Electromagnetic Fields for EEE Students
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Lecture – 20
Stokes Theorem

Good Morning. The last two lectures I have been talking about the magnetic field and I hope I have sufficiently justified it is existence. So, today I am going to do some mathematics to make it possible to manipulate this magnetic field when we make use of it. So, let me go back to the starting equation.

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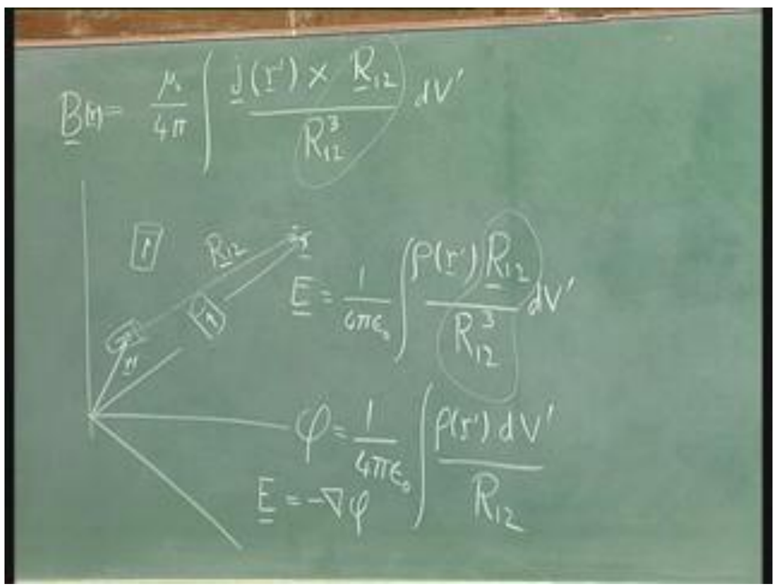

The equation we had was that the magnetic field was defined as mew not over 4 pi, I still have not really discussed why mew not over 4 pi. The reason I have not discussed it is it pretty arbitrary integral, the current at any other position. So, this is B at a position r, it is an integral over other points. Whereas current j of r prime-cross R 1 2 divided by R 1 2 cubed dV prime. So, we are talking about various current elements and we are talking about a point r it is all coordinate system.

So, this is r. This is r prime and this vector is what I called R 1 2. So, you take the current take the cross product where the R 1 2 divide by the cube of the magnitude of R 1 2 integrate. The result over all the current elements multiplied by normalizing constant and that is a BC. It is a relatively complicated expression, but it is not that complicated because you can compare it with the electric field which is one over 4 pi epsilon naught again integral over all other points the charge density of that point times R 1 2 divided by R 1 2 cubed.

So, you see the same R 1 2 over R 1 2 cubed appears both in the electric field and in the magnetic field. Now what I would like to do is to go from this equation is quite useful as it is do something that is more easier to this. For that first let us look at this equation and look at the associated equation which is the potential pie. We already worked this out the potential was integral row of r prime dV prime divided by R 1 2. Now if you look at these two equations and you look at the equation that electric field is equal to minus of the gradient of pie. You can write this this equation out. Let us do that.

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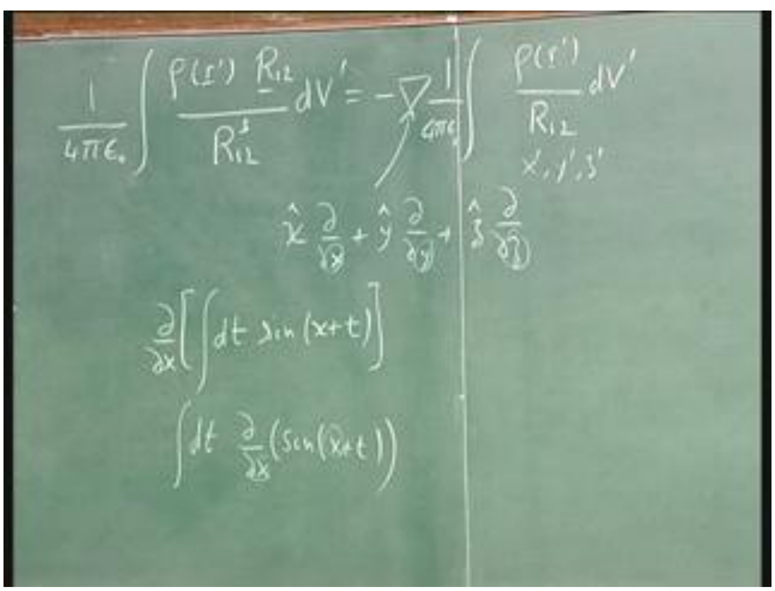

What it says is 1 over 4 pi epsilon naught integral over other points row of r prime R 1 2 vector divided by R 1 2 cubed dV prime is equal to minus of the gradient of integral 1 over 4 pi epsilon naught outside row of r prime over R 1 2. This is the same equation as E is equal to minus grad pie I have just written out what E is written out what pie is. Now, this gradient if you look at what it is it is x hat del del x plus y hat del del y plus z hat del del z. That is to say if you go back to this picture the gradient represents shifting this point r because this r is x x hat plus y y hat plus z z hat. This point r prime if you root out its coordinates it would be x prime x hat plus y prime y hat plus z prime z hat.

So if I am doing a derivative with respect to x it means I keep all the charges where they are and I am move the point where I got my measuring instrument where I am measuring potential or I am measuring field. So, the gradient operator there means move this points slightly there see the change in potential divide by delta L do it in all direction and construct the gradient operator. But if you look at this integral the integral is over where the charge is, it does its not over where the instrument is. So this integral is not even involving x, y or z it involves x prime, y prime, z prime.

So, I can take this derivative inside its like if I did an integral dt of sin x plus t and then I had del del x. Now this derivative with respect to x does not is not the same thing as the t. So, I can take this del del x inside. Now, I can write this as an integral dt del del x of sin x plus t and then I can make this x act on here and do whatever you want to do. So, we are going to do the same thing here. The gradient is acting on a coordinate that is not what we are integrating over. I am going to pull that gradient inside and make it act on all the parts of the integrand that depend on x y z rather than x prime, y prime, z prime, what do I get?

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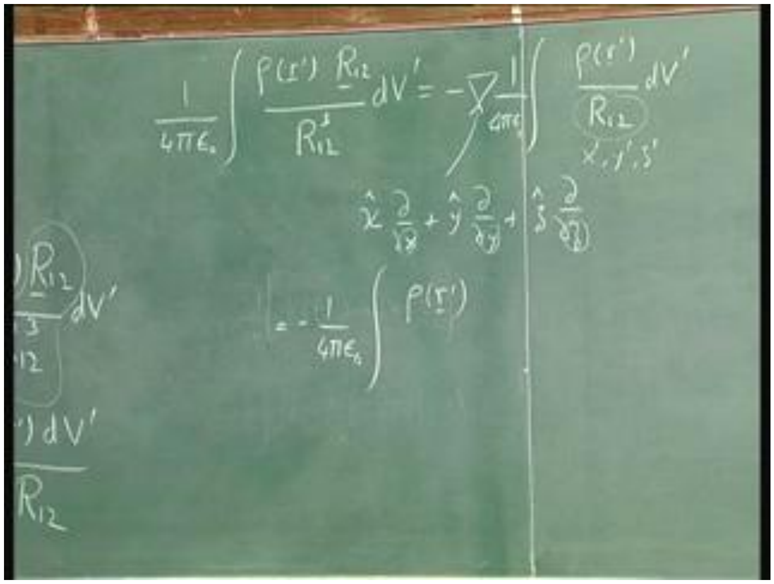

This electric field becomes equal to minus one over 4 pi epsilon naught integral. Now row of r prime the location whether the charge is not a function of x, y, z; x, y, z, is where I got mine meter by meter is not where the charges. So, these are two separate piece of information. So, I can feed the row of r prime out dV prime before that but the gradient does attack this piece. We go back to the figure.

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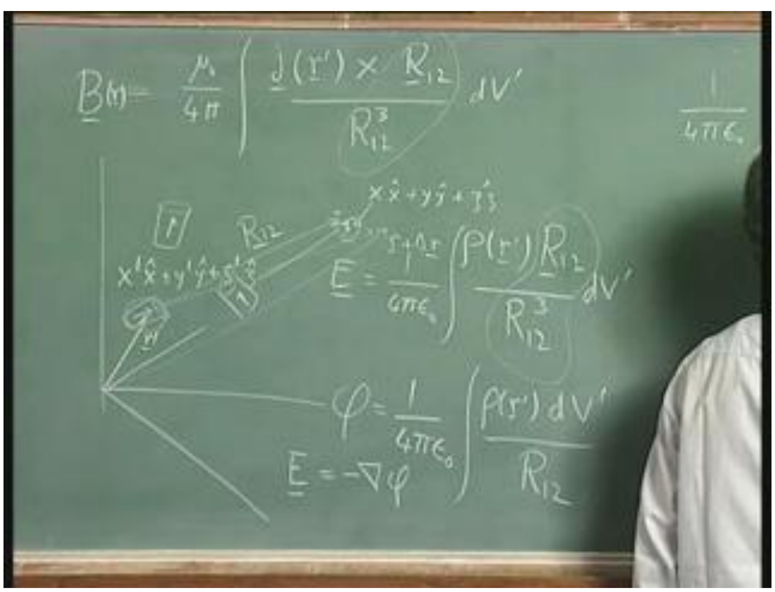

If I change the location here to another point there r plus delta r then R 1 2 will change R 1 2 will become R 1 2 plus delta R 1 2. Therefore the gradient does affect the value of R 1 2 it does not affect the value of charge. But it affects the distance of the charge from the measuring instrument.

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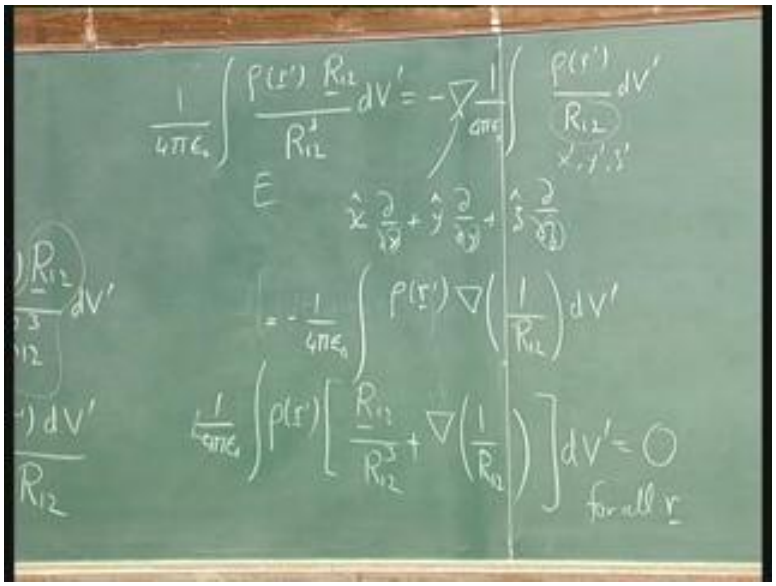

So, I can put gradient acting on one over R 1 2. Now look at this I have got I can take both of these together and I get that 1 over 4 pi epsilon naught integral row of r prime R 1 2 divided by R 1 2 cubed minus gradient of 1 over R 1 2. I think from plus because I have got a minus sign here and I took it to other side dV prime is equal to 0. What I have done is as said the electric field is equal to minus grad pie I have rewritten minus grad pie in this form.

So since both of these involve one over 4 pi epsilon integral over row of r prime dV prime I took that common. What is remaining R 1 2 over R 1 2 cubed and with this minus sign when I take it on this side it will become plus gradient of R 1 R 1 2 1 over R 1 2 this is equal to 0 and this is equal to 0 for all r. Now when this kind of integral happens that, this is equal to 0 for all r, you might in some very special cases find some non-zero answers.

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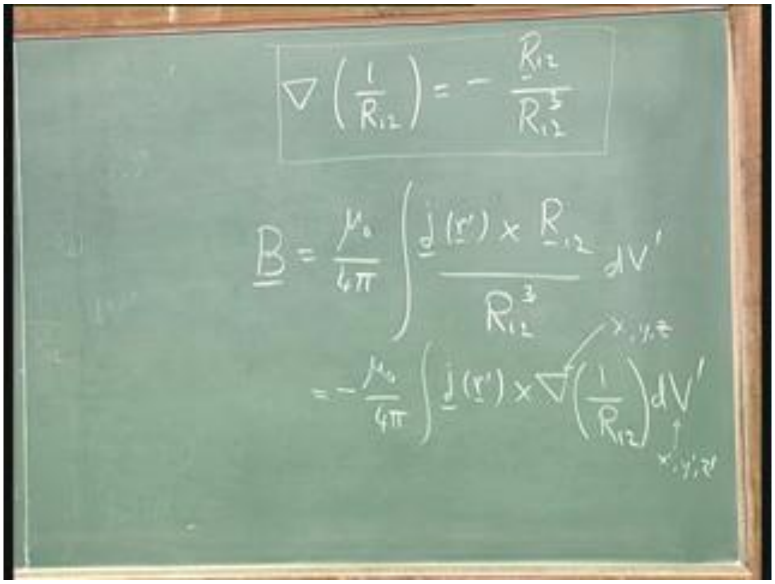

But invariably what it means is gradient of 1 over R 1 2 is equal to minus R 1 2 over R 1 2 cubed. So it is a very important result. This is in fact what allows us to have potential electrostatic potential derived from coulomb's law. This is coulomb's law 1 over r squared. This is electrostatic potential 1 over R and the reason why coulomb's law and electrostatic potential are related is because gradient 1 over R is this vector operator. So, this is all electrostatics you already had it.

But now look back at the magnetic field equation. Now, this magnetic field equation of course involves current now rather than charge that involves mew not rather than epsilon naught. But it still involves R 1 2 over R 1 2 cubed. The very same operator that electric field uses. So, we can use this relationship between R 1 2 over R 1 2 cubed and gradient of R 1 2 one over R 1 2 in here. So, what do you get you get the magnetic field is equal to mew not over 4 pi volume integral j of r prime cross R 1 2 over R 1 2 cubed dV prime becomes equal to minus mew not over 4 pi volume integral j of r prime cross gradient of one over R 1 2 dV prime.

Have you used the same affects? This is R 1 2 over R 1 2 cubed. So, it must be equal to minus gradient of R 1 2. But, this gradient acts on x, y, z. This integral is over x prime; y prime, z prime, in the current is also over x prime, y prime, z prime. So, it means when I vary the position of the instrument the currents do not change therefore I can pull this gradient right out. So, I get its equal to well let me before I do that I have to prove to you what that becomes.

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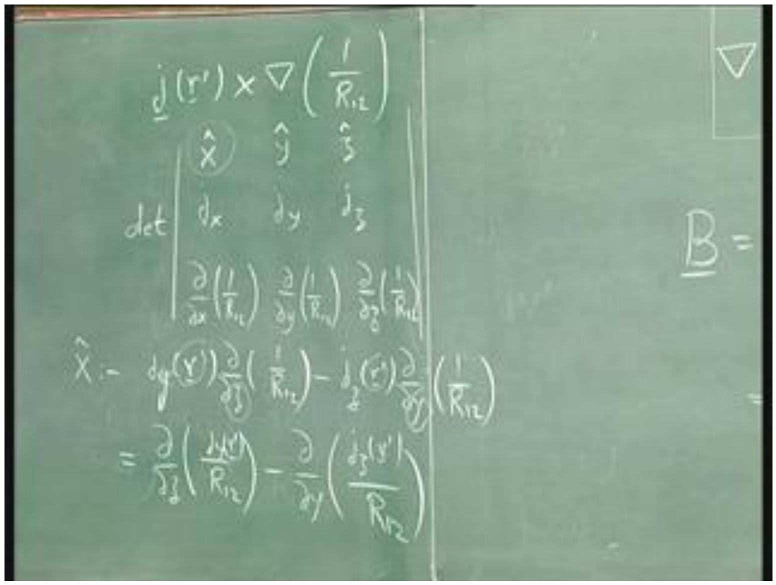

Look at the integrand here it is j of r prime cross gradient of 1 over R 1 2. Now in Cartesian coordinates we write this as unit vector along x, unit vector along y, unit vector along z, jx, jy, jz, del del x of 1 over R 1 2, del del y of 1 over R 1 2, del del z of 1 over R 1 2 and determinant of. Now this del del x, del del y, del del z, are actually not acting on jx, jy, jz. So, if I root out the x component of this of the determinant, the x component part would give me jx of r prime del del z 1 over R 1 2 minus. Sorry, jy jz of r prime del del y of 1 over R 1 2. That is, when you take the determinant you choose a row first entry times this, the product of these two minus the product of these two.

Second entry times product of these 2 minus product of these two third times this. So the first entry the component multiply x hat would be jy del del z 1 over R 1 2 inverse minus jz del del y 1 over R 1 2. But, z and y do not attack R 1 2 prime. They only act on x and y and z. They do not act on x prime y prime z prime. So, I can pull this out. So, I get this is equal to del del z of jy over R 1 2 and jy is a function of r prime minus del del y of jz of r prime divided by R 1 2. Now, this is still looking like a curl like a cross product. But what kind of cross product is it? Well, let us write it out. We have to write it out.

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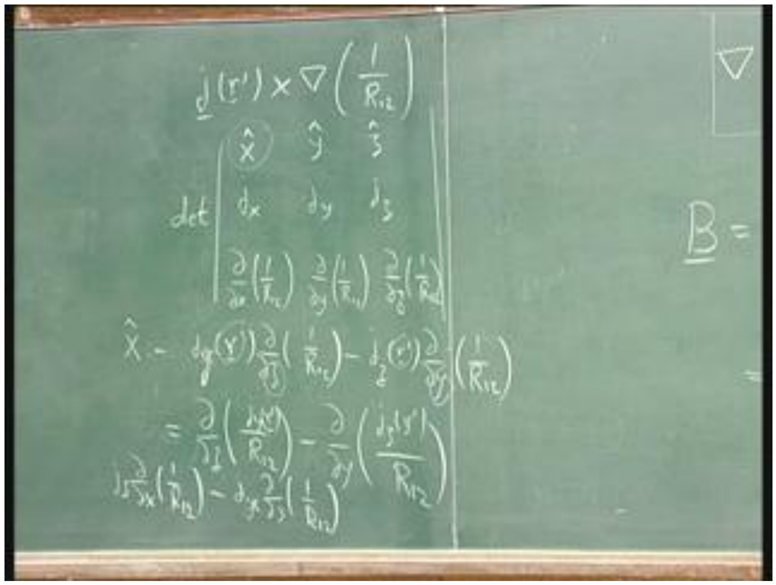

I would say this is looking like the determinant of I still have my x hat. But now I want del del z of jy R 1 2 okay. So I would say that I will put a minus sign so that this becomes minus and this becomes plus. So del del y of jz over R 1 2 minus del del z of jy over R 1 2. So, you look at the y component the same thing will happen will give you um jz times del del x of 1 over R 1 2 minus jy sorry minus jx del del z of 1 over R 1 2 once again put a minus sign there.

So, you will get y hat times del del z of jx over R 1 2 minus jz del del x acting on jz over R 1 2. And similarly z hat times del del x of jy over R 1 2 minus del del y jx over R 1 2. So it looks like a cross product again. Let us started with the cross product this it was the cross product of j and gradient of 1 over R. What I have ended up with is a cross product that is strange kind of cross product. This if I have to write what it is minus the gradient operator itself. That is, this cross product the function j divided by R 1 2.

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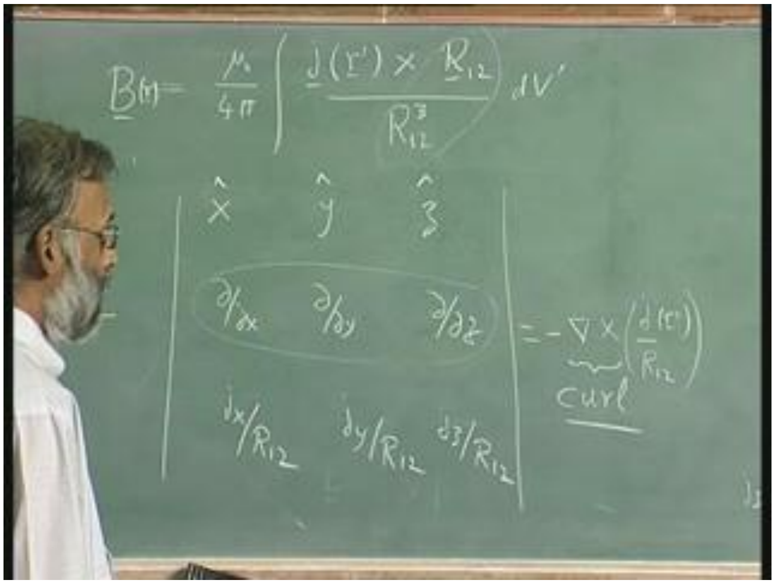

The first element is not really a quantity. It is an operator because you can see just del del x del del y del del z. It is a very strange kind of cross product in fact it is not a true cross product at all. It is a differential operator and this gradient cross is called the curl. And it is the most important operator, once you have learned gradient and divergence. Along with these other two operators, this completes the set which will require for vector analysis. So, let me repeat what I did.

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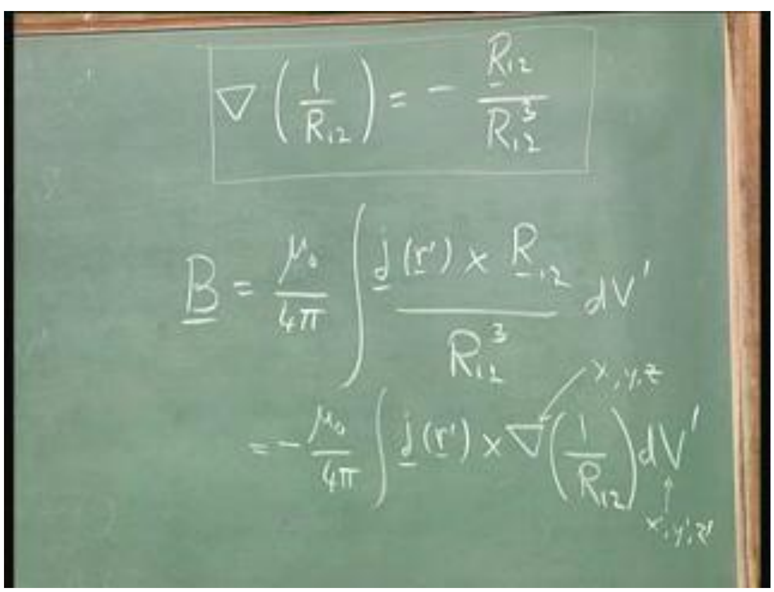

I started with this magnetic field equation. This equation was derived two lectures ago and I repeated it last lectures. Then I said that from electrostatics have already seen that R 1 2 over R 1 2 cubed is nothing but minus gradient of one over R 1 over R 1 2. So, I can replace this piece by minus gradient of one over R 1 2 then I looked at what is in here it is a cross product. So, this cross product can be written out as the determinant of a 3 by 3 matrix that you must have done quite a few times.

So you write down unit vectors on x, y, z, the components of the first vector the components of the second vector. So, perfectly reasonable cross product but the operators here del del x del del y del del z and you write it out they can be pulled out because jy jx jz do not depend on these coordinates. So, you can write it this way and when you write it this way, you can re-cause this determinant as another determinant.

But the determinant that still uses unit vectors x y z. But instead of jx jy jz I am putting derivatives del del x del del y del del z and instead of del del x one over r components and I am putting j over R components. It is still a cross product like operator except that this first vector field is, appear operator it is just gradient. So, if you have to write it out you would say gradient cross field this gradient cross is a very important operator and that is what we called the curl alright. Given that you have got the curl what do we do with it?

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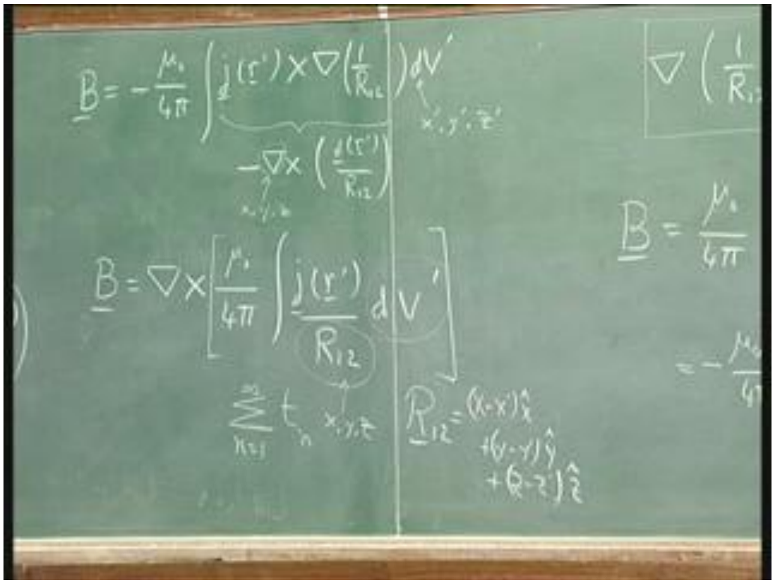

So, I will write out the equation from B again B is equal to minus mew not over 4 pi volume integral j of r prime cross gradient of 1 over R 1 2 dV prime. Now I am going to take this gradient out and rewrite this as minus curl of j of r prime over R 1 2. That is what we just proved. We proved that this cross product is minus of the curl. But this curl is acting on x, y, z. The integral is on x prime, y prime, z prime. That is the curl is acting on where the instrument is where the measuring instrument is this integral is over where the current elements are.

So, I will pull this right out so I get the magnetic field this minus sign and this minus sign cancel, it is equal to curl of mew not over 4 pi volume integral j of r prime divided by R 1 2 dV prime. This whole thing it depends on x prime, y prime, z prime through j and through R 1 2. But, that dependent on x prime, y prime, z prime is integrated over. So, is a as I had a sum n equals one to infinity of various terms okay t sub n once you have summed over all the n’s this sum no longer depends on n.

Similarly, this integral no longer depends on x prime y prime z prime. What it depends on through this R 1 2 it depends on x y z. Because R 1 2 is the vector R 1 2 is nothing, but x minus x prime along the x direction plus y minus y prime along the y direction plus z minus z prime along the z direction. So R 1 2 depends on 6 different coordinates it depends on where the current elements are it depends on where the measuring instrument is depend on both so depends on six elements six coordinates three of which get integrated over its summed over. The remaining three coordinates are what I have left and they are what defined the value of B.

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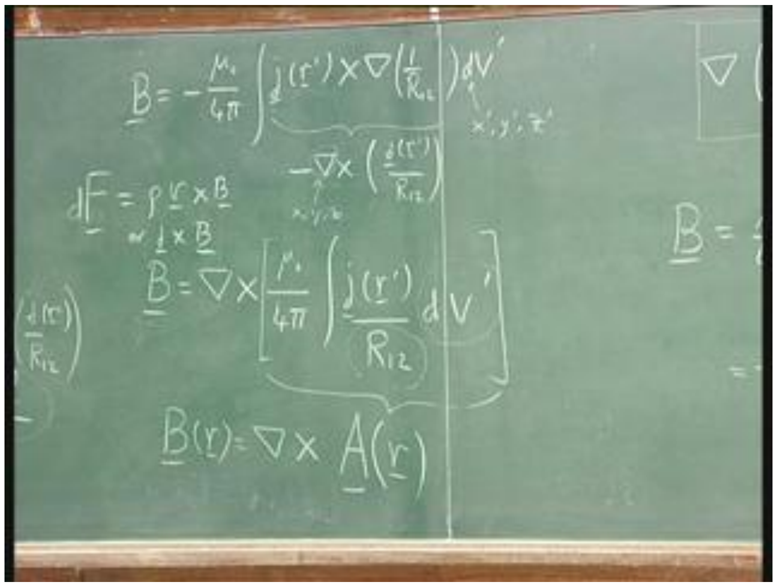

So, this square bracket is now a function only of x y z and I can call its some function I call it A vector it's a vector because j is a vector. So it has direction some vector A which is the function of r. So I can finally write this result out which is the magnetic field B at any positions r is equal to the curl of some other field A of r and what other field is it. That field is mew 0 over 4 pi integral over all the currents at any other position divided by distance from the current to the inter point where we want to calculate the field.

So this is what A is and we take calculate A the curl of A will give you B and as before let me remind you once you have B the force is nothing but row v cross B or j cross B I will call it dF. So, the force per unit volume is nothing but current cross B. So, that is why we want B otherwise you would not care about what B is. Once you have the magnetic field you can calculate magnetic forces. Now, there is a very important equation probably the most important equation you can calculate about magnetic field. But, an immediate conclusion you can draw.

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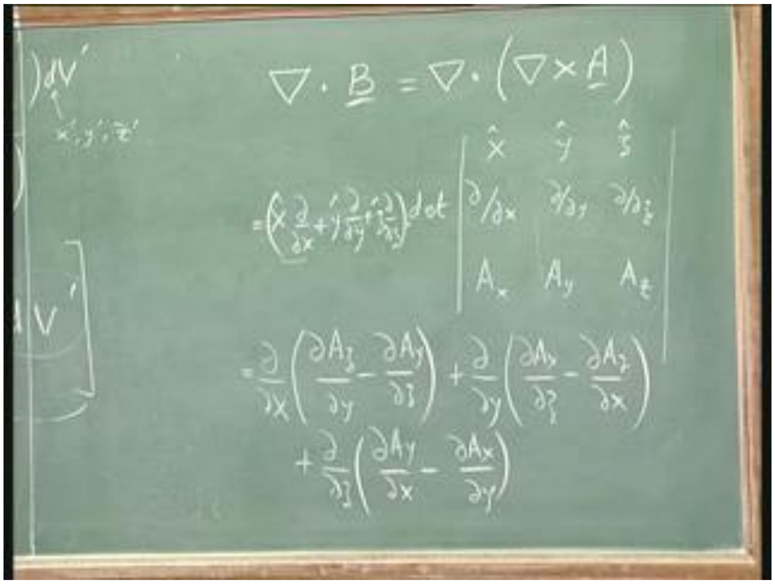

Supposing, I calculate divergence of B. Now divergence of B would be divergence of curl of A because we just worked out B is curl of A. Let me write this out. It is equal to divergence of determinant of x hat, y hat, z hat, del del x del del y del del z Ax Ay Az. That is what it is divergence of curl of A. Now what this means is x hat del del x plus y hat del del y plau z hat del del z this vector dot the determinant. Now, if you look at what this is we can write out x y and z components separately. So, the x component of this vector will multiply the x component of this vector.

So, it will become del del x of x component of curl of A. The x component of curl of A is the minus corresponding to 2 2 2 3 3 2 3 3 which is del Az del y minus del Ay del z. The y component will be del del y of now I have to take the y component of the curl which is del Ax del z minus del Az del x. Then, the z component del del z acting on this which is del Ay del x minus del Ax del y. So, there are six terms. Now, if you look at this there is a del del del squared Az del x del y. It is also del squared Az del y del x. So I am going to collect terms out now.

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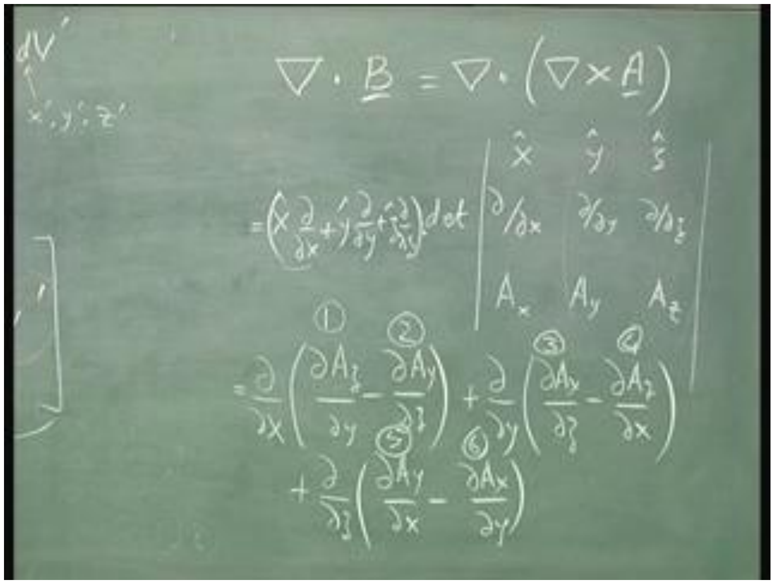

What I will get I will number this terms 1 2 3 4 5 and 6.

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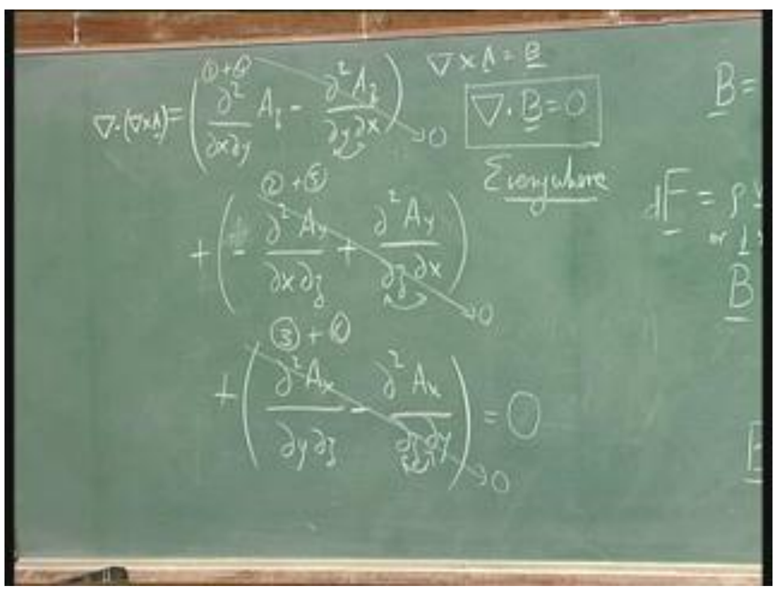

So taking terms 1 and 4 what I will get is del squared del x del y of Az minus del squared Az del y del x. If I take terms 2 and 5 then I get del squared Ay del x del z minus del squared Ay. Sorry, this is minus this, is plus del z del x and then if I look at terms I have taken care of the Az I have taken care of A Ay. So, I will take care of Ax terms 3 and 6 I get del squared Ax del y del z minus del squared Ax del z del y. So, this is what divergence of curl of A is equal to all I did was I took I wrote out what divergence was I wrote out what curl was and I combined all terms.

Now you look at each of this, if you look at this one, you are taking the second partial derivative of Az with first with respect to y then with respect to x or you are taking the second partial derivative first with respect to x second with respect to y now you know from your theory in partial derivatives. The second partial derivatives commute. What that means is that, I can interchange del y and del x so I will do that one set of these. What does that give me? Gives me del squared Az del x del y minus del squared Az del x del y. So, this is 0 gives me minus del squared Ay del x del z plus del squared Ay del x del z 0 del squared Ax del y del z minus del squared Ax del y del z.

So, this whole thing is zero which means because curl of A is be divergence of B is equal to 0. This is always true at it comes out of simple the fact that curl of A is equal to B. If you can find a function whose curl is B then its divergence the divergence of B is automatically 0. We proven it and there is no assumption I put in to it. It is a very general statement if any field is the curl of another field its divergence is 0. Now, why do we care we care for a very important reason. If you look at electrostatics, the divergence of the electric field is extremely important.

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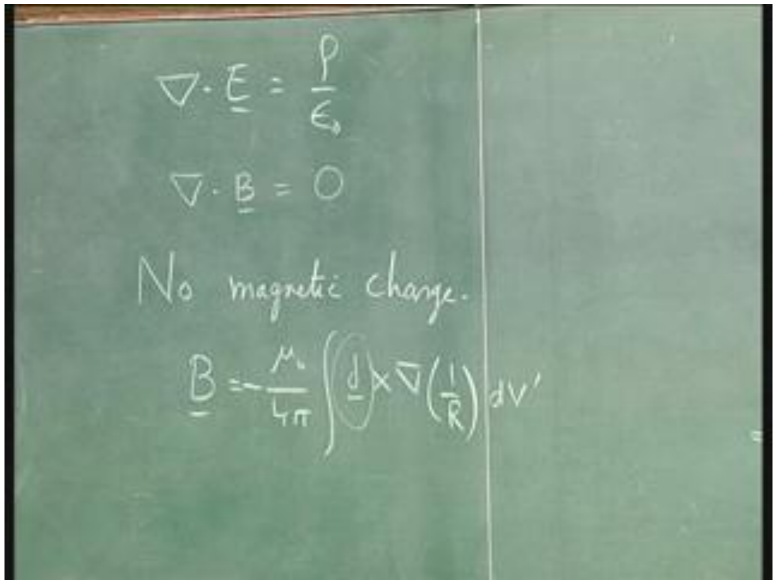

The divergence of E is equal to row over epsilon naught. The divergence of the electric field that the source of the electric field because row is what creates the electric field and you take the divergence of the electric field you get back to the source. The divergence of B is equal to zero which means there is no charge like source for the magnetic field. There is no such thing as magnetic charge.

So it is very important statement and it purely comes out of the fact we wrote B is equal to mew not over 4 pi integral j cross gradient one over R dV prime. That is we said B is derived from currents and not from charges. Moment you say that all the rest follow and this is been verified by observation that the magnetic field is derived from currents and its also understandable because magnetic field is due to relativity which means is due to moving charges. Therefore it is due to currents. But there is something surprising about this statement.

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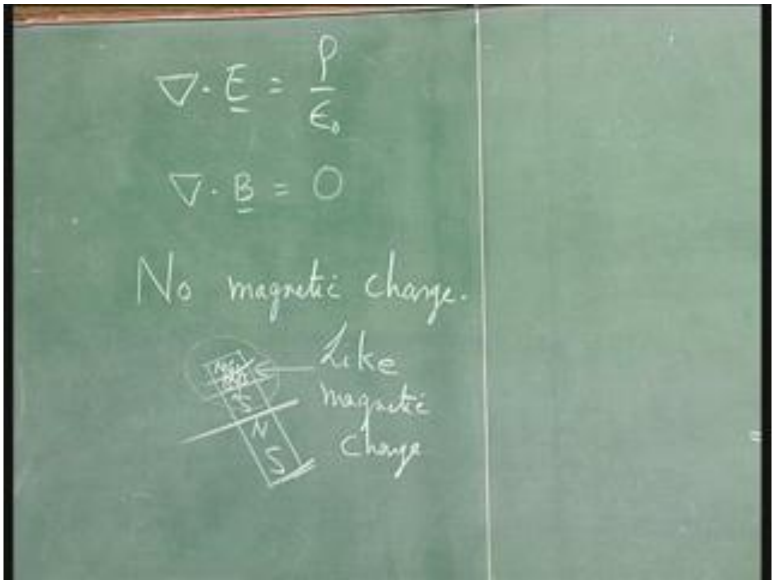

Because the magnetic field was first discovered because of permanent magnets and in permanent magnets we always said north pole south pole. And if you look at what we mean by a north pole and what happens when a north pole is placed in a magnetic field. A north pole is like magnetic charge. So, I am saying there is no magnetic charge it is intrusively true yet the most basic magnetisable material has what looks like magnetic charge. The answer is a little quickly answer is if I cut this magnet in two I am going to find I cannot just get the north-pole in one piece and the south-pole in the other piece.

Rather what will happen is I will get a south pole at the bottom of the first piece and a north pole at the top of the second piece. No matter of how much I cut them I will keep getting north and south and this is what it means. It does not mean that we do not have north like south behavior like behavior it means you cannot extra just one charge out. Anytime you try to save up a little bit of this magnet and say I will take only the south-pole, you will find that you look a little bit a north pole as well so much. So, it is still a magnet with the pair of poles attached to it. That is what is meant by divergence B is 0. We will never find a pure north pole.

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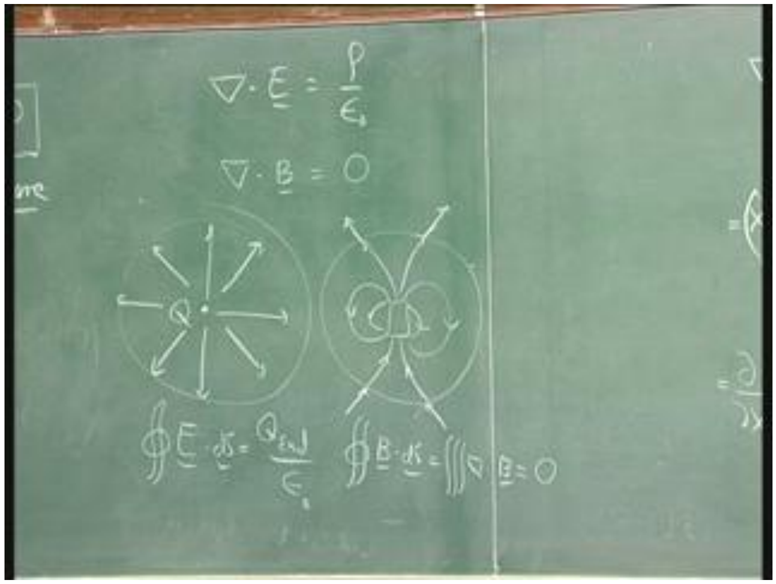

Now, in terms of drawing lines of force, what that means is if you drew for electric field we drew a charge Q and then we drew arrows. And so we had Gauss's law and we could say surface integral E dot dS was equal to Q enclosed over epsilon naught. Now divergence B is 0 which means you can never have this situation the kind of situation you will have from magnetic field would be something like this you have a magnet. Now, if you surround this by a surface, there are many lines which never touch the surface there are few lines where the magnetic field is going out but corresponding each of this there is a line where the magnetic field is going in.

So, when you integrate over the whole surface you get surface integral B dot dS which is volume integral divergence of B is equal to 0. Because we have divergence B is equal to zero. So, no matter of how you surround a piece of magnetism you will always find that the total amount of flex leaving that volume is zero its no such thing as magnetic charge. This is almost the most important thing that you have to know about the magnetic field and if you know one more thing, you know everything there is to know about it.

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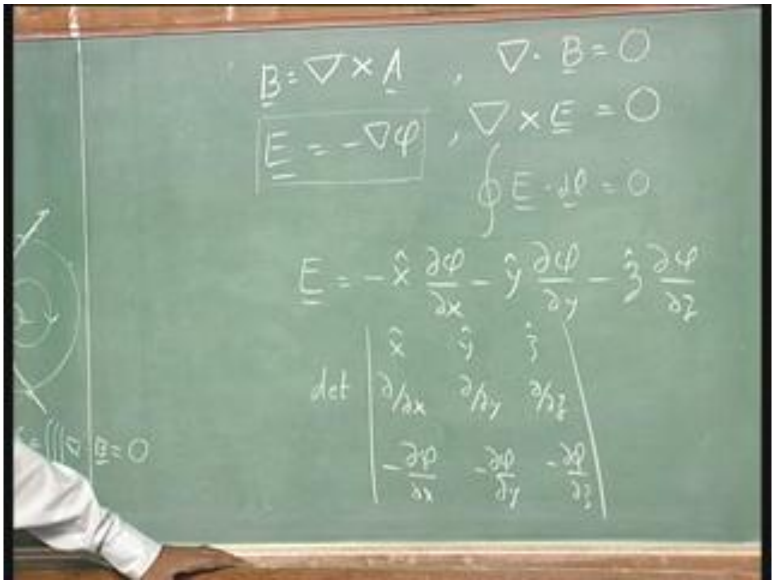

So, we have B is equal to curl of A and divergence of B is equal to 0. Similarly, we have from electrostatics E is equal to minus gradient of pie curl of E is equal to 0. Now let me show this have actually up to now only shown you loop integral E dot dl is equal to zero. But now I introduce a new vector and I want to talk about. I have E is equal to minus grad pie. So, it means electric field is equal to minus x unit vector along x del pie del x minus unit vector along y del pie del y minus unit vector along z del pie del z.

Now, I know what it means to take grad cross it means take the determinant of x hat y hat z hat del del x del del y del del z and then a mistake Ex Ey Ez which is minus del pie del x minus del pie del y minus del pie del z. So, this is what it means we talk about curl of E. Let me work out what one of the components is.

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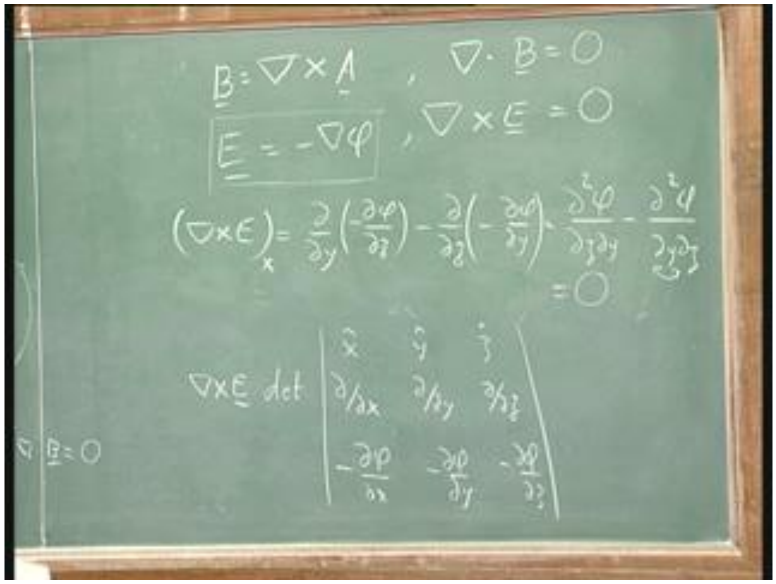

Curl of E x component is equal to del del y of minus del pie del z minus del del z of minus del pie del y. So it is again equal to del squared pie del z del y minus del squared pie del y del z. Again we know from our mathematics that I can exchange these two derivatives. So, these two terms are identical one has a minus sign. So, it is equal to 0. Similarly, the curl of Ey will be 0. Similarly curl of Ez will be 0. So, that is why the moment you have anything call grad pie its curl becomes zero anytime you have something, that is curl of a vector field its divergences. Now I am going to introduce a vector operation a vector law called Stokes theorem.

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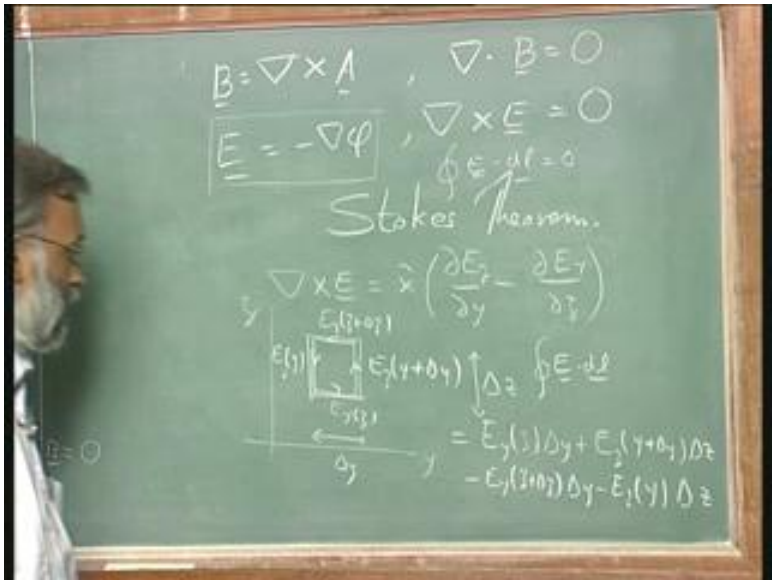

It is a very easy theorem provided you approach it in the right attitude. But because it involves this thing called a curl most people get very scared of it. If, you look at what curl is, if you look at the x component it is del Ez del y minus del Ey del z. Let us draw this out this is the y z plane and I have electric field with components in x and components in y. What I am saying is if I keep z component z fixed go along y if I keep y fixed go along z there is a relationship. Let us look at what it is trying to say.

Supposing I have electric field in some general direction, so I have a certain amount of Ey and I have Ey of z plus delta z. Similarly, I have Ez at y Ez at y plus delta y. Supposing, I do an integration round this square. I will assume that this length is delta y and this length is delta z. So, my loop integral E dot dl we shown by surprised that we are talked about E dot dl. Because, earlier I had written down loop integral E dot dl equals 0 before I wrote down curl of E equals 0. They are saying the same thing.

Loop integral E dot dl becomes the bottom loop is Ey at z times delta y then it becomes plus Ez at y plus delta y delta z minus Ey at z plus delta z delta y minus Ez at y delta z. So, when I go in this direction increasing y which plus sign increasing z is plus sign decreasing y minus sign decreasing z minus sign, so on. This first leg Ey at z, second leg

Ez ay y plus delta y, third leg Ey at z plus delta z, fourth leg Ez at y. Now, I will combine the Eys and I will combine the Ezs. So, what do you get?

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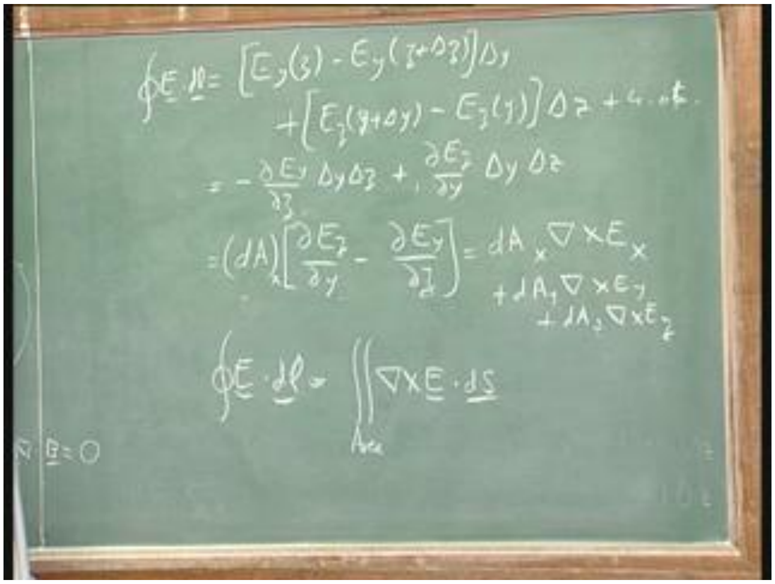

You get that that term is equal to Ey at z minus Ey at z plus delta z times delta y plus Ez at y plus delta y minus Ez at y delta z. So I have combined this term and this term and written out the first square bracket. I have combined this term and this term and written out this the second square bracket. Now each of this can be written out as a derivative. So, this can be written out as minus del Ey del z delta y delta z plus del Ez del y delta y delta z. So I have loop integral E dot dl around this tiny loop is this.

But I can identify delta y delta z as nothing but the area of this loop. So, it is equal to dA the area multiplied by del Ez del y minus del Ey del z and if you remember back to what the definition of curl was this is nothing but curl of E along x. Now, this area is the area in the y z plane. So, you can actually write this is dAx. So if you do this you take the next other parts also this is plus four other parts other terms, you will get plus dAy curl of Ey plus dAz curl of Ez. So, we add it all up you get a simple answer you get that for the tiny loop very small loop.

Loop integral E dot dl and this is any E I call it E but I have not used the properties of E here any E its equal to surface integral over that small area curl of E dot dS. So it is true for very tiny loop rectangular loop that is still tiny in general direction. The x component will give you dA projected along x times curl of E in the x direction. The y component will give you this. The z component will give you this. So, in general it can be written out is curl of E dot dS. Now, if you look at the general surface you can generalize this idea.

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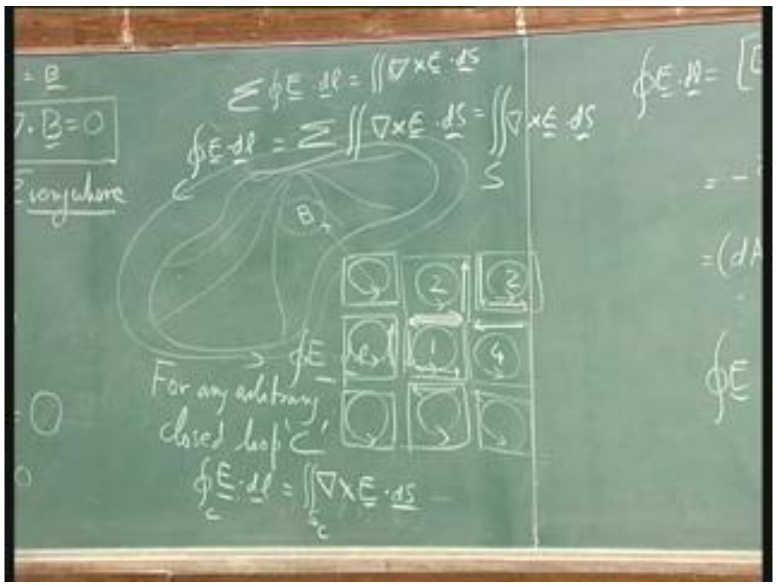

Supposing I have some closed loop and I want to calculate E dot dl on this loop, what I will do is, I will take this loop and connect it through some surface I deliberately draw on the surface so that it is bulging out. This surface does not have to be flat it can be any surface at all. Then on this arbitrary closed surface I can draw little rectangles on those tiny rectangles my theorem will hold namely loop integral E dot dl is equal to surface integral curl of E dot dS. Then I can take the next rectangle and do it again.

But now let us look at this region bigger I have a loop in integral then I have a loops around it. I am going to draw nine of them. Now, I am going to do my loop integral this way I do my loop integral the same way in all the nine rectangles. Now, if you look at this particular leg when I am doing it when I am doing E dot dl from rectangle one I am moving in this direction. When I am doing E dot dl from rectangle two I am going this direction.

So, the E dot dl calculation if I add them up will cancel out. Similarly the E dot dl calculation in two will cancel E dot dl calculation in 3. The E dot dl calculation in to at the bottom will cancel the E dot dl in 4 and so on and so forth. So, every interior edge will cancel out. The only thing that one cancel out it is the outside edge because there is no, there is no rectangle outside to cancel it. So, when I add up all this little rectangles by some them all up I take this entire surface tile it with lots and lots of little rectangles and sum them all are. What I will get is, loop integral over the our full curve E dot dl because all the internal little integrals canceled out.

Only thing that is left is the outside integral. But what is on the other side? On the other side is sum on all the little rectangles curl of E dot dS. But this is nothing but a summation and integration is nothing but summation anyway. So this is surface integral over the whole surface curl of E dot dS. So, this is what it calls Stokes theorem. It says for any arbitrary closed loop C I am just going to call it C. The loop integral on c of any vector I am going to call it E dot dl is equal to the surface that connects c. So, usually I call it S sub c to mean that it is the surface whose edges correspond to c curl of E dot dS.

It is an extremely important theorem and if you did advance mathematics, you will learn this theorem is a special case of divergence theorem itself. However, in our use of electromagnetics this theorem is exceptionally important; we will use it in paradise law.
We will use it in Ampere’s law. We will use it all over place. So, it is a theorem you should become very familiar. I will continue on next time and go for use this theorem and gets some interesting results.