Electromagnetic Fields

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Lecture – 22
Field due to Current Loop

Good Morning. The entire previous lecture was spent in explaining what exactly curl meant and I hope you have got a good understanding of what that operator is. It is the most complicated of the three operators we have studied.

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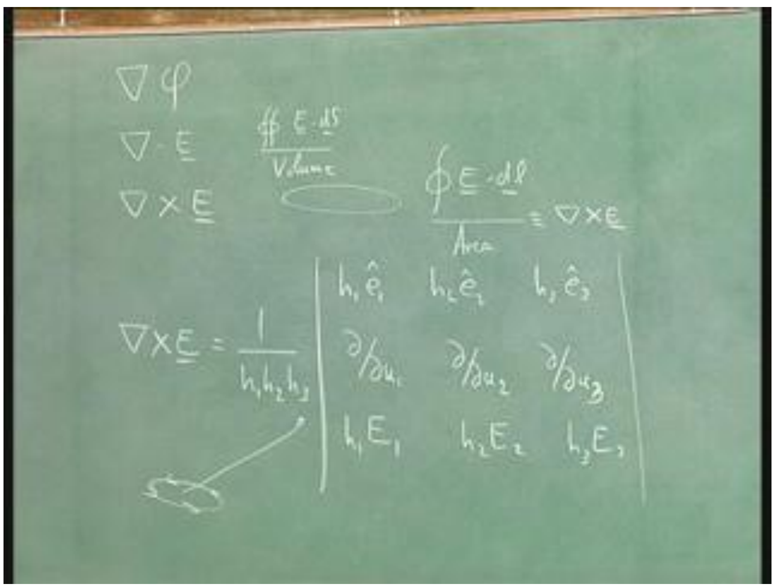

We have looked at the gradient operator, we have looked at the divergence operator and we have looked at now the curl operator. And we know that the curl operator is basically the average of the loop integral of any vector field divided by the area of the loop and if you take the area to be small that is you take the limit as this loop becomes tinier and tinier you get it is nothing but the curl.

Just as for divergence we had that it is equal to the surface integral of e dot d s divided by volume enclosed by the surface as you take the volume to 0. So, both of these look very similar and in fact in a more sophisticated mathematical treatment you can show that both of them are saying the same thing. In a certain sense this is the divergence theorem in a different dimension, at the end of last lecture I talked about what curl was in non cartesian systems and I wrote down the answer.

I did not derive it. It is derived in the appendix of your textbook e 1, e 2, e 3 are unit vectors, u 1, u 2, u 3, are the coordinates. For example r theta phi or r theta z or x, y, z and so this is the expression for curl. It is general in the sense that it is true for any orthogonal coordinate system and last time we worked out what it looked like in cylindrical coordinates. Now today I am going to do a problem. Just one problem. So, it is going to you take the lecture which will use curl in spherical coordinates. The problem is to find out the magnetic field far away from a loop of current.

This is an extremely important problem because anyone in electrical engineering has to do with magnetic materials and the entire theory of magnetic materials has to do with the magnetic field due to loops of current. So all of material properties of magnetics comes, from this one problem. So, we need to tackle it with some intensity and that is what we will do in this lecture.

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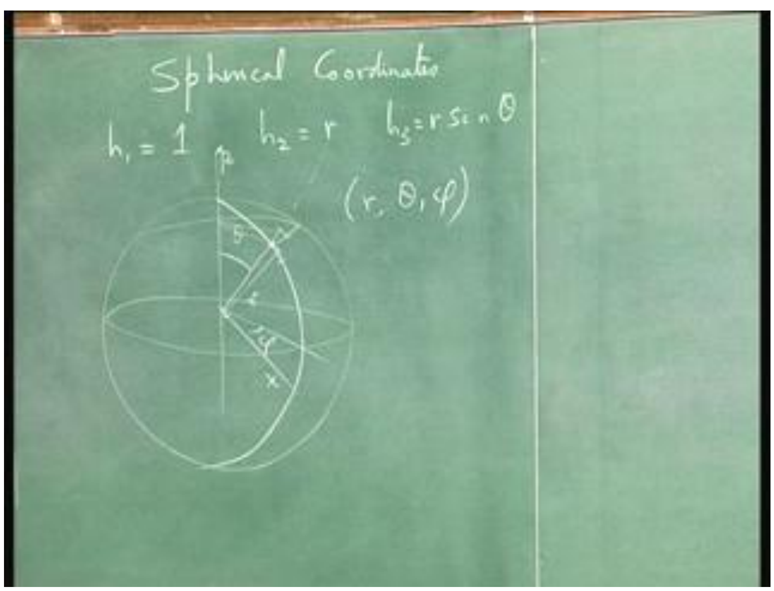

So, for spherical coordinates I have already explained where we get h 1, h 2, h 3, the values are r r sine theta and 1. The coordinates this is the z axis this is any point, the angle made with the z axis is theta. The distance from the origin is r and if you draw a great circle through that point and see where it intersects the equator and find it is angle with the x axis, this angle is phi.

So, the coordinate system is r theta phi and what this is saying is that if you move a distance unity in r the length is r itself d r itself. So, there is no scaling factor. If you move a unit distance in theta the distance you moved is r d theta. So the scaling factor is r and if you move a distance unit distance in phi the scaling factor is this length d phi and this length is nothing but r sine theta. So that is where these three coefficients come. Now, let us substitute this in to the expression for curl.

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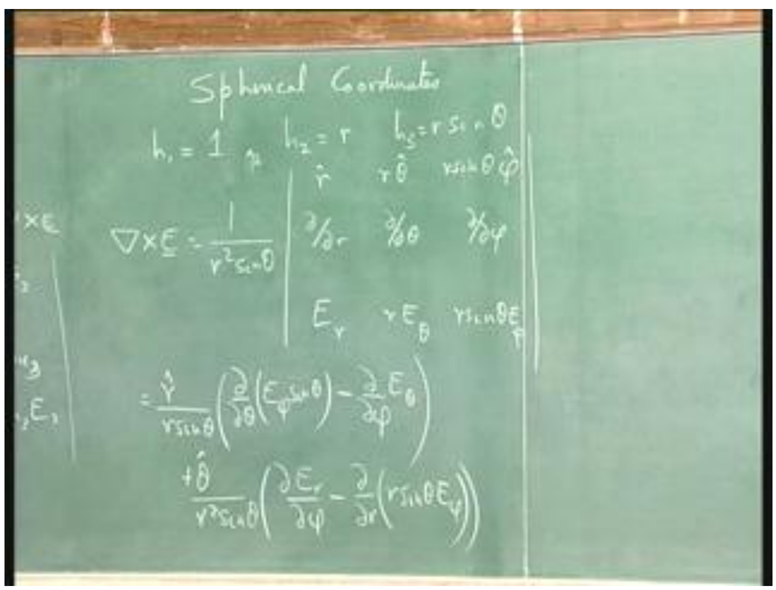

So curl of e becomes 1 over h 1 h 2 h 3 which is one over r square sine theta determinant of r hat r theta hat r sine theta phi hat del del r del del theta del del phi e r r e theta r sine theta e phi. I have just substituted for h 1, h 2 and h 3. Now we take the determinant by taking the top row and expanding the determinant of the minor. So, I get first r hat times del del theta of this minus del del phi of this. We can see that a r is there which does not get acted upon by theta and phi.

So, one r cancels out. So, r sine theta of del del theta of e phi sine theta minus del del phi of e theta. Then you have the theta hat component. There you have to keep both the r's but you do not have any derivative with respect to theta. Therefore the sine theta will cancel out when you take the phi term r square sine theta of del e r del phi minus del del r of r sine theta e phi. I am going to write the last term at the top itself so that I do not leave this board.

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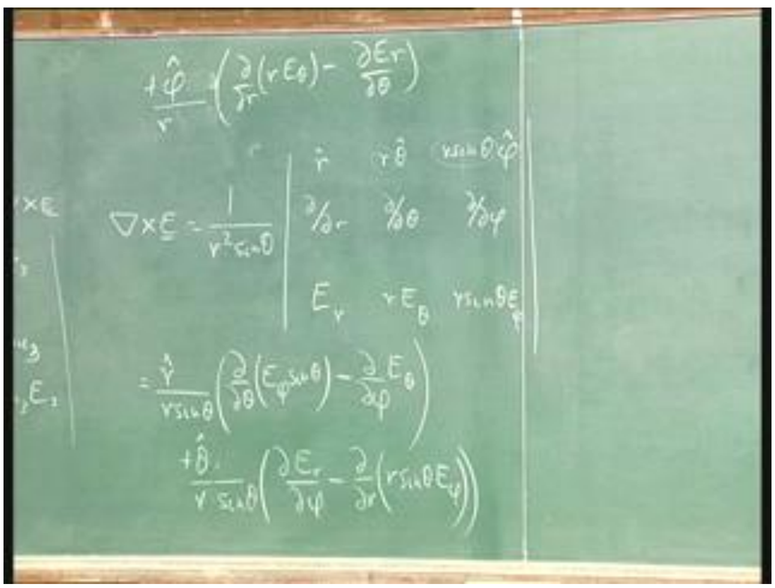

Plus phi hat again divided by r square sine theta del del r of r e theta minus del e r del theta. Let me just check that I have not made any mistakes. I have forgotten these two terms. So, this will come as r here and r sine theta here. So this is the result, r sine theta will cancel out here and r will cancel out here. So if you, so this is what you get. So it is a fairly complicated looking expression but it is not as bad as it looks.

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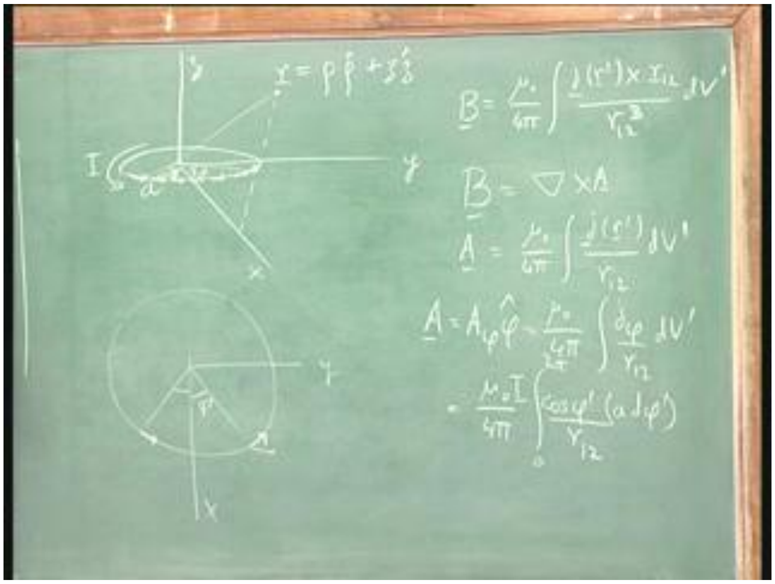

Now what we are going to do is we are going to take a loop and I am going to put the loop in the origin. So center of the loop is at the origin. The loop has a radius a, it has a current flowing of i. This is the x axis, this is the y axis and this is the z axis. Now I am interested in a point r and because they are there is symmetry present I can always rotate it so that this point points along x. So this point r is what I will call row, row hat plus z z hat in cylindrical coordinates. That is it has got a distance row and height h of z.

Now different current elements on this loop are going to cause magnetic field here and we know the expression. The expression is b is equal to mu naught over 4 pi integral over the whole volume j of r prime cross r 1 2 divided by r 1 2 cubed. What that means is, I take each of these pieces connect the line joining them to r then take the cross product of that line and this j, the j and this line and integrate over it.

Now you can see that is going to be a very complicated operation because the solid angle between j and this line is going to continuously change and I have to work out that angle work out the resultant direction and then integrate over them. Luckily, we have a better way. We know that b is equal to curl of a and a is equal to mu naught over 4 pi volume integral j of r prime divided by r 1 2 d v prime. Now at least we do not have to do the vector cross product that vector cross product is done later, when we take the curl to calculate a we only sum over j's. But we still have to divide by this distance r 1 2.

Now if you look at this case consider that i take two current elements which are the same angle with respect to x 1 is at minus phi and the other is at plus phi. Now if you look at it from above this direction is x. This direction is y. I have a current flowing this way and I have a current flowing this way. Both of them are the same distance from the observer because you can see that they are symmetrically placed. So, the distance from this element to r is the same as this element to r.

So it is the same r 1 2 but the j points in different directions. But because r 1 2 is the same when I do integration, I can add up these 2 j's integration is nothing but sum and if I add these 2 j's. Vectorially I get a j that is horizontal and this coordinate system a horizontal j corresponds to a j phi. So this j had both in the x direction and in the y direction. So did this but the net was only in the y direction and in spherical coordinates that y direction would be nothing but the phi coordinate.

Since for every one of these elements there is a corresponding element, I can take them in pairs. I can add them up. So the final answer is a is equal to something that is only along the phi direction. So, if i want something only along the phi direction, obviously only j phi can contribute. It is equal to mu naught over 4 phi integral j phi over r 1 2 d v prime. Now this j phi is actually coming out of a current I which is only present on this loop. So I can replace this volume integral by mu naught over 4 phi integral 0 to 2 phi a d phi prime that is the d l times the current I.

But this current if this is angle phi prime, this current is flowing not in this direction, but at some other angle. So how much of the current is flowing in the phi direction. Well, all of it is flowing in phi 0. None of it is flowing in phi 90. I can put down a cos phi prime all right. So what I have got divided by r 1 2. What I have got is a simplified expression for calculating the vector potential, if I can calculate this vector potential, then I can take the curl to get the magnetic field. It may seem that this is not terribly simple and it is not, but it is a great deal simpler than calculating this expression. This expression is a torture, all right. So now I want to calculate, what r 1 2 is.

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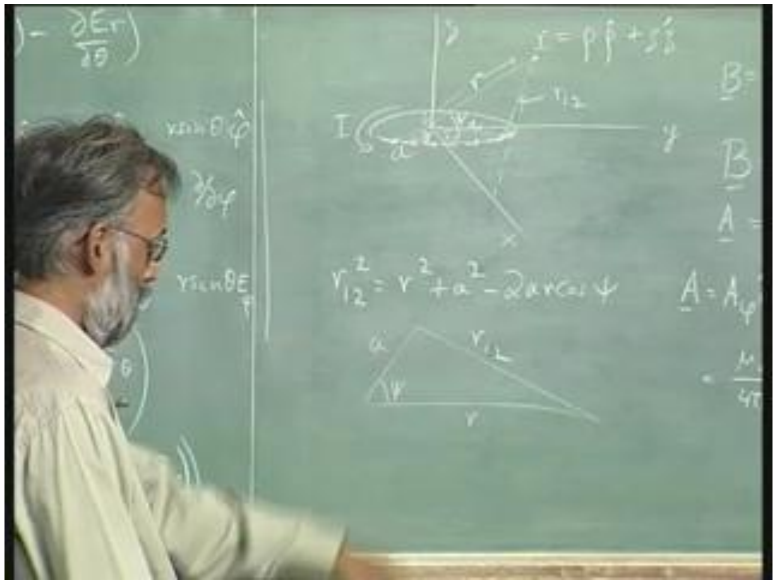

If you look at this point and I draw a line joining the place where the current element is to the observer. This is my r distance because that is nothing but the length of the observer’s vector. This is, this length is a, because the loop has a radius a and this is r 1 2. So, it is a triangle and I know that if I want r 1 2, I can write down r 1 2 square is equal to r square plus a square minus 2 a r cos psi where cos psi is referring to this angle psi the angle between the r vector and the a vector.

This is what is called the triangle relation. You would have derived it in school. For a right angle triangle this goes away and you get Pythagoras’ theorem. For a straight line you will get this becomes unity and then that just says that r 1 2 is r plus a or r minus a. But in general if you have any triangle and this angle is phi this is r this is a this is r 1 2. This is the relation you must satisfy.

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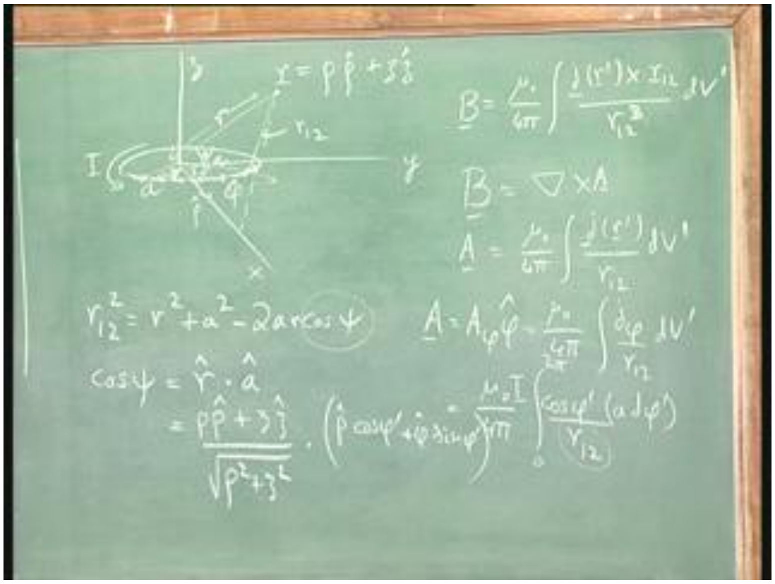

The problem in this is very straight forward except we need to know cos phi. If we can calculate cos phi we can substitute in there and we can write down the integral but what is cos phi? Cos phi is the angle between this radius vector and this observer vector and the radius vector of the loop. So, in order to calculate cos phi, what i am going to do? I am going to say cos psi is equal to unit vector along r dot unit vector along a. I know that because if I take the dot product of these two, it is magnitude of this unit vector times magnitude of this unit vector times the cos of the angle between them.

Let me write down what these are. This r hat is nothing but I have taken the observer position itself written in cylindrical coordinates actually and divide it by row square plus z square. What about a? Well, a is nothing but this vector which can be written in terms of row hat and in terms of phi hat. So this becomes row hat cos phi prime. Remember this is phi prime plus phi hat sine phi prime. May be if i draw a picture it will make it clearer.

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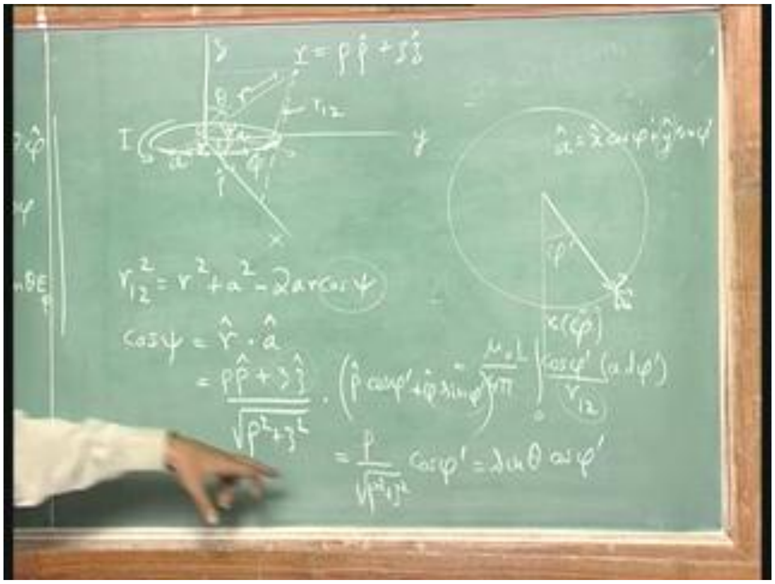

This direction is x which is also the direction phi hat-, it is also the angle phi for the observer. Observer is in the x, z, plane. I have rotated the coordinates so that he is in the x z plane. I am talking about a current element that is at an angle phi prime. So, if I want to know what this direction is I want unit vector on along the radial line in that direction. I can build it out of these two vectors. That is the component along x and the component along y. So I can write this as a hat is equal to x hat cos phi prime plus y hat sine phi prime.

But this x hat is nothing but the cylindrical coordinate r direction and y hat is nothing but this direction which is the phi hat direction. So, that is why I have written it in this fashion. Now, the purpose of doing all this is that when I take the dot product there is only one common direction in all this. The z hat appears only here and the phi hat appears only there. So, the dot product is only between these two terms. So I get it is equal to row over square root of row square plus z square times cos phi prime because there is no z hat component in that vector.

There is no phi hat component in this vector. This row over row square plus z square. What is that? It is this length divided by this length which is nothing but sine theta cos phi prime. It is a very standard result. It is just that I thought it is better to derive it. So the angle between the observer vector and this radius vector this psi is sine theta cos phi prime. Let me put that in and let us see where we get. What I want is the value of a.

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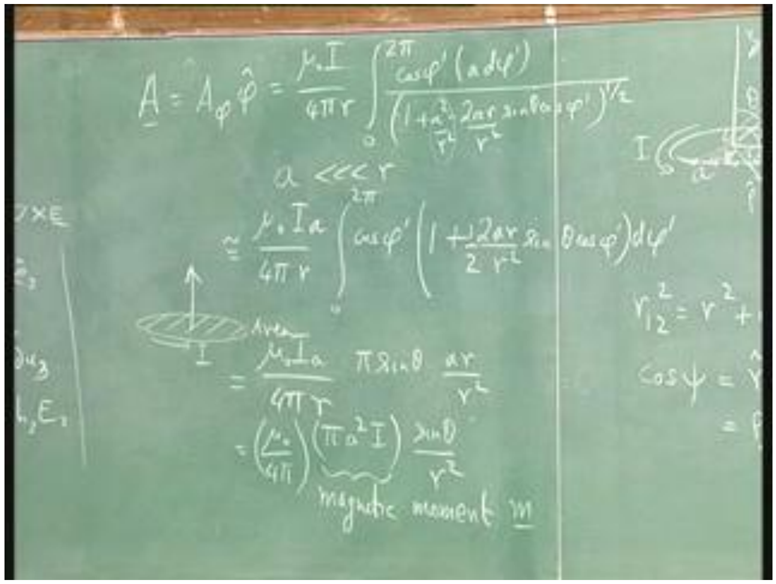

The vector potential a is a phi phi hat which is equal to mu naught i over 4 phi integral 0 to 2 phi cos phi prime times a d phi prime divided by r 1 2. What is r 1 2? It is equal to r square plus a square minus 2 a r times sine theta cos phi prime to the power of half. I have taken this cos phi and substituted sine theta cos phi prime. So that is the expression we have. This is r 1 2 and this is d l. This is a very complicated integral except that wherever we are interested in this integral we know that a, is much much much less than r. We are talking about loops that are the radius of an atom.

So they are angstroms in size. We are talking about the distances that are of the order of macroscopic distances. So a is tiny compared to r and so we can expand this denominator. If we expand it, I can pull a r out. I am going to doing it in place to save some place. One I will divide by r square divide by r square and pull out a r. So you know that 1 over square root of 1 plus something is equal to 1 minus that same something divided by 2. That is 1 over square root of one plus x is equal to approximately 1 minus x over 2.

This is your binomial expansion for 1 over square root. So if I use that there, I get that equal to mu naught i over 4 pi r integral 0 to 2 pi, I will put this as approximate. I will pull this a out cos phi prime times, 1 plus now this is a very small number. So I am not going to keep it. I am only going to this term, 2 a r over r square with a half sine theta cos phi prime d phi prime. So I have taken this and I have applied the assumption that this is small and brought it to the numerator. The minus sign has become a plus sign and I have got a half.

Now what happens? I am doing an integral from 0 to 2 pi integral of 0 to 2 pi of cos phi prime d phi prime is 0. So, this first term cannot give me anything, but the second term involves cos square phi prime and I know that integral 0 to 2 pi of cos square phi prime d phi prime this is the average half. So it is equal to pi whereas integral 0 to 2 pi cos phi prime d phi prime equals 0. So since I have this, I can simplify this integral completely. I get this is equal to mu naught i a over 4 pi r. This is going to give me pi sine theta times this factor a r over r square. The factor of two cancels out.

Let me collect terms. So, I get mu naught over 4 pi. That is still my normalization constant for the magnetic field. Then I have pi a square i and then I have a sine theta divided by r square. So, I have taken care of mu naught over 4 pi pi a square I sine theta and there is a r cube in the denominator and r in the numerator. So I am left with r square. This quantity is the area of the loop pi a square multiplied by the current and it is called the magnetic moment. It is given the symbol m.

It is a vector in the sense that if I have a loop with the current i and an area a the sense in which the current is rotating it take my right hand screw driver and I tighten that screw driver in the direction, the screw is rotating and see how that screw driver is rotating. That direction gives me the direction of my vector. The magnitude is the current into the area. So, the final answer is the vector potential is equal to a phi along the phi direction. It is equal to my normalization constant mu naught over 4 pi times the vector times the magnitude of that magnetic moment times, sine theta over r square.

This can be written in a simpler form conceptually simpler form which is its mu naught over 4 pi times the vector magnetic moment cross gradient of one over r. That is because we know that gradient of one over r is nothing but minus r over r square and if you take z cross minus r over r square, you will end up with the desired expression. I am not sure of signs. You can check the sign of that result, all right?

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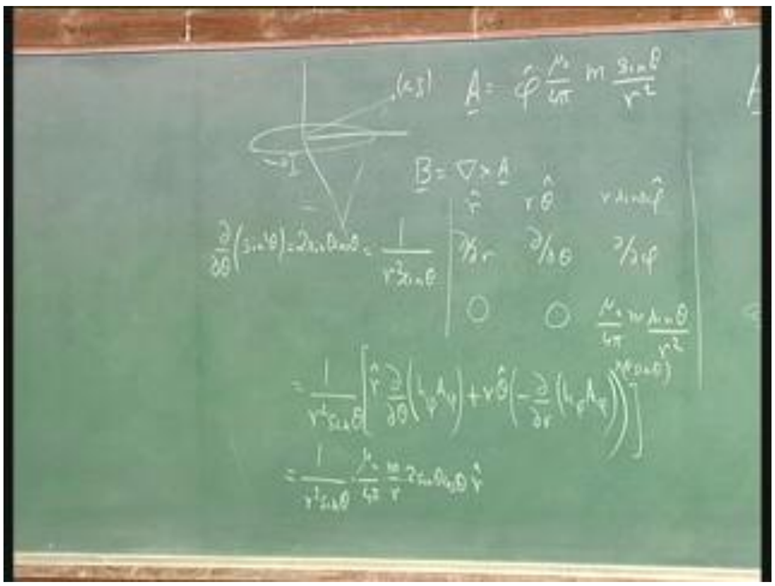

So now, I have a result for this loop, the loop is at origin. The observer is also at some point in the x-z plane and for this observer with the current i. I have that the vector potential a is equal to phi hat mu naught over 4 pi m cross, sorry m into sine theta over r square. But this is only the vector potential. I want the magnetic field and there is no point calculating the vector potential if calculating the magnetic field after that is even harder. The magnetic field b is equal to the curl of a and we worked this out.

But let me do it again. It never hurts to repeat. It is r square sine theta times r hat r theta hat r sine theta phi hat del del r del del theta del del phi, a is only along phi. So 0 0 mu naught over 4 pi m sine theta over r square and multiply it by r sine theta that is my same r sine theta that comes into h phi. Now it has to be seen that I cannot have any derivative of phi because there is nothing here. The only term that is present is a phi. So I get derivatives with respect to r and derivatives with respect to theta. So, it will be straight forward to write down what the curl is. It is one over r square sine theta r hat times del del theta of this.

I will call it as h phi a phi and then plus r theta hat times minus del del r of h phi a phi, h phi a phi is nothing but this term. So what I have done. I have just expanded along the first row r hat has del del phi of this minus 0, r theta hat has 0 minus del del r of this and phi hat has 0. So there are only two terms that I have to worry about. When I take the theta derivative, I am taking theta derivative of sine square theta. Let me write this out del del theta of sine square theta is equal to twice sine theta cos theta. Is that all right?

Because at first I can apply my chain rule, so d d sine theta of sine square theta is twice sine theta times d d theta times sine theta which is cos theta. So the first term has a 2 sine theta cos theta coming out of it. So, it is one over r square sine theta times again mu naught over 4 pi times m, over r times 2 sine theta cos theta and this is along r hat. The second term has a minus sign and there is a factor of r.

Now I want the r derivative of h phi a phi. This is one over r because there is a r in the numerator and r square in the denominator. If you take del del r of 1 over r it is minus 1 over r square. So, you just write out mu naught over 4 pi m sine theta times minus one over r square and this is along theta. Now the r square sine theta the sine theta part cancels out in both these cases. So I will remove the sine theta sine theta sine theta r square and r make it r cubed and here you have an r in the numerator and r to the power of 4. So, again r cubed. So we combine everything.

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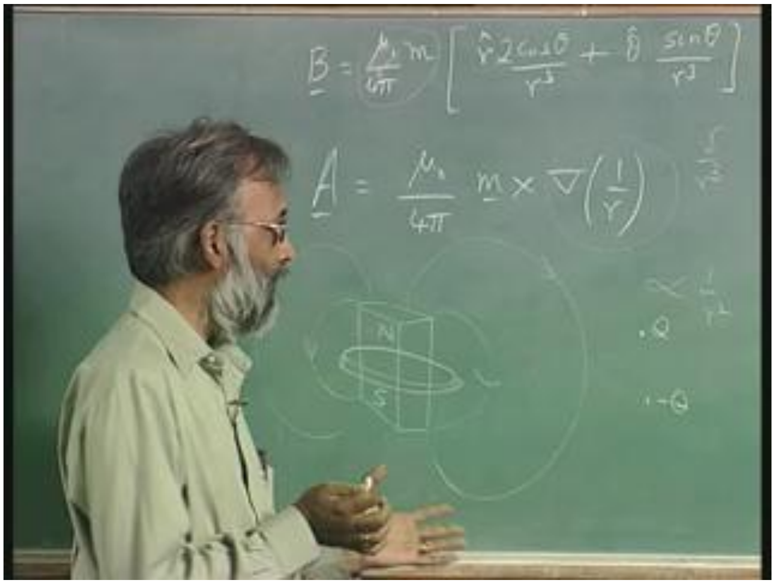

The final answer you get is the magnetic field b is equal to r hat times we can pull out the common factors mu naught over 4 pi times m times r hat. So I get cos theta over r cubed to the factor of 2 plus theta hat sine theta over r cube. The factor of 2 comes out here, here, there is a cos theta, there there, is a-, there should be a sine-, this is sine square theta. So one sine theta goes, other sine theta comes here the r cubed is common. So this is the expression for the magnetic field and let me recapitulate how we got there.

We started with a loop of current. This loop of current this picture here is producing a magnetic field far away. Now we can directly try to calculate the magnetic field but that is very difficult. The reason why it is very difficult is, you have to do a cross product inside the integral and it is bad enough doing a curl but doing the cross product and then integrating is much worse. So, instead we work with the vector potential which directly integrates current and because it directly integrates current we can add up symmetric components of current and conclude that there is no component of the vector potential along x.

That is all the components is along y and in spherical geometry the y direction is nothing but the phi direction. So we calculate a phi and a phi involves mu naught i over 4 pi. It involves this cos phi prime simply because the current is not pointing along phi, it is pointing in a slightly different angle that is taken care of by cos phi prime and then there is a one over, r. Using the triangle inequality, we worked out what 1 over r should be and we got this expression which is r square plus a square minus 2 a r cos psi and by dot producting two directions. We were able to obtain that cos psi is sine theta cos phi prime expanding the denominator we get the answer. What does this answer tell us? Let me write down the vector potential as well.

These expressions are only for very tiny loops when you look at the field very far away. What the vector potential tells us is that once again it is the r over r cubed factor that is present. You can see that the magnetic the vector potential is proportional to 1 over r square which means the magnetic field is proportional to 1 over r cubed. This is very similar to the dipole, if you go back to your electrostatics notes and look at the electric field due to a dipole it will be exactly the same exactly the same dependents.

So in a certain sense if you properly change the multiplying constants the magnetic loop is exactly like an electrostatic dipole and that is why if you look at a bar magnet and you had a north pole and south pole and you looked at the fields, due to that magnetic, due to that magnet it is essentially the same field that you would have got if you put a charge q and charge minus q.

This is an electrostatic dipole. This is a magnetic dipole, they have the same structure and now what we are finding is you took a loop and put that loop in the middle of the magnet and removed that magnet you will still get the same pattern. So a loop of current or an electrostatic dipole or a bar magnet produces the same field far away. The field decays faster. It decays as 1 over r cubed. So, you would not see it once you go very far away. Now there is one other very interesting thing that you can see directly from this.

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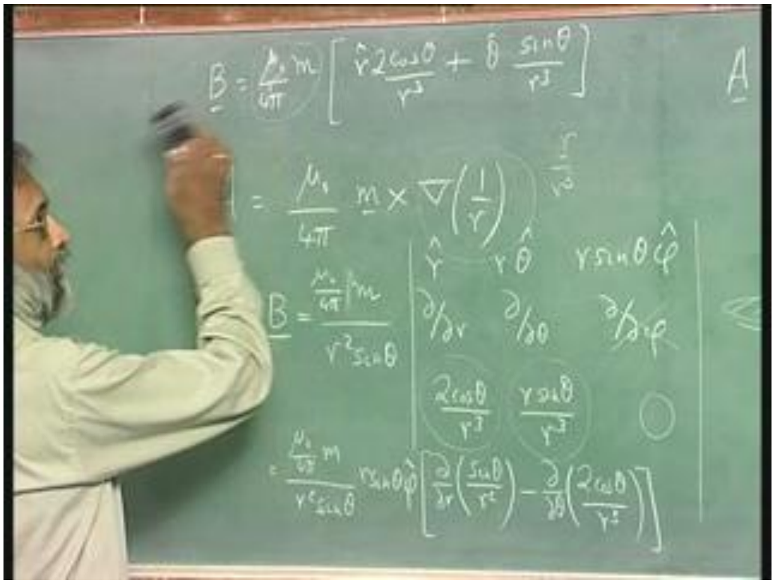

I am sure you do not want to see it, but you can take the curl of b. This is probably going to sound very painful to you because we have doing a lot of vector analysis. But let us do it, h 1, h 2, h 3 is 1 over r square sine theta r hat r theta hat r sine theta phi hat del del r del del theta del del phi and then these two components. There is a constant. So I can pull the constant out. So I can call this mu naught over 4 pi times, m.

So what is down here is twice cos theta over r cubed and the theta component is r times, sine theta over r cubed and then there is 0 here. Now if I want to take this determinant it is important to realize this term and this term are not functions of phi. So, this term goes away. There is no derivative with respect to phi that is possible here because the field does not depend on how you on which phi angle you measured it. Because, it is a loop it looks the same at all angles.

So it is only these two derivatives that can make any difference. If you look at the r component del del theta acts on zero and del del phi is not present. So there is no component along r. If you look at the component along theta del del phi cannot act and del del r acts on 0. So, you are left only with the phi component. So it is equal to mu naught over 4 pi m, over r square sine theta times r sine theta times phi and then you have these two terms which is del del r of sine theta over r square minus del del theta of twice cos theta divided by r cubed del del theta of cos theta is minus sine theta. So, I will do it. Well I am going to erase the middle of this board.

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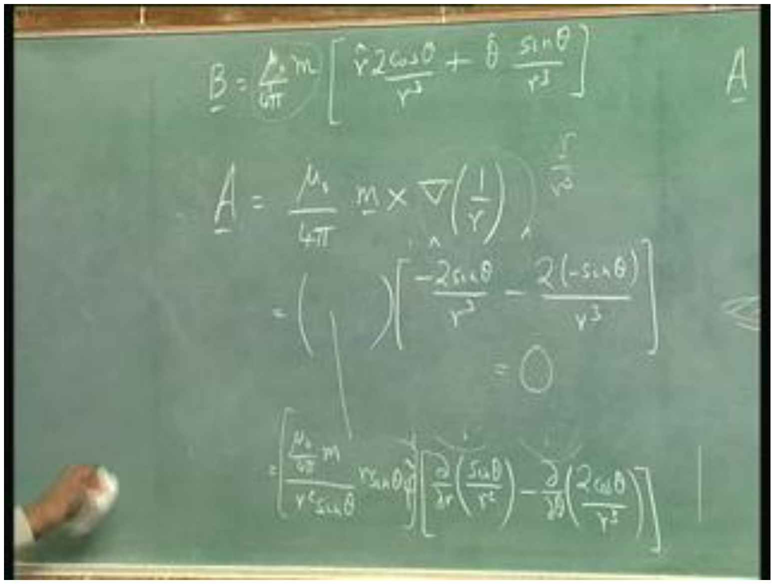

Del del theta of cos theta is minus sine theta, del del r of 1 over r square is minus 2 over r cubed. So it becomes this common factor times minus 2sine theta over r cubed minus twice minus sine theta divided by r cubed. So derivative of 1 over r square is minus 1 over r cubed, derivative of cos theta is minus sine theta. Well it is 2 sine theta and r cubed in both places. There is a minus sign here there is a plus sign here. This is 0. So not only do we have these expressions, we have also proved something very very important.

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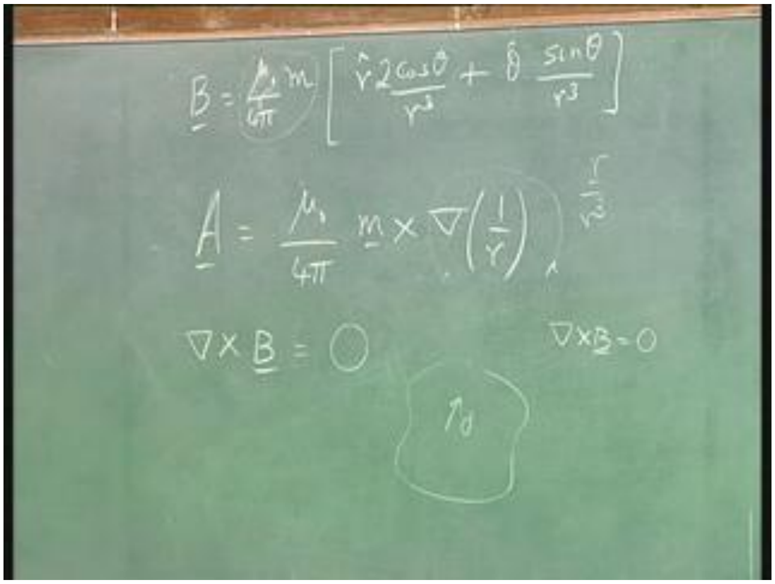

If you take the curl of the magnetic field due to a loop far away that curl is 0, now why is this important? Supposing I take any general current distribution, I know that I am looking at divergence j equal to 0 which means the current must go somewhere current cannot just vanish. Since the current cannot just vanish, I can build this current out of little, little loops. So for example any general current can you build out of little loops at every point this current and current cancel except at the outside. It is Stoke's theorem in reverse. We did the exact same construction for Stoke's theorem.

So any current distribution can be built out of tiny loops and we know the magnetic field due to a tiny loop and that magnetic field has 0 curl. What that means is that, if I have any general current distribution j away from that current distribution curl of b is equal to 0. What happens inside the current distribution? Well if I am looking here whatever currents are there outside where I am looking curl of b is 0. But where I am looking itself curl of b will not be 0. It takes a little bit more mathematics to prove the final statement.

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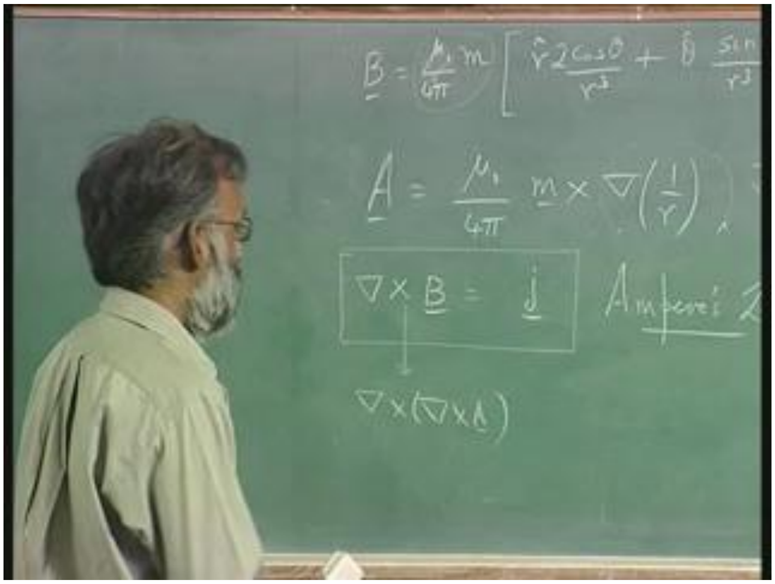

The final statement is if you consider currents everywhere curl of b is 0 for currents far away and it is equal to j itself nearby and this is called ampere’s law. There is a mathematical way of proving it and that is again given in appendix in your text book. What you do is you say curl of b is curl of curl of a. Apply a few vector identities, then do a lot of mathematical gymnastics and you can prove the same result. Now we cannot actually prove that it is equal to j but what we can prove is it is not equal to j at any other point. Utmost it is equal to j at that point and we know that b is derived from j.

So it must be a function of j in some way. So the purpose of doing this exercise was just to motivate the answer that it makes sense that curl of b should be related to j at the place itself. It cannot be related to j anywhere else because we know that form the loop curl of b is 0, all right? So we have done this rather painful derivation, but I think the lesson to be learnt from this derivation is that the vector potential is useful. When you first encounter the vector potential the question that should occur to you, even though I did not mention.

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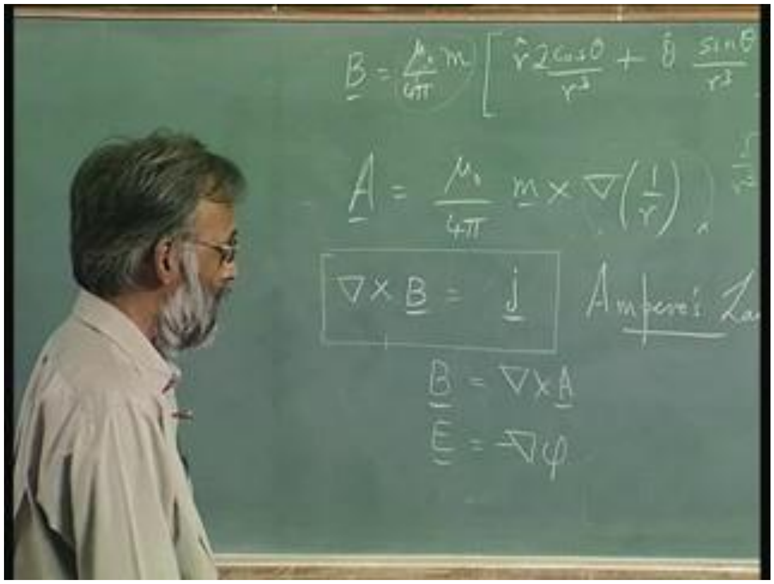

It is that I am having a vector field b and I am saying b is curl of another vector field a, and then I say what a great thing I have done. I am saying I can get this vector from another vector field that is hardly anything. I mean the electric field I got from a scalar field. So, instead of three components I was dealing with one component. But when I went to the vector field three components came from 3 components, so I do not see any saving by working with a vector potential rather than working with a vector field with the magnetic field. Well, the saving is there. The way to look at it is not to count the number of components. Rather what you have to do is to look at what is inside the integral and if you that here is what you see.

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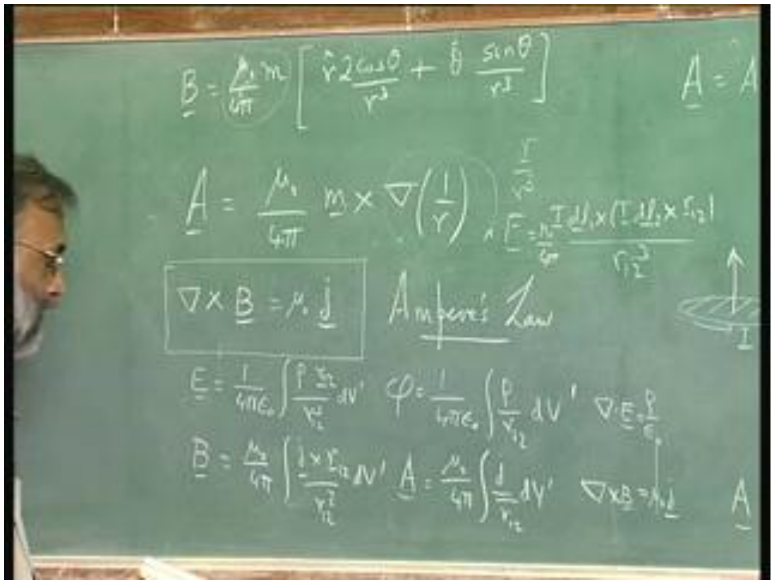

The electric field is one over 4 pi epsilon naught volume integral row r over r cubed r 1 2, I will say d v prime whereas potential was equal to 1 over 4 pi epsilon naught integral row over r 1 2 d v prime and because of this construction we had that divergence of e was equal to row over epsilon naught. Actually there should be a mu naught here. Now b is equal to mu naught over 4 pi volume integral j cross r over r cubed r 1 2 d v prime a is equal to mu naught over 4 pi j over r 1 2 d v prime curl of b is equal to mu naught j.

Look at the symmetry of these equations. The electric field is derived from a charged density, the potential involves only that source divide by r 1 2 and if you take the divergence of electric field, it is related to the source at that same location at the observer's location. The magnetic field is derived from a cross product of its source the current density and r over r 1 r cubed. The vector potential simplifies that by being just the source divided by r 1 2. No cross product no vector operation at all and the curl of b is connected to the source at the observer's location.

So there is a complete symmetry between electric field and magnetic field. It is only that we are very used to thinking of vectors that keep going in the direction that we push them in. So, we are very happy with the electric filed because if the electric field is in a certain direction the force is in that direction. Whereas the magnetic field if it is in a certain direction, the force is 90 degrees to that direction. The source of the magnetic field is again 90 degrees to its. The direction in which it points but that complication all came from the fact that we had that force was equal to i d l 1 cross i d l 2 cross i one 2 divided by r 1 2 cubed.

Once you had this kind of structure and i 1 cross i 2 cross r gradient of one over r, you have to have these kinds of ninety degree relationships. It is force is i d l cross b. Therefore already b has to be 90 degrees to force b itself is i d l cross r over r cubed which means, b is 90 degrees to its own source. So that is why you keep seeing cross products everywhere. But cross products not withstanding there is a complete symmetry between electric and magnetic fields and as long as you do not get scared by the cross products, you can in fact use all the techniques that are used in electrostatics in magneto statics and we will do that next time. We will basically use Laplace's equation to solve some simple magnetic problems.