Electromagnetic Fields
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Lecture – 21
The Curl

Good Morning. Last lecture was a very heavy lecture. We were introduced lots of new mathematical concepts. So this lecture I am basically going to rework through those concepts, make them a little more sensible hopefully, so that you feel comfortable with all the things I did. So, let me put down what we did.

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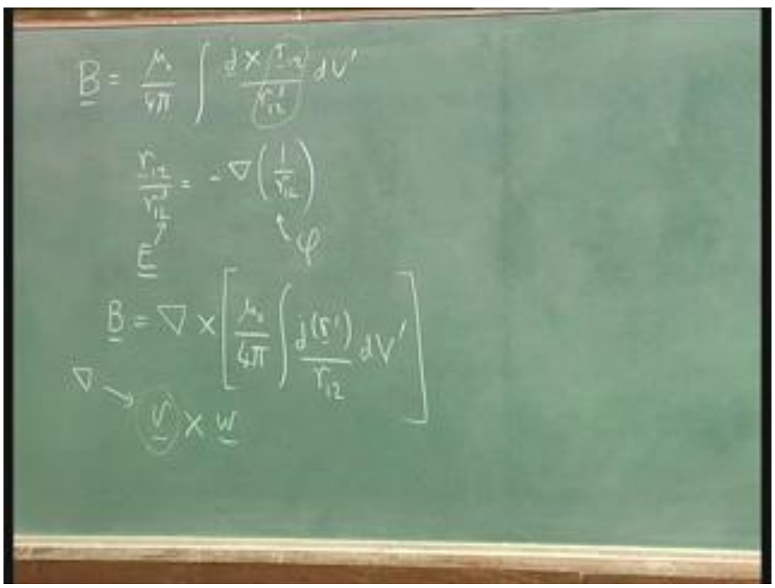

We talked about the magnetic field and we said that the magnetic field which is earlier, we have got is mu not over 4 pi, volume integral j cross r 1 2 over r 1 2 cubed d v prime. We took this expression and we identified that this piece r 1 2 over r 1 2 cubed is nothing but minus the gradient of 1 over r 1 2. This came from electrostatics. Because, this was found in electric field; this was found in potential. So, minus of the gradient of the potential is electric field and the only difference between the two expressions is this. Then it must be true that it is all coming out of the fact that minus of the gradient of one over distance is distance over cube of it is amplitude.

So it is actually mathematically, this is the most important relationship that derives everything else in electro magnetics. So, you take this you put it in there and you come up with a new result. You come up with a result that B is equal to a new operator that B is the curl of something else and what exactly is this curl. This curl is nothing but you take the vector product cross product of two vectors and you replace one of these vectors the first one by the operator gradient.

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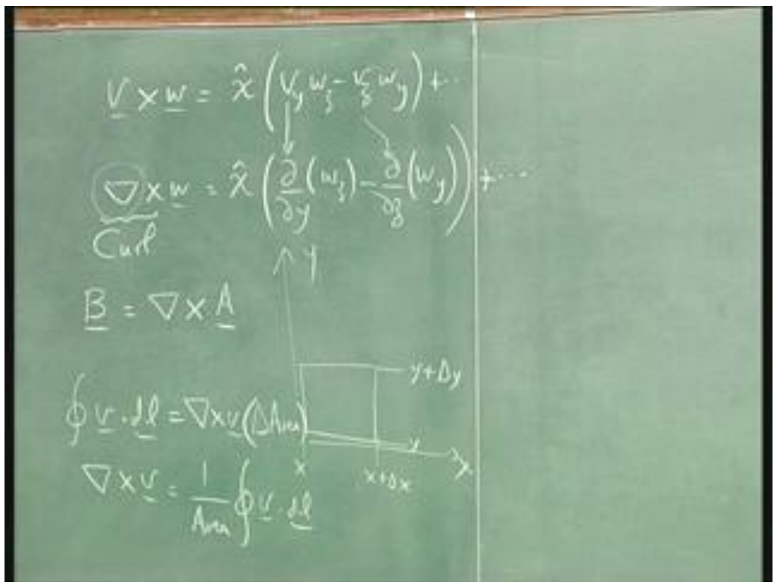

So for example instead of saying v cross w which would be x hat v y w z minus v z w y plus other terms, now we replace it as curl of w is this x hat. This v y becomes del del y of w z minus this v z becomes del del z of w y plus dot, dot, dot. So we are treating gradient the gradient operator as a vector and when you do that, this is now only a function of one variable one vector field. The other vector field is being replaced by a differential operator.

So, whereas this was a function of two vector fields this is now only a function of one vector field and it gives you back a vector field. So, it is a vector operator, it takes vectors and gives you vectors and this operator is called the curl and what you have seen is that the magnetic field is the curl of something of a vector field. So, obviously curl is a very important concept. Now we also saw the last time that if you took this curl concept and you looked at it closely what you could say is supposing i go to a very small loop in say x y this is at x this is x plus delta x.

This is, at y, this is at y plus delta y and if you worked out what was loop integral v dot d l for any vector v any vector field v what you found. We worked it out was that it is equal to curl of v times the area. So, in other words a definition of curl of v is 1 over area loop integral v dot d l where this loop is a very tiny loop, it is size is delta x delta y. I am not going to repeat the derivation because we did that. Now why this is interesting is, we have already seen this somewhere else. We have seen loop integrals in connection with the electric field.

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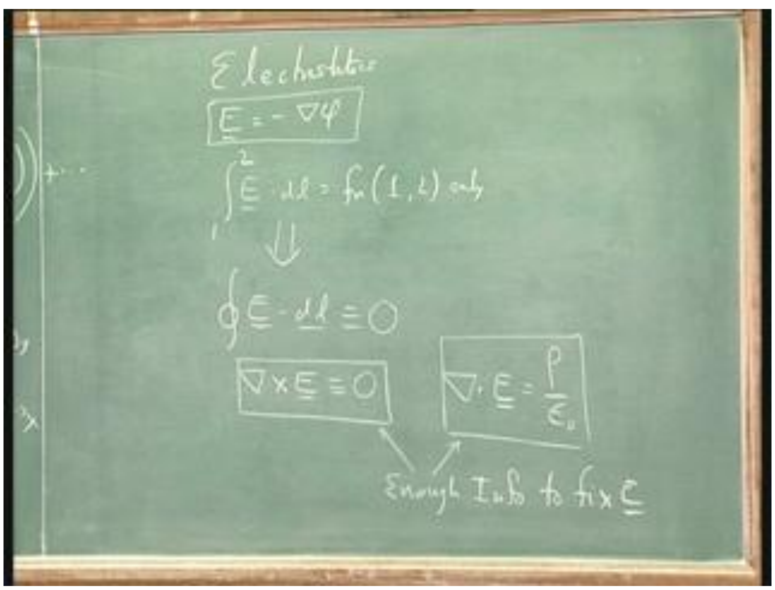

So, in electrostatics we had that e is equal to minus gradient of phi and the very same derivation that allowed us to write this which was that integral e dot d l from position 1 to position 2 is a function of 1 and 2 only. That is it does not depend on how you went from 1 to 2. It depended on where you started and where you ended. So, if you took 1 to 2 and you took 2 to 1 and combine them together you could prove that loop integral e dot d l was always 0.

No matter what loop you took e dot d l is always 0 which means you took a very tiny loop e dot d l is also 0 or curl of e is identically 0. So, in electrostatics we have e is minus the gradient of phi curl of e is 0 and divergence of e is equal to row over epsilon naught. So, this expression which I had been putting several times can now be put more mathematically. It is saying the same thing. Saying this is equivalent to this because the definition of curl is nothing but one over the area loop integral of e dot d l.

So saying curl of e is 0 and saying loop integral e dot d l is 0 is to restate the same facts. But this is a differential operator. This is an integral and there are times when the integral is more useful and there are times when the differential equation is more useful. So we will use both of these, so you can see that in electrostatics we have got two equations. We have got an equation saying curl of e is 0 and we have got an equation saying divergence of e is row over epsilon and we also know something else.

We know that there is a unique solution to this problem, given these pieces of information given some boundary conditions there is only one vector field which gives you these answers which means curl and divergence are enough information to fix e all right. This is a very crucial statement. We will come back and make it more precise but it is worth looking at it. We found out that divergence e was something we found out curl is something.

We have also found out e is pinned down which sort of means that, if you have these two pieces of information you have the vector field and the statement to that effect is the theorem we will prove later, all right. So what I want to do today is to take this idea of curl and make it more sensible. It is all very well to write down curl, but what does it mean? So, I am going to take some examples and I am going to try and work out those examples and show you where curl is coming from and of course I will also derive curl for non-Cartesian conditions.

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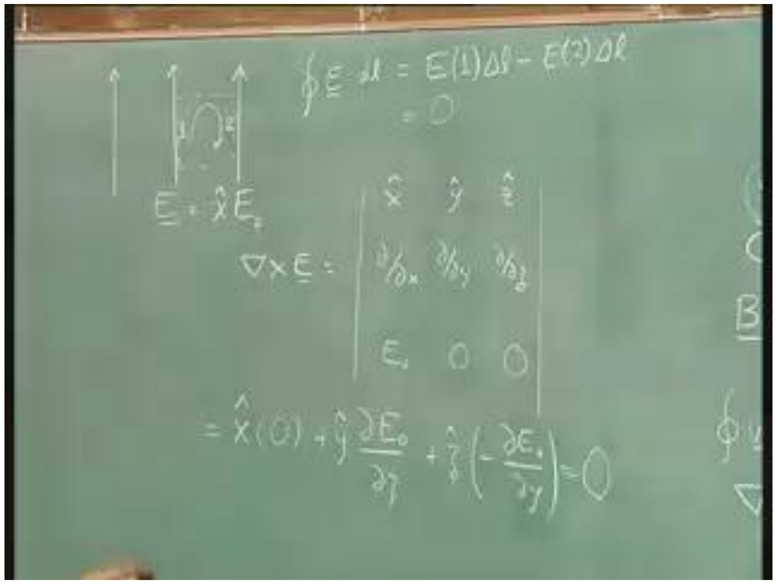

So let us take simplest example. I have a uniform vector field. Let us say the electric field. Electric field is uniform. So, I know that if I drew a little loop the loop is tangential to the field only on these two edges and these two edges it is normal. So, d l dot v, or d l, or d l dot e is 0. So, I know that loop integral e dot dl let me call this side one. This side two and let us say that I am going through this loop in this direction. So it is equal to e, at location one times delta l minus e at location two times delta l.

But e is the same at both places, it is a uniform field. So, it is equal to 0. What does curl tell us? Well, to write down curl I would want to actually give a form for this. I will say the vector field is along the x direction and it has some value e naught. So, let me take the curl of that. Curl of e would be equal to I will do the determinant x hat y hat z hat del del x del del y del del z, e x is e naught, e y is 0, e z is 0. So, I have to take the determinant of this matrix and that should give me curl.

Well, if I take x component I have 0 and 0 in 1 row so that 0x hat times 0. If I take y hat, then I have del del z of e naught. But this side is del del x of 0. Then, I have a z hat del del of 0 minus del e naught del y. But e naught is constant. So, it has no derivatives. So this becomes 0. So, this is an obvious case. We knew that there was going to be no curl and we found that the formula works out all right. Now let us try to do a slightly more complicated case. It is again a case where we know the answer.

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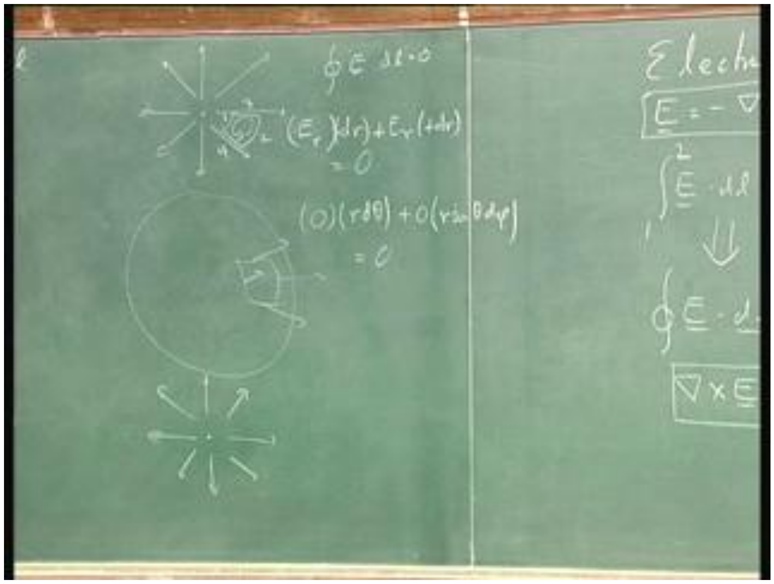

I am going to take the field of a point charge. If you take a field of the point charge what do you expect the curl to be, I am not going to work out yet it is spherical coordinates because we have not worked out what curl is in spherical coordinates. But I am going to argue for you that curl should be 0. We already know that in a certain sense because we know that loop integral e dot d l is always going to be 0 for electrostatics. But let is work out that loop integral. Supposing I had a loop that did this, so the loop is pointing out of the board.

That is it is, face it out of the board and I am going to try and go round it in this direction the counter clockwise direction. Well, the electric field is normal to this and this. So, to edges one and two the electric field is normal. So, it does not give me any contribution. Besides 3 and 4 there is a contribution. So what is the contribution? It is minus e r times d r because the electric field is pointing outwards the loop is going inwards. So there is a minus sign all right. We should say e r times minus d r plus here also e r times d r.

So, it is equal to 0. So obviously if I take a loop of this type this, electric field cancels this electric field. The loop is going in opposite directions but the electric field is the same. If I take a loop that is on the surface of a sphere well the electric field comes out of this sphere. So it is 90 degrees to the loop at every point. So clearly it is just zero times r d theta plus 0 times r sine theta d phi. So it is 0 so that particular loop does not give me anything. Now I can look at the loop that corresponds to a vertical direction rather than a horizontal direction. But because this is spherical symmetry if I look in the vertical direction rather than the horizontal direction it is still the same picture.

That is, if I cut the three dimensional picture vertically, I will still find that I have got a charge in the middle and I have got electric field pointing in all directions. So, no matter what I do? I am getting integral e dot d l is 0. As I said this is what I expect. Electrostatic fields cannot give me e dot d l non-zero. But it is good to always draw these pictures and make sure. Now let us go for a case where there is curl. Let us try and understand what sort of field can give me curl. First let me again look at a case where it does not give me.

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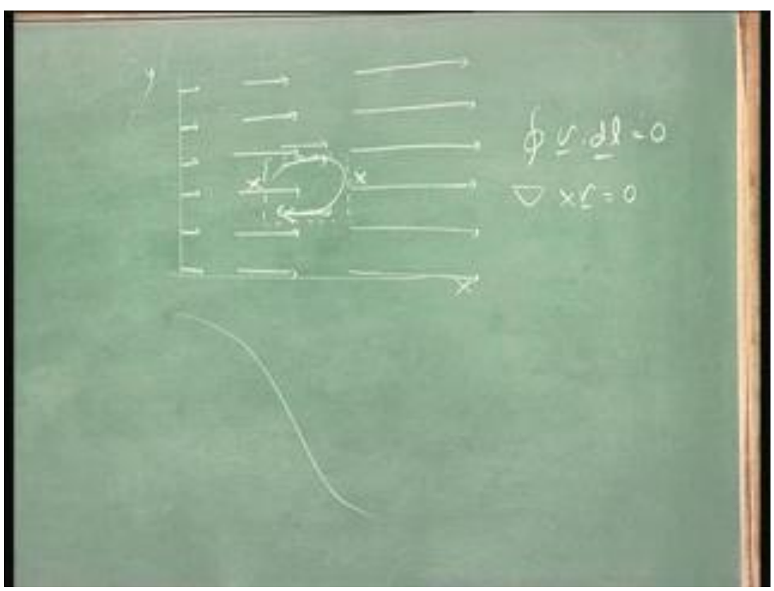

I am going to go to Cartesian. This is x, this is y and let us say I have a field which is uniform in y and growing. What sort of field can this be? For example, supposing you have a mountain and there is water flowing and you are looking this water velocity of the river. So here is slow, here it is speeding up, here it is very fast. So the water is flowing faster and faster is there curl. Well you can draw your dotted loop. There is nothing on this side. there is nothing on that side because they are 90 degrees. What about these two sides? Well in these two sides you can do integration. That is there is a net velocity here, there is a same net velocity here.

So, when you do the integration over this side, it is equal in magnitude to the same integration done on this side because it is the same vector at both points. But the loop is going this way here and it is going this way here. So, whatever you gain from your e dot d l on this edge, you lose on your e dot dl on this edge. So, loop integral v dot d l equals 0 which means curl of v equals 0. Now from the various examples we have seen so far it should be fairly obvious what you require for curl? What you require is a variation in the field which is at ninety degrees to the field itself? So that is the kind of example i am going to draw next. Let us look at it.

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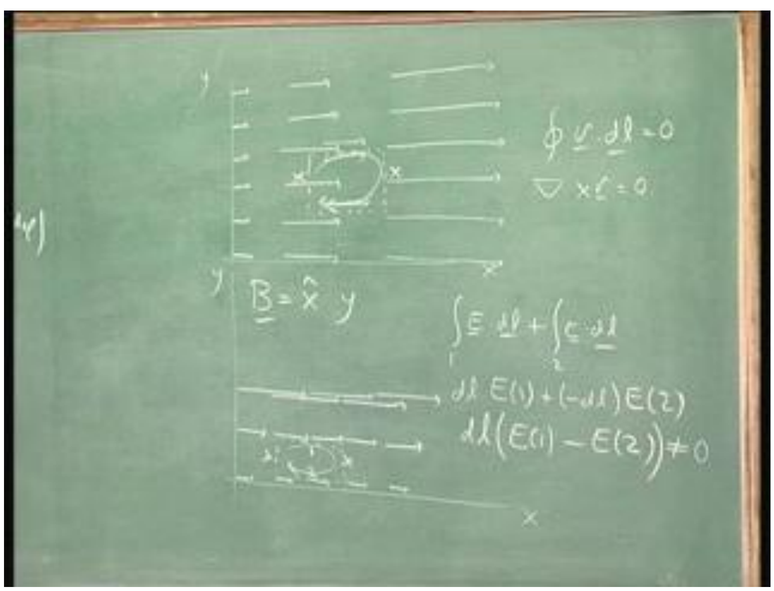

Again x y I am going to take a field which is weak and it is getting stronger and it will get stronger and stronger as you go further. Now what happens? Well, let me draw my loop as usual this side and this side cannot contribute because the field is 90 degrees to d l. So, e dot d l goes to 0. But now I have sides 1 and side 2. Let us say that I am doing my integration this way. So, I have integral over one e dot d l plus integral over 2 e dot d l.

Now clearly in the region one e is larger right. So it is d l times, e in 1. Here it is plus minus d l e at 2. But e, at one is not equal to e, at 2, e, at 1 is a large vector, e, at 2 is a small vector. So these two are not equal, this is bigger than this. So, even though they are tending to cancel, the cancellation would not happen. You will get d l times e of one minus e of two and it is not equal to 0.

This is a case where you have curl. There is a kind of an experiment you can do which is mentioned in every text book. What you can do is, imagine that you made a paper boat and you put it into this vector field. Just imagine it in your mind and you watch what happens to that paper boat or better than a paper boat put a flower. You can see that the side of the flower at the top is being dragged fast. The side of the flower here is being dragged back slow. It is being dragged back slowly. So what happens is, one side of the flower gets pulled further.

The result the flower starts rotating and the rotation of the flower is nothing but the presence of curl. In order to fix this in your mind I am going to take an actual expression for this field. I am going to say that I have say, an electric field. Why an electric field? I will choose a magnetic field which is equal to it is in the x direction but its magnitude is y. So, it is zero at y equals 0 and it keeps growing and gets bigger and bigger. So, that is very similar to this field. Now, I want to take the curl of this field. So, what do I get?

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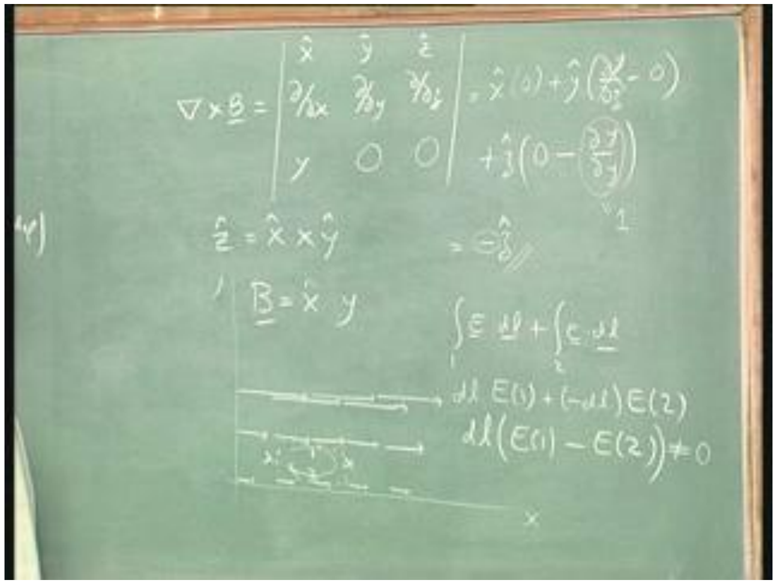

If i take the curl, it is the determinant of x hat y hat z hat del del x del del y del del z. The x component of b is y, the y component of b is 0, z component of b is 0. So, this is the determinant I want to calculate. Let us do it; x hat is the determinant of this 2 by 2 piece is 0, Y hat the determinant of this 2 by 2 piece is del y del z minus 0 and the z component is the determinant of this piece which is 0 minus del y del y. Well, del y del z is 0 because if I move in the z direction, the y coordinate does not change. That is the meaning of coordinate systems.

If I move along one coordinate the other coordinate is not changing. So this is zero. But del y del y is equal to 1 if I 1, if I move unit distance along y the value of y changes by unity. So, this slope of y with respect to y is 1. So, it is equal to minus z hat. So, curl of b is not 0 and in fact curl of b is minus z hat does that make sense. Well, you can see take a screw driver in your right hand and try to screw in the direction that you think the rotation is going to happen.

Imagine your flower put it into this field. You know the flower is going to rotate this way. It is not going to rotate the other way. It is going to rotate clockwise. So if you take a screw driver rotate it clockwise, it will go into the boat. But now what is z hat? Z hat is x hat cross y hat. That is the unit vector along z is the unit vector along x vector product unit vector along y. So, I take my screw driver again, but I now need to turn it from the direction x to the direction y.

Can I do it this way? No, I have to do it this way because I have to bring it from x to y. So, the z direction is coming out of the boat, curl is going into the boat and that is why we have the minus sign. So, this is a case of the vector field that has curl. Now of course, the common examples of curl are not in Cartesian fields at all. The commonest example that you would have seen is you wash basin.

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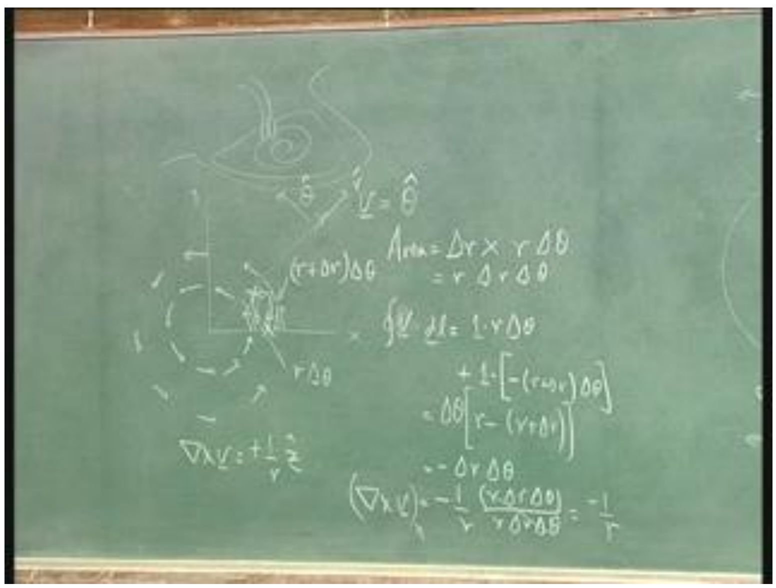

You have water coming from a tap and the water fills up and then you allow it to drain. So, what does the water start doing? The water starts circulating and then comes out of the, comes through the pipe and goes to the drain. This is a classic case of circulation. So, you just these fields they were first the reason for calling it curl because you could see that the vector field is curling round and round. There is not curl in a Cartesian field. It just so happens that the mathematical operation gives you curl, but when people first invented the operator they were thinking of eddies, they were thinking of whirlpools.

So, let us look at a whirlpool kind of field and see what kind of whirlpool field has curl and what does not. So I am going to draw now in r theta z is out of the board. This is my x y. So some general direction is r and theta is this direction. Now if I want to draw a circulating field I am going to draw arrows like this. Now the question is what kind of field of this type would have curl and what would not. So, I am going to guess well what about if I look at this. I will say what about a vector field v which is equal to theta hat.

So it is not growing in amplitude but it is still always rotational. So, for this we have to draw our e dot d l or b dot d l. So we make a loop. I have made a loop where the sides have 45 degrees and 0 gong from r to r plus delta r. Well, you can see that pieces at constant r are the ones that are parallel to the field. Pieces that are along constant theta are 90 degrees to this field. So, these two sides will not contribute exactly opposite of electrostatics. In electrostatics the field was radial.

Now we have got fields that are in theta. But, what about these two fields? This distance is our delta theta where delta theta happens to be 45 degrees in this diagram. This distance is r plus delta r delta theta and now I am going to do, I have to choose a direction in which I will around. So let us say I choose that direction. So, what will my loop integral e dot d l be sorry v dot d l be? There is a piece at r, the magnitude of v is 1. So, it is 1 times r delta theta and there is the other piece which is plus one times minus r plus delta r delta theta. The minus sign comes because theta is in this direction, but d l is in the opposite direction.

So, I get delta theta time's r minus r plus delta r; r's will cancel. So I get minus delta r delta theta. Now, to get curl I must divide by the area of this object of this loop. What is the area? Area is equal to delta r times this length which is r delta theta. So it is r times delta r delta theta. If you look at the curl it has only a delta r delta theta. So, curl of v is equal to well I put a minus sign. So, curl of v in the direction that is normal to this which is into the board n is equal to minus one over r times r delta r delta theta divided by r delta r delta theta.

What did I do? This is the value I have. I multiplied and divided by r. The area is r delta r delta theta. So, these two cancel out. So I get minus over r. So, the amount of curl is minus one going into the board. Well, as it so happens if you look at r hat and theta hat this way z hat is coming out of the board. So, if you wrote out what curl of v is, it is actually equal to plus 1 over r in the z direction. So this is a field that does have circulation.

If you put a flower in here what is going to happen? The flower will as it moves will see this velocity, will also see this velocity, but it is move much less here moving much more there as it moves it is going to start rotating and what you find is that this has a much greater effect because it is applied over a much longer distance. This has a much less effect and therefore the flower starts rotating. As it rotates, it shows up that there is a curl. So, we have seen there is circulating fields with curl are there circulating fields without curl.

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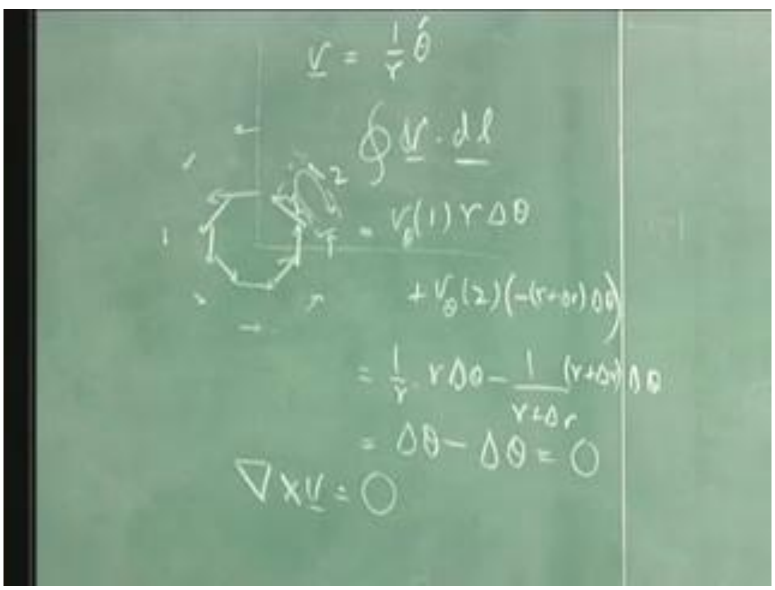

I am going to take a field which is large near the origin. It gets smaller as you go further away from the origin and gets smaller still. So, the particular field which I am going to look at is v is equal to one over r theta hat. Now what happens? If I do my loop integral, the same thing that I did earlier works except that I get, If I draw my loop, I find this is, let us say 1 and this is 2 v of times r delta theta plus v theta of 2 times minus r plus delta r delta theta. That is just the definition.

The sign comes from the fact that v is always pointing this way but d l id pointing the opposite way. But v theta is now one over r, it is equal to 1 over r times r delta theta minus 1 over r plus delta r r plus delta r delta theta. You can see the r cancels out the r plus delta r cancels out. So you get delta theta minus delta theta equals zero. So, this is the case a field that is getting weaker as you go out specifically as one over r. This is the field that has zero curl.

So it is not true that every circulating field has curl. It is not true that every straight field does not have curl. Curl comes from your imagination that we put in a round object and does it start rotating. If it starts rotating, it is curl. Curl is present if it does not rotate curl is absent. Now you can keep on drawing e dot d l's. While actually you can. But, mathematically you would like to formulize and find out a mathematical way of calculating what curl is. Your appendix a in your text book will give you the answer. Supposing you have I have introduced this idea before, but I will repeat it.

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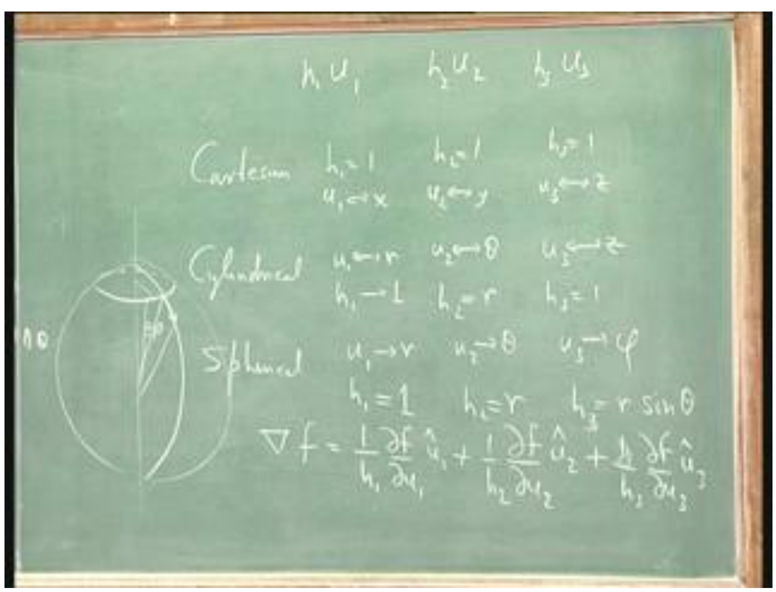

Supposing you have instead of x, y, z, I am going to call them u 1 u 2 u 3 and d l along u 1 is h 1 delta u 1, this will be h 2 delta u 2 and h 3 delta u 3. Let me give examples for Cartesian h 1 equals 1 h 2 equals 1 h 3 equals 1, u 1 is nothing but x, u 2 is nothing but y, u 3 is nothing but z. For cylindrical u 1 will be r u 2 will be theta u 3 will be z. If you look at distance along r distance along r is nothing but d r. So, h 1 is 1 distance along theta is not d theta. It is r d theta. So, h 2 is equal to r distance along z is d z. So h 3 equals 1. We go to spherical, you u 1 is r, your u 2 is theta, your u 3 is phi.

Distance in spherical coordinates along r d r. So, h 1 is unity. If you go along theta, so that means this is my axis. I go on a great circle. So, if I go on a great circle by a distance subtended by delta theta this distance is r delta theta. So, h 2 is equal to r. If I go in the phi direction, then I am going on a circle that is fixed latitude and this is the angle delta phi. This distance is delta phi times this radius which is r sine theta. So, h 3 is equal to r sine theta. So, you can do it for any coordinate system but these are three common coordinate systems.

So these are the three things you should know in terms of this notation, we have already written out what gradient was gradient of any, I should not use phi. Gradient of any scalar field is equal to 1 over h 1 del f along the direction u 1 plus 1 over h 2 del f del u 2 along u 2 plus 1 over h 3 del f del u 3 along u 3. In Cartesian, these three are missing. But, in cylindrical for example you have 1 over r del f del theta. So that is where the 1 over r came from. Similarly we have done it already for divergence. But, now I want to talk about curl. The derivation is easy. But I am not going to give it. It is more useful to know the answer. Answer is, you draw you write down what curl is.

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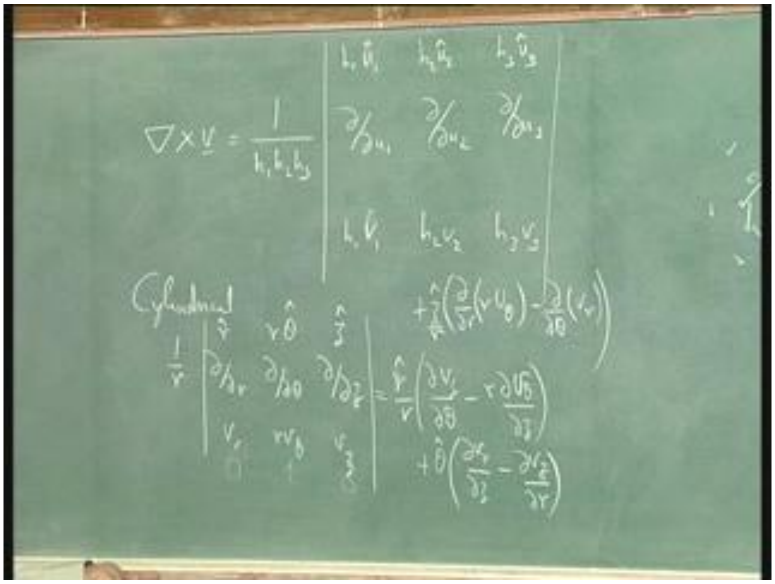

It can be written as one over h 1, h 2, h 3, determinant of h 1 unit vector along u 1, h 2, unit vector along u 2, h 3, unit vector along u 3, del del u 1, del del u 2, del del u 3 and h 1 v 1, h 2 v 2, h 3 v 3. So all these metric coefficients is things that relate change in coordinate to distance along the coordinate line come in this fashion 1 over h 1 h 2 h 3. This is like a volume factor. Then your unit vector now has a h associated with it because what does it mean top go 1 unit in that direction and you function itself is also multiplied by h 1, h 2, h 3.

Let us work it out for cylindrical coordinates and spherical coordinates. So, what you will get is for cylindrical 1 over h 1, h 2, h 3, is 1 over r determinant r hat r theta hat z hat del del r del del theta del del z v r r v theta v z. That is what it is I just substituted h 1, h 2, h 3, to get it. So, what does it become? If I take the r hat component, it becomes one over or I will say r hat over r del v z del theta minus r times del v theta del z. Then if I take the theta component the r and the r cancels out, I get del v r del z minus del v z del r and I am going to write the last component up here plus z hat divided by r times del del r of r v theta minus del del theta of v r.

So this is just the expression. There is nothing magical about it. You can in fact derive it. What is useful is, it in fact allows you not to keep thinking of e dot d l. I do not know if that is a good thing or a bad thing because the e dot d l, the v dot d l is the correct fundamental definition of curl. So, if you can keep little loops in your mind, you can never go wrong. And this expression is nothing but little loop with it is face pointing along z little loop with it is face pointing along r and little loop with its face pointing along theta.

Each of these is nothing but loop integral v dot d l divided by area. That is all it is. So, if you could directly do it, you do not need this expression. But you can take it as an problem and just verify with the previous examples we have done. They all fit and give you the correct answers that we have already derived from loop integral. For example if you take 1 over r theta hat, so what happens? This v theta is 1 over r. So, 1over r times 1 is 1. There is no v r, there is no v z, obviously, if you have a constant vector and you are taking derivatives answer is going to be 0.

So, velocity vector with one over r theta hat has zero curl. If on the other hand your v theta was constant then this would have been r because constant times r. Now if you do the expression what you get is d d r of r. This term survives. This goes away this goes away and all the others are 0. So, you get one term which is d d r of r and it is along the z direction. So what you get when you work out this term survives and you get z hat over r times the r derivative of r which is one, which is what we got earlier.

We found that it is in the plus z hat direction and its magnitude is 1 over r. There is nothing surprising about any of this because the loop integral concept is behind this formula. The earlier derivation was more fundamental than this. This is only a formula. If you actually put loop integrals and do it that is the most correct way of it, all right. I hope now you are a little more comfortable with what curl is, now let us get back to electro magnetics.

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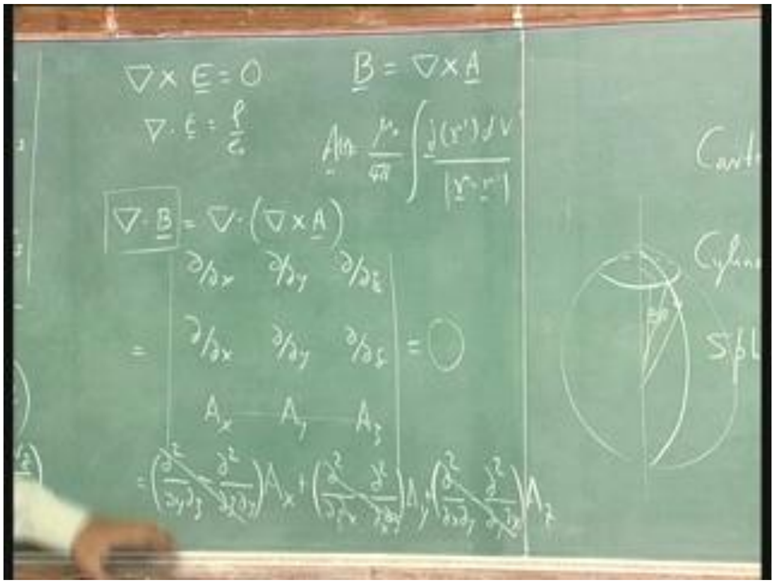

We had already seen electrostatics told as curl of e is 0 and we had worked out that b is equal to curl of a where a was equal to mu naught over 4 phi volume integral j r prime d v prime divided by mod r minus r. This is what i call r 1 2 and this is a at r. Now, we would like to know the properties of b. We say that b is equal to curl of a. So, the first thing we would like to do is to work out what the divergence in curl of b, are because we already know that when we wrote down divergence of e equal to row over epsilon naught somehow we managed to fix e.

We got a unique answer. So we would like to know what the divergence and curl of b is. So, when we take the divergence of b, it is actually the divergence of the curl of some vector a and you put it into your matrix and i am working in Cartesian coordinates x hat y hat z hat dot divergence. So the x hat, y hat, z hat gets replaced by del del x, del del y, del del z. This is what you get. It is the determinant of the matrix with two identical rows. Now those two rows are actually differential operators.

So it does not really mean anything to say they are two equal rows. But if you look at what will happen? You will see that if you choose to expand along the third row, then you will get a x is acted upon by del square del y del z minus del square del z del y.

Similarly you take a y you will get you will have to check the sign. I think it is del square del z del x minus del square, del x del z d y plus for z it is del square del x del y minus del square del y del x acting on a z.

And from your mathematics you know that del square del y del z is nothing but del square del z del y. These two are the same operators. So, this goes off this goes off, this goes off. So it is equal to 0. So, the divergence of b is 0 whenever b is a curl and we have derived that b is a curl. So we have got one of the two relations we want. You know what the divergence of b is. Now we have one other fact which is that, if you look at how we derived for electrostatics that loop integral e dot d l was 0.

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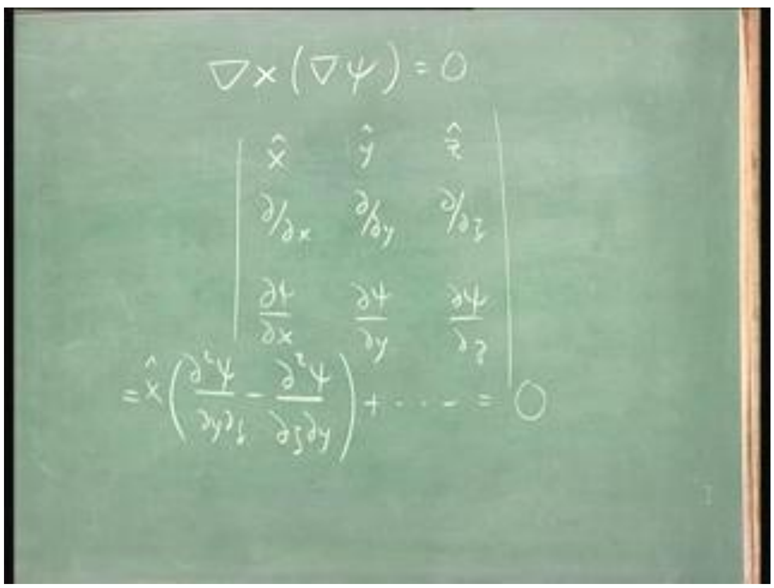

Actually you can easily show that curl of the gradient of anything is 0. It comes from the same idea. That is the curl is nothing but x hat y hat z hat del del x, del del y, del del z, del psi del x, del psi del y, del psi del z. So when I take say the x component, it is equal to del square psi del y del z minus del square psi del z del y plus 2 other terms and once again because del y del z is the same as del z del y this is 0. So there are two important statements that are there in vector analysis and they are crucial to electro magnetics.

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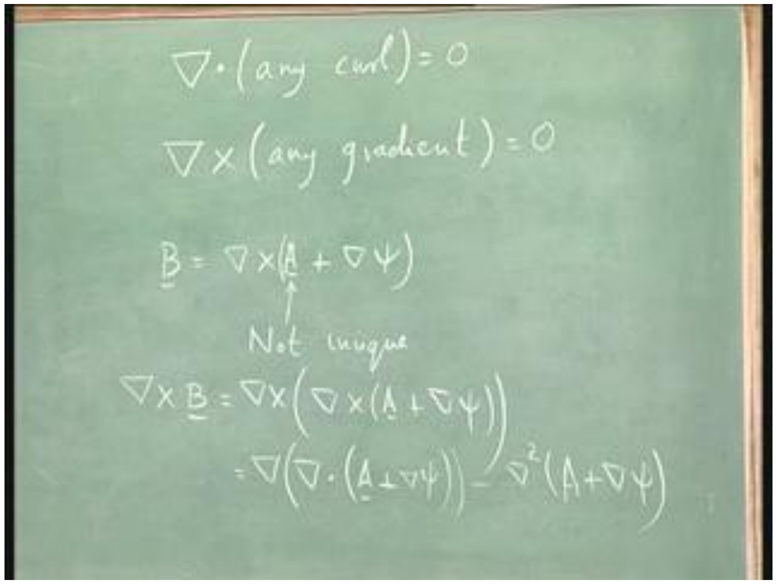

One is divergence of any curl is 0 and curl of any gradient is 0. This is why curl of the electric field is 0. This is why divergence of the magnetic field is 0 because b is curl of a, and e is gradient of phi. Now, what does this mean? We already know that b is the curl of something. So, supposing I said b is curl of a, and then I look at this. I can say b is also the curl of a plus gradient of something because curl of the gradient of anything is 0. This will just give me 0. So the value of a, is not unique. We have given an expression for a, but that expression is not unique. In fact there are many vector fields a, whose curl give me b.

Now if I take the divergence of b I get zero because it is curl. I mean it is already a curl, but if I take the curl of b, it is going to be equal to the curl of the curl of a plus grad psi. I am not going to give you the vector algebra right now. But it turns out that this is equal to gradient of the divergence of a plus grad psi minus del square of a plus grad psi. And because of this you can use grad psi to adjust the values that are present here and by doing this we will be able to put this, a, in the correct form where curl of b will have a useful value. Divergence of b is 0 which fixes b as a unique vector field. It is a bad place to stop. But I will continue next time and complete this topic.