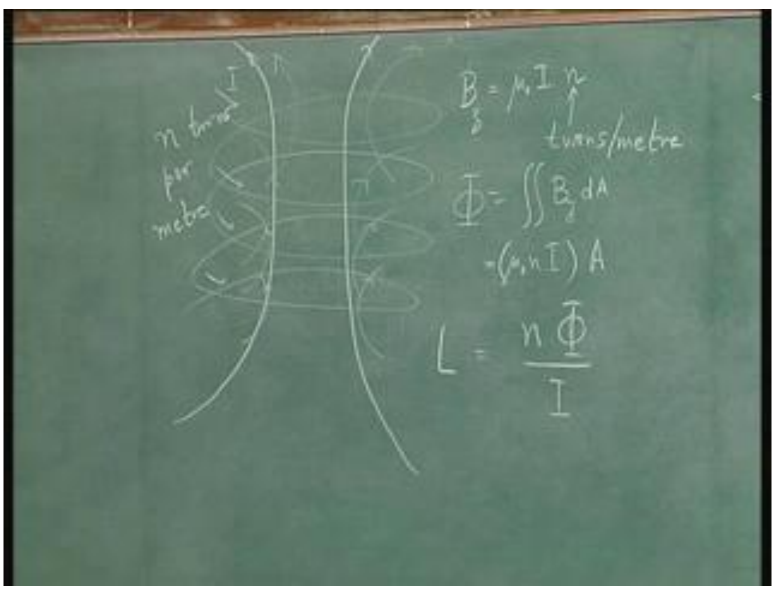

So, we assume that there are many coils. They are all connected up. So, you have a current i. This current is also flowing the other coils. Since it is a very common arrangement you always work with coils in electrical engineering, so we have built the concept of inductance in the concept of inductance we have built into it the number of turns per meter. We could actually have defined inductance per turn and then generalize it for a n turn system. But it is so common to have multi turn coils that we just built the number of turns per meter into the definition of inductance. So let us say we have n turns per meter. Now in each of these turns there is a current and that current is trying to create a magnetic field. So, if the current is going this way, the magnetic field is like this.

Now the next turn is trying to produce its magnetic field. Third one is trying to produce its magnetic field. Fourth one is trying to produce its magnetic field and from this picture you can easily see if you can added them all up vectorially, you are basically going to have a magnetic field that goes like this which is what we derived last time. A uniform magnetic field inside the solenoid diverging magnetic field on entry and on exit, it converging on entry and diverging on exit.

Now what is worth knowing is that when we derived it we found for a solenoid that the magnetic field in the interior of solenoid b z is equal to mu naught i times n. That is the magnetic field is stronger if you have multiple turns. But it is not the total number of turns in the solenoid that matters. It is the number of turns per meter per unit distance. So, we tightly wind this solenoid we get a strong magnetic field. A solenoid could have 400 turns and the b field is not 400 times, not if the 400 turns are stretched over 10 metres. Then it is only 40 turns per meter. So, the magnetic field is only 40 times.

So, this n is turns per meter and not total number of turns. Now the definition of flux you would imagine would be phi is equal to surface integral b z d a which is nothing but mu naught n i times a. Now we define the inductance as the relationship between phi and I, but not really between phi and i we say that this magnetic flux is seen not just by one coil but it is seen by n coils. So we define the inductance l as the flux that links n coils per unit current, as I said this concept of putting in the number of turns is more for convenience. You could have defined flux per coil and inductance per turn and then you would have worked out that there will be an n square factor coming into the answer.

But because it is so common to have coils we have put the number of turns per meter into the definition of inductance itself. Now if you put back the definition of phi into this what do we get. It is equal to n over i times mu naught n i times area. Since the flux is proportional to current if we divided by current the current has to go away. So, you are left with mu naught n square times the area. The inductance is proportional to the square of the number of turns per unit length that you wind around your solenoid. This is generally true. You will always see a n square. It is because the magnetic field you produce is higher because of the number of turns and you are linking n coils with that same magnetic field and therefore you get n twice or n square. You can work out the inductance due to many other geometries.

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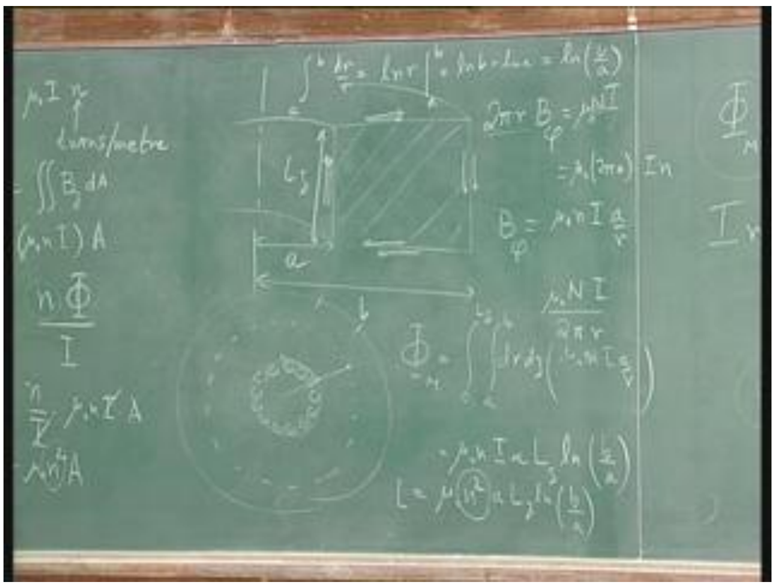

For example supposing we had our torus. Then I am going to choose a square cross section. That simplifies the maths. So, my inner radius to the center of symmetry is the distance a outer radius the distance b the height is l z. We worked out that if you take this solenoid then the magnetic field is purely in the phi direction. Therefore you can apply stoke’s theorem and say 2 pi r b phi equals mu naught i. But there are n turns. So, I put a n because when it goes round it is all these currents are coming out. So, the number of turns is the total number mu naught capital n i. Let me show it from above. You have your inner wall outer wall of your solenoid.

You are looking at magnetic field somewhere some radius. So, you take a loop like this. This loop has a length 2 pi r times b phi. Coming out of this inner wall are all these coils. So, the total number of turns not the turns per meter. The total number of turns is what is getting linked by this loop. So I have put a capital n and this capital n is equal to mu naught times 2 pi a n I; 2 pi a is the circumferential length around the inner wall and n is the number of turns at the inner wall per meter okay. So, what do I get? I get b phi is equal to mu naught n i times a over r. The two pi will cancel out. If we kept it in terms of capital n, it would have been mu naught capital n i over 2 pi r, both are okay. You have to decide which form of the definition you want.

Now this is the magnetic field. We need the magnetic flux. Now it is important here to understand what surface we are talking about. The current is flowing like this and then it is coming back to the next coil and so on. So, the current is flowing in the r z plane. The magnetic field is in the phi direction. So, the surface we are talking about is the surface that links this coil. So, phi magnetic is equal to surface integral of z going from say zero to l z r going from a to b d r d z of b phi which is mu naught n i times a over r.

The integrals are trivial. It gives you mu naught n i times a times the z integral the integrant is not dependent on z. So I can just integrate it trivially l z. The integral in r has a one over r in it. The integral of one over r is log. So, the answer becomes l n b over a. That is because integral a to b d r over r is equal to l n r between the limits a and b which is ln b minus ln a or ln of b over a, all right? So I have got l n of b over a. So, this is the flux per turn. So inductance is now going to be n times this phi divided by i or it is equal to mu naught n square times a l z l n b over a.

Now it depends very much on how you define your n because I have defined my n as number of turns per unit-, per meter on the inner wall. Since the dimensions of such a system is varying as you in r i could have defined my n as number of turns per meter on the outer wall. Then I would have got different expression. But whatever expression I use, I will always have n square once again. Inductance depends on the square of the turns per meter. Let us take a third example a coaxial cable.

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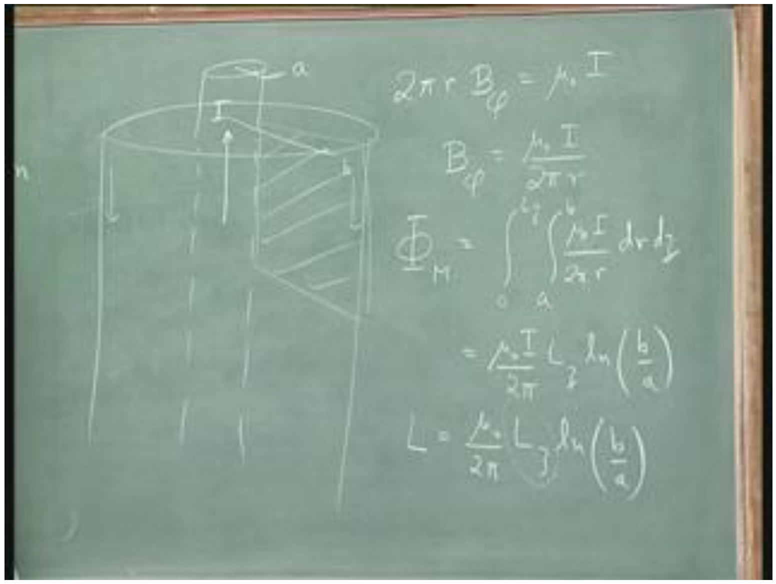

I have my inner radius of my wire is a outer radius is b. So the current goes let us say up i z in the inner wire and comes back down on the outer wire because there is a symmetry I can again calculate b. Stoke's theorem tells me 2 pi r b. Since the current is in the z direction b is in the phi direction is equal to nu naught i. There is only one turn. This is not a case with n turns, a single turn single wire is going up and coming back bringing everything back.

So, this is the total current single coil. So, b phi is equal to mu naught i over 2 pi r. What is the flux? The flux therefore is the cross section. I have to take flux is what intersects this cross section because this is a phi direction b phi. So, the cross section is in r and z. So, phi magnetic is equal to integral say 0 to l z integral a to b mu naught i over 2 pi r d r d z. So, same integration so we get mu naught i over 2 pi times l z times ln b over a. So, this is the magnetic flux. It is proportional current.

So, I can define an inductance. Inductance I define will be phi over a phi over i because there is no n, n is one. So, I get mu naught over 2 pi l z ln b over a. So, the inductance in this example is proportional to the length. That is not surprising because the more length of wire I take the greater amount of flux I am linking and therefore the more inductance, because inductance is nothing but flux per unit current. Okay, we will come back to the concept of inductance later on. There is no point in going further till we define faraday's law. I want to turn to another topic right away which is the topic of matching conditions for magnetic field when you have a boundary.

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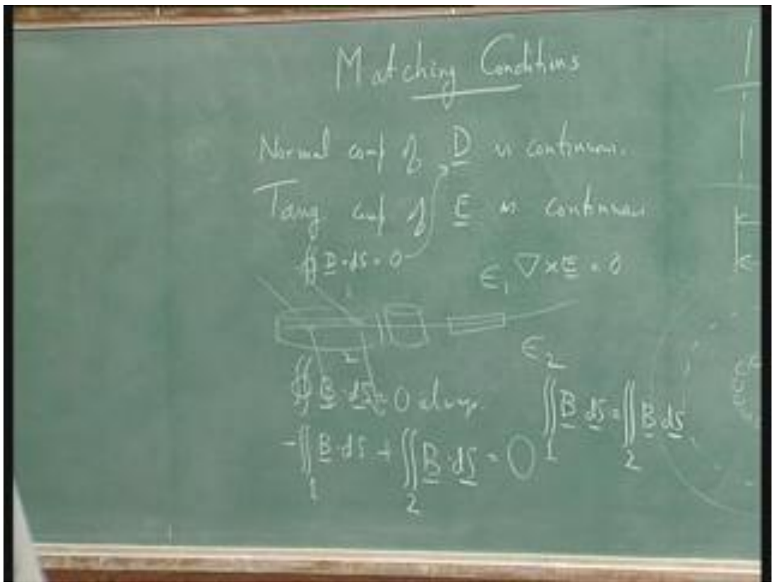

In electrostatics we had developer matching conditions. What were the conditions we had? We had that normal component of d is continuous and tangential component of e is continuous. We proved this using Gauss' law and stoke's law. Gauss' law and stoke's theorem namely we had some surface. There was let us say epsilon one on one side there was epsilon two on the other side. I will put an imaginary cylinder cylinder that was very, very thin. So even this picture is not very correct, a better picture would be like this. It is a small cylinder but its surface area at the top and the bottom is much larger than its side area.

Now I try to do integral d dot d s. There is no free charge enclosed. So, I got zero and using that I came up with continuity of normal component of d and for continuity of tangential component of e. I use stoke's theorem and I used curl of e is equal to 0. We have to do the same thing now. We have to use corresponding equations in magnetic field. Well, we do know that volume integral b dot d s is equal to 0 always. This is fundamental because divergence b is identically 0. Using this condition which is the same as this we were able to say normal component of d is continuous, so apply surface integral b dot d s equals zero to this cylinder. What will happen? Your magnetic field is doing something. The sideways fields will contribute to the sloping sides. But we are taking a cylinder so flat the sloping sides have almost no area.

So, all the contribution is coming from the top and the bottom. So I get this side is one and this side is 2. So I get surface integral I am assuming the directions the way they have drawn. So, minus surface integral b dot d s along surface one plus b dot d s on surface on surface two is equal to zero. Because in one case it is entering, the other case it is leaving. What does that mean? It means surface integral over surface 1 b dot d s equals surface integral on 2 b dot d s. But this can be written as integral over 1 b normal d a is equal to integral over 2 b normal d a, areas are the same. It is a small enough cylinder b is uniform over that surface.

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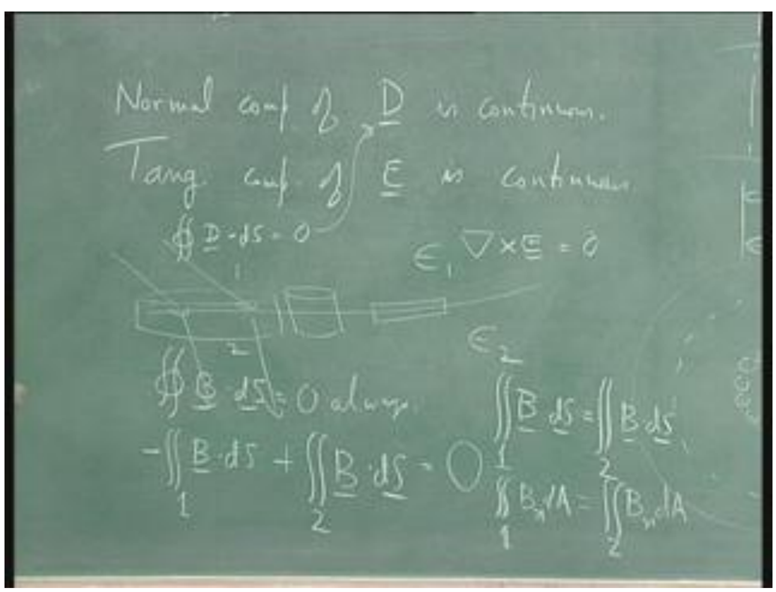

So, this finally gives me my result which is that b normal one times the area is equal to b normal two times the area or b normal is continuous. It is the exact same derivation as what we did for displacement vector. But you will see that in displacement vector we had to make an additional approximation or additional assumption which was there is no free charge on the surface. Here this is absolutely true. It does not depend on whether there is current or charge on that surface. It is always true that divergence b is 0. It is always true that b normal is continuous at any surface. Now what about the other condition?

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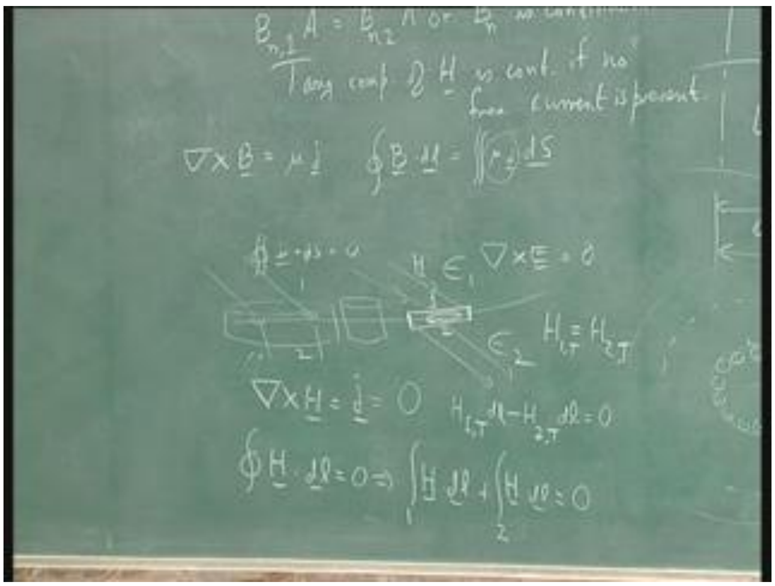

Well we used curl of e continuous to prove that e is continuous. Let us use curl of h. Curl of h is equal to j ampere's law which means that I can now look at this surface. Hence assume that there is no free charge free current. This is the equivalent of what I did for d? Supposing this surface is only a surface between dielectric. So, there is no free current there. It is equal to 0. If curl of h is 0, then I can use the same argument. I used for proving e is continuous. So, I take a long loop so that the vertical sides are negligible in length. Again I have my field. This is h and the field does something in the other side. So, I call this side one I call that side 2 and I go round this loop in some direction. Then I have this statement is equivalent to saying loop integral h dot d l is equal to 0. That is, integral on side 1 h dot d l plus integral plus integral on side. 2 h dot d l is equal to 0. But if I do integral on side one h dot d l that is h 1 tangential times d l.

On side two it is minus h 2 tangential same d l is equal to 0. The minus sign came because I am integrating in the reverse direction. What does this mean? It means that h one tangential is equal to h 2 tangential or tangential component of h is continuous if no free current is present. Now why did I use h here? Why could not I have used b? Let us try to use and let us see what goes wrong. So, I am going to follow the same steps and see what goes wrong when I use b. I will use curl of b is equal to mu j.

Now when I do this, there is a problem. Let me show you what the problem is, I do a loop integral. A loop integral b dot d l is equal to surface integral mu j dot d s. Now j free may be zero but any time I have two magnetically sensitive materials the different permeability mu 1 and mu 2. What is going to happen is at the surface boundary between these two, there is going to a surface current. The surface current represents the fact that the amount of induced spinning in this material and induced spinning on that material are different. So, on this surface alone there will be a large surface current which means that mu j cannot be set to 0 or if you like I should not call it mu j i should call it mu naught j total.

So then it will be mu naught j total which can then be written as surface integral mu naught j bound. So it is this term j bound which is not zero. I do not have any free current on this surface. It is a non-conductor. But I do have induced current on this surface and this induced current is going to cause a contribution and because it is going to give a contribution, it is not true that tangential component of b is continuous. In fact if tangential component of h is continuous what does it imply.

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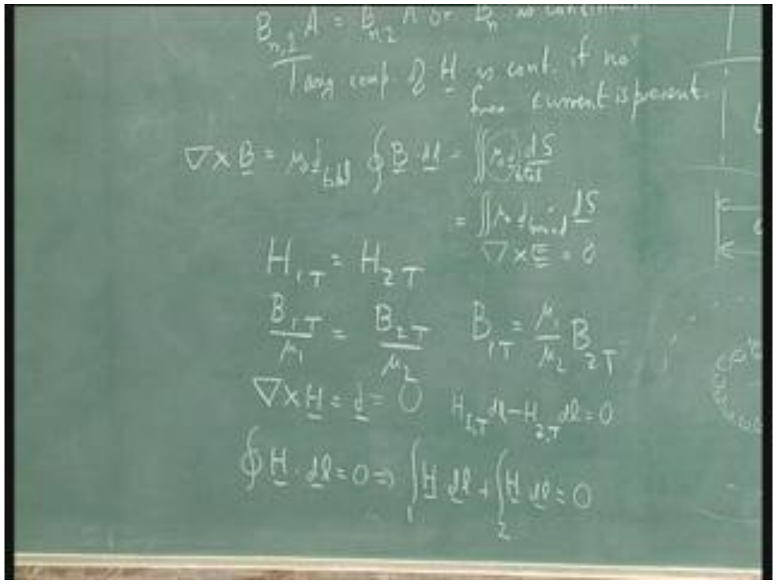

It implies that tangential component of b is not continuous because h 1 tangential is equal to h 2 tangential but h is nothing but b over mu. So b one tangential over mu one is equal to b 2 tangential over mu 2 or b 1 tangential is equal to mu one over mu 2 b 2 tangential. So, there is a jump in b tangential, b normal is continuous, b tangential is not continuous but h tangential is continuous. We will use all these matching conditions little later when we come to dealing with magnetic circuits.

Talking about what happens when you enter a magnetic material and when you leave it. Now I want to just touch upon one important topic in the rest of this lecture and that is the topic of what happens to free charges when they are in the presence of electric and magnetic fields. Up to now what have we done? We have talked about charges in the presence of electric fields and currents in the magnetic fields. But what happens if I put a free charge and apply electric and magnetic fields.

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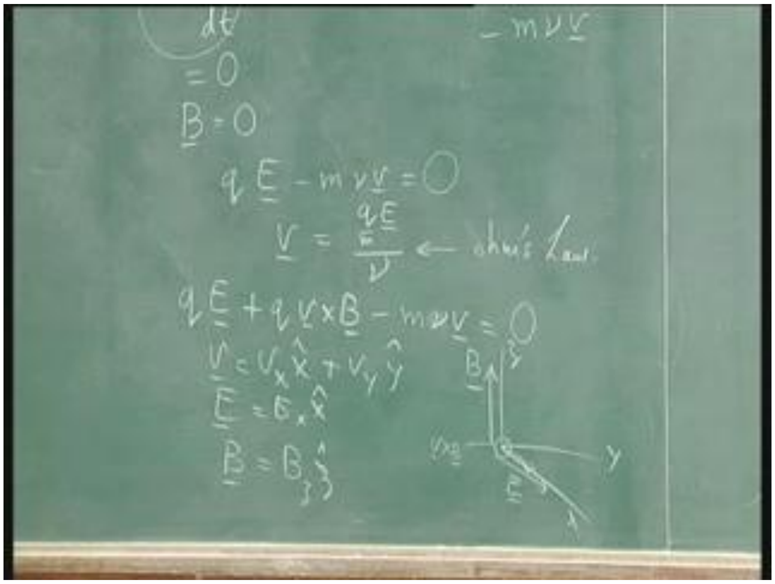

I know that my force equation according to newton is m d v d t the rate of change of momentum is the applied force. Now what is this applied force? It is the electric force q e plus the magnetic force which is q v cross b, q v is like j. So this is nothing but j cross b and this if it is an electron or a piece of the fluid also suffers from the viscous forces. So, you can put a drag minus drag minus m nu v.

I am talking about a particle. So this particle let us say it is a you can doing the millikan oil drop experiment or you can be talking about the hall effect. Any of these will satisfy an equation similar to this. The rate of change of momentum is equal to the applied electric force plus the applied magnetic force minus the drag on the particle. The drag here is modeled as a nice simple model where I am assuming the frictional the coefficient of friction is a constant. Now I am looking only from steady state problems. I do not want to look for time dependent solutions because that is a more complicated situation. So I am going to say steady state the left hand side is 0.

So, I am looking for constant velocity solutions. Let us look at some simple solutions. Supposing I say b is 0. We have already done this problem. What does it give us? It gives us q e minus m nu v is equal to 0 or taking m nu v to the other side I can solve for v. V is equal to q over m e divided by the coefficient of friction. So q over m, e is the acceleration and acceleration is balanced by a slowing down friction and when these two forces when these two accelerations balance that gives us the steady velocity.

This is nothing but ohm's law. Because inside a material what is happening is you have a steady electric field and you have a steady frictional force. Of course you have there is also a statistical component to it because each electron that is moving is moving with a different velocity and it is only on average this equation is satisfied. But, if you asked what is that average velocity that the electron has it is given by the balance between acceleration due to electric field and drag due to collisions. So, this is our ohm's law which we are very familiar with.

Now what happens if I have a magnetic field present as well. Well my equation becomes q e plus q v cross b minus m nu v is equal to 0. Now I am going to simplify this problem by assuming certain directions. This is the x y z direction. I will assume the magnetic field is along z. I will assume the electric field is along x and I want to solve for the velocity. So, the stationary electron or the stationary particle feels only the electric field. But as it starts moving it feels the magnetic field through the v cross b force. So let us try and imagine what will happen if I start with a particle at rest let us say there.

The first thing it will do it will start falling in the x direction because there is an electric field present. It needs to catch up till its steady state velocity is reached. But even as it starts moving there is now a v cross b force and the v cross b force is in this direction because of the v cross b force now the particle starts moving in the minus y direction as well. So it starts bending over. But when it starts moving in the y direction once it has a v y then it has a minus force in the x direction as well. Because v y cross b z is in the x direction.

So, there is a force in the x as well as a force in the y and these two forces together. When they are both present is what will give a final balance of forces in this equation. So velocity is equal to v x x plus v y y. The electric field is only in the x direction. The magnetic field is only in the z direction. Let us see where it gets us.

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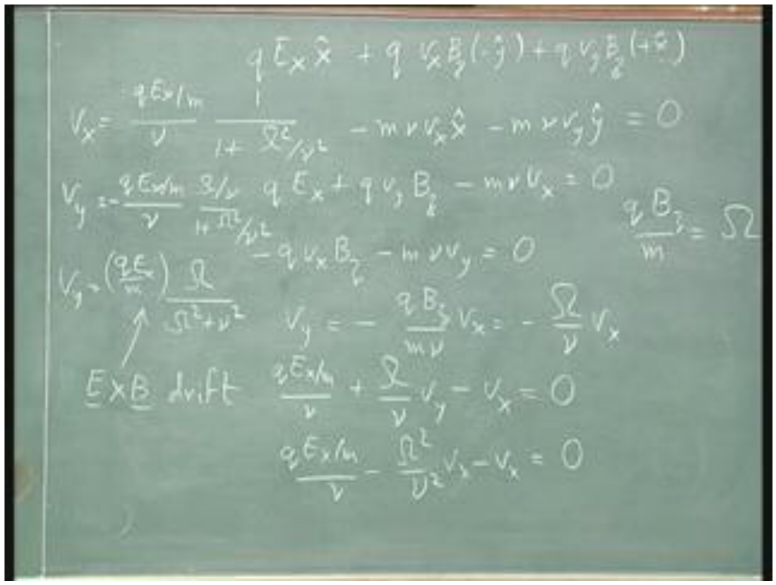

So I have q e x along x plus q v x b z x cross z is minus y plus q v y b z y cross z is plus x. So the v cross b term gives me two terms. The v x cross b gives me a minus y direction force. The v y cross b gives me a plus x direction force. Then minus m nu v x along x minus m nu v y along y. The whole thing is equal to 0. So, these are two equations because there is an equation connecting all the components along x and an equation connecting all the components along y. Let me write them down, q e x plus q v y b z minus m nu v x is equal to zero and minus q v x b z minus m nu v y is also equal to zero. So, this is the x direction force balance this is the y direction force balance.

Both of them must hold and I have two unknowns. I do not know v x, I do not know v y, two equations two unknowns. Let us solve this equation first. It allows me to solve for v y in terms of v x or the other way round, v y is equal to minus q b z over m nu v x. I just take the m nu to the denominator. Now I have a very important quantity called the gyro frequency. This gyro frequency is the speed at which particles go round and round the magnetic field. We will come back to this later, right? Now, just note that this is a well-known frequency, so I can write this as minus gyro frequency omega over nu v x. Similarly I can take this equation and what do I get?

I get that I will divide through mu m nu q e x over m divided by nu plus q b over m. So omega over nu v y minus v x is equal to 0. But I already know v y in terms of v x. So I can substitute here. So I get q e x over m divided by nu minus omega square over nu square v x minus v x is equal to 0 or if I solve I get the answer v x is equal to q e x over m divided by nu. This is the force balance of forces that in the absence of magnetic field would give me a stationary velocity but it is multiplied by one over one plus omega square over nu square.

So the magnetic field is 0. This is the drift terminal velocity. If the magnetic field is very strong, then what happens is it is the magnetic field that acts as a friction. It is not the collisions that act as a friction and the x velocity basically becomes 0. What does v y look like? We know that v y is minus capital omega over nu v x. So it becomes q e x over m over nu omega over nu with a minus sign divided by 1 plus omega square over nu square. So v y is much larger than v x when magnetic field is strong.

So it is in a case of strong magnetic field what happens is there is nu square that cancels out and it is only magnetic field that gives me drift. Let me work out that case, v y is equal to q e over m times the nu square has cancelled out omega over omega square plus nu square. This is called the e cross b drift. This e cross b drift is present in any magnetic field with or without friction. This drift is really a modified ohm's law. It is a ohm's law in the presence of magnetic field and it is from effect like this that you get things like hall effect and image de power generation and other effects alike that are quite interesting to electrical engineers.