Electromagnetic Fields
Prof. Dr. Harishankar Ramachandran
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 17
Field near Sharp Edges and Points

Good Morning. Last time I had introduced a new technique for solving Laplace's equation and I had gone half with through the problem. So, I will complete the problem right now and then go on and apply it to a very interesting and relevant example, that we will see how this problem works out in practice.

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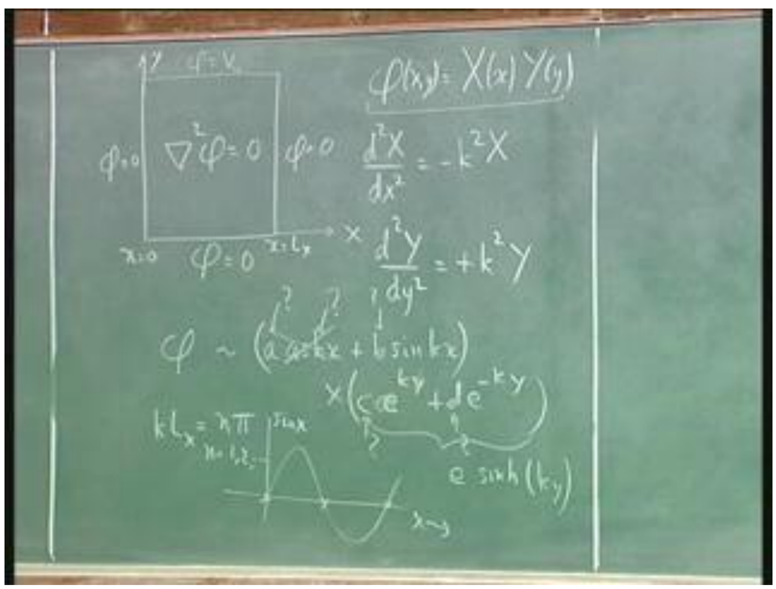

What we are looking at was the solution of Laplace's equation? When you applied to a two dimensional box? So this direction is x, this direction is y and I had the condition that the potential was equal to 0 at the bottom, 0 at the side, 0 at the other side. But it was equal to V naught at the top. Now we are worked out that if you assume pie of x y is equal to some capital X of x, capital Y of y and route out the equation we get certain solutions. So, this is an assumption. There is nothing to say it works.

However, if you put it in, what we found was that is solution that takes this form you get two independent equations. One equation you get is d squared capital X dx squared is equal to minus some k squared capital X, but k is unknown just a constant. And you got the second equation which was d squared capital Y, dy squared equals plus k squared capital Y. This came out of just substituting this equation into dell squared pie equal to 0, namely Laplace's equation.

So now, if you take this equation and solve it which we did last time, we get straight forward answer because this is nothing but the pendulum equation and this is nothing but the unstable pendulum equation. So, we already know the answers to this and what we worked out was that the potential looks like from this equation, I get something a cos kx plus b sin kx. And from this we get into c e to the kx plus d e to the minus sorry ky.

This was the general solution you can get. And we still do not know k and we also do not know a b c and d. So, we have five pieces of information to be fixed. But we have the boundary conditions to help us in fixing this. Now, we look at the boundary condition in x, we have that the potential goes to 0 at x equals 0 and the potential goes to 0 at x equals Lx. From that we are able to eliminate cos kx, because at x equals 0 cos kx is 1. So it would not go to 0 and by requiring at x equals Lx sin kx goes to 0.

We have the condition k times Lx is equal to some n times pi. Because sin, if you plot sin of x verses x we know that sin goes to 0 at 0 goes to 0 at pie 2 pie etcetera. So if this function goes to 0 at x equals Lx it must be the argument must go to 0 pi 2 pi etcetera. So k times Lx is equal to n pi the n is some integer we cannot take k n equals 0 because we do not have any solutions at all sin of 0 is 0.

So, n can be equal to 1, 2, 3 and so on, n equals minus one is not important because sin of minus x is nothing but minus sin x. So it can be absorbed in to the amplitude b. Similarly, when you are talking about the y dependence we could replace this piece by some e times, sin hypopolic of ky because, this can be written as something times cos hypopolic plus something times sin hypopolic and cos hypopolic goes to one just like cos goes to 1.

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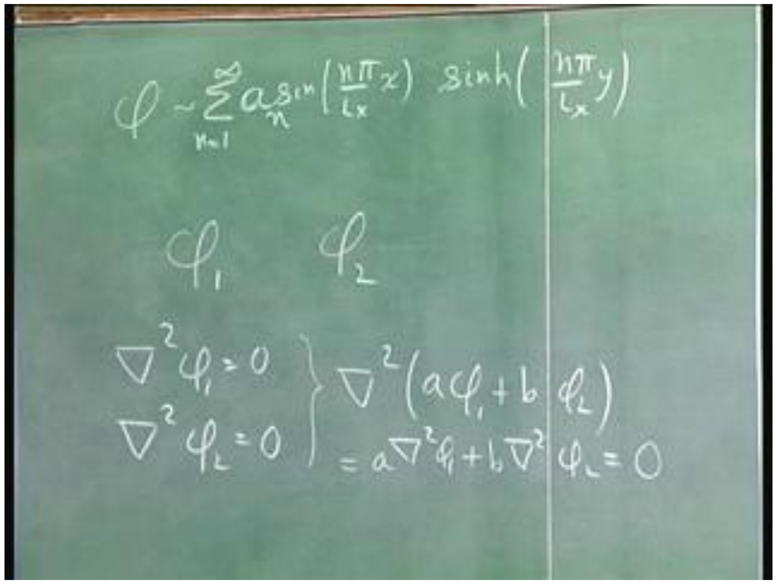

So, the solution we have is, pie goes like sin n pi over Lx x times sin hypopolic. The same quantity n pi over Lx y times, some unknown coefficient call it a with a problem is this side depends on n this side does not depend on n. So what is this n, which n can we choose? Frankly, we can choose any of them because all of them satisfy the boundary conditions. And since Laplace's equation is linear, if I have one solution pie 1 and I have another solution pie 2, both of which satisfy Laplace's equation. Then again take any combination of this.

And because it is a linear operator it will give me a dell squared pie 1 plus b dell squared pie 2 so therefore 0. So, any linear combination of solution also a solution, so I am going to use the most general linear combination I can use which is I am going to sum over all the n's. Now, as you see, I went from having five unknowns you having one unknown. But now, I have gone back to having infinite number unknowns. So, I am not really progressed at all. I have got more unknowns than I had.

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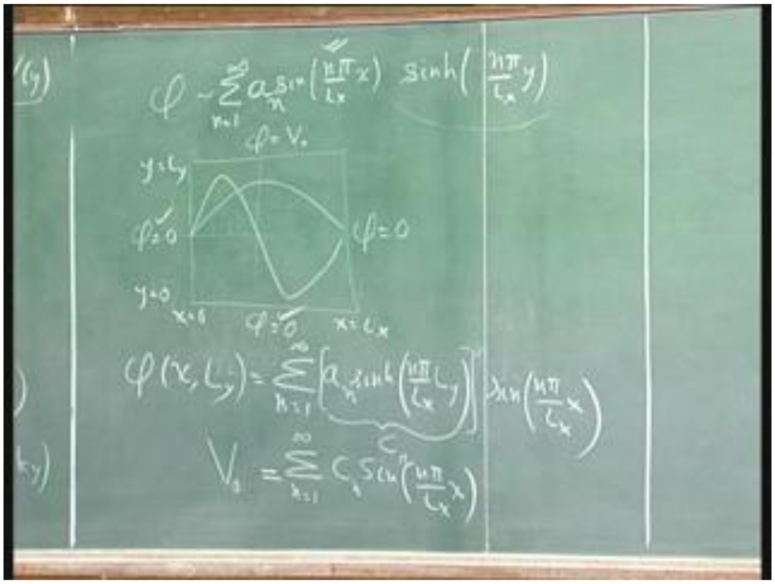

But I have used up quite a few of my boundary conditions, pie equals 0 has been implemented because this is 0. Pie equals 0 at x equals 0 has been implemented because of this. Pie equals 0 at x equals Lx has also been implemented because of this. So, all that remains is the top pie equals V naught. So let us put in this function and see what it is equal to at the top which is y equals Ly. This y equals 0 x equals 0 and x equals Lx. So at the top we have pie of x Ly is equal to I am going to use the summation sum n equals one to infinity some unknown coefficient a n.

Let me write out the sin hypopolic sin hypopolic n pi over Lx times y. But y is equal to Ly. So I can substitute Ly there times sin of n pi over Lx times x. This is not dependent on x is just a number. So I can call this some cn. So, I can write this so that the form of this equation is clear. Pie at the top is equals to V naught is equal to n, equals 1 to infinity summation some coefficient cn times sin n pi over Lx x. You have studied Fourier series in your maths courses and you can immediately see this is nothing but a sin fourier series. So, we have to solve it.

Now what is, how, what is the method of solution. For the method of solution is simply to recognize if you look at the sin Fourier series, for n equals 1 it is a function like that, for n equals 2 it is a function like this. Which means, that over this period both are positive but, over this period, this is negative and this is positive and since the integrals are equal but sin is changed, the product of these two would have 0 integrate.

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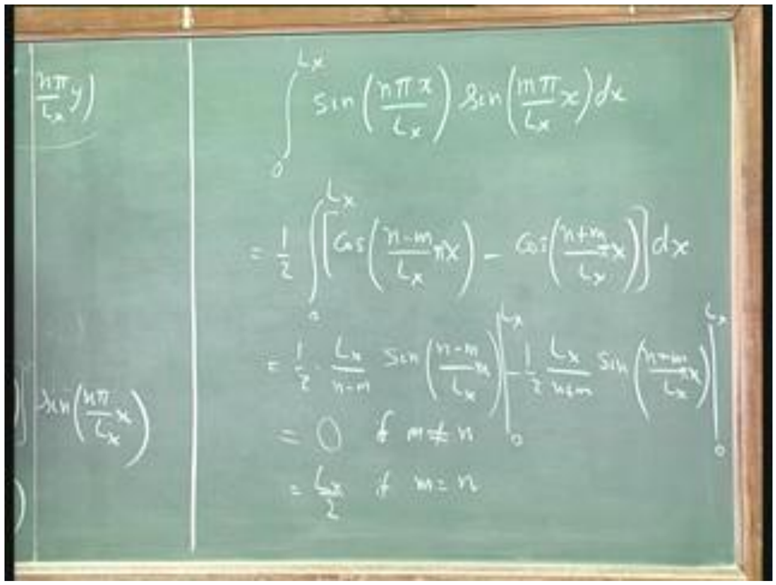

More generally, you can show let us show it 0 to Lx sin of n pi x over Lx sin of m pi over Lx x dx. Well, sin times, sin is a well known formula we just take when we have the product of two trigonometric functions. It is nothing but the sum or difference of two trigonometric terms. This becomes one half integral 0 to Lx cos of n minus m, over Lx. I can never remember that m minus n n minus m it does not matter minus cos of n plus m, over Lx. You can correct any algebra errors here. Now, m and n are not equal or if n and m are not opposite sign, but in this case we do not have opposite sign because, m must go from 1 to infinity.

So, n plus m is always going to be a positive number, n minus m as a chance of being a 0. But it can be 0 only n is equal to m. For any other case when you do this integral, what will you get? One half times Lx over n minus m times sin n minus m, over Lx x between 0 and Lx minus one half Lx over n plus m sin n plus m, over Lx x between 0 and Lx. When you substitute 0 of course both sin terms goes go to 0. When you substitute Lx x equals Lx then it Lx cancels with all of this have pi. The Lx cancels with the denominator.

So you are left with some integer number times pi and you know that sin of any number times pi sin of n pi is always 0. So this whole thing becomes 0. So it is equal to 0 if m is not equal to n. If m is equal to n then this term is cos of 0 cos of 0 is a constant it is 1. So, the integral of cos of 0 will become Lx over two if m equals n. Because, the factor of two comes out the integral of 1 from 0 to Lx is nothing but Lx, so that gives me Lx over 2. So very powerful statement it means that if I integrate any sin n pi x over Lx multiplied by any other sin n pi x over Lx the answer is non 0 only if m equals n and if m equals n then the answer is Lx over 2.

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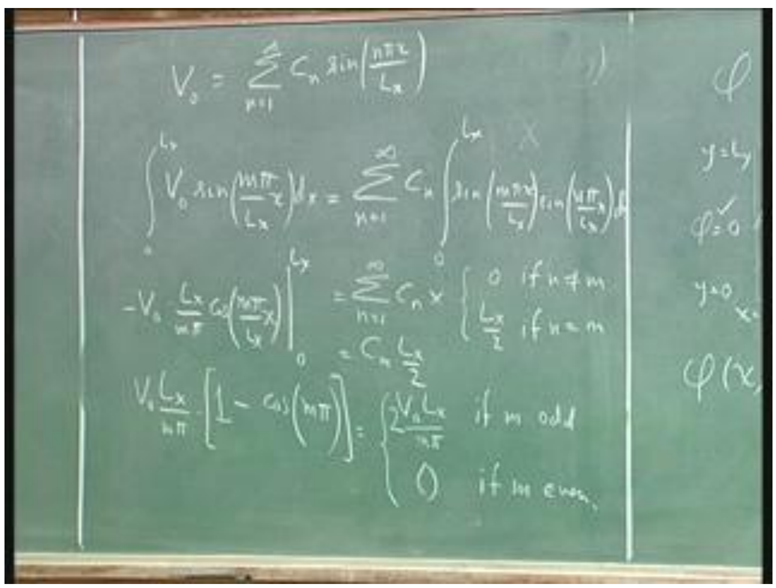

Let us use that here. I have my equation V naught is equal to sum n equals 1 to infinity Cn sin n pi x over Lx and I am going to multiply both sides by sin m pix over Lx and integrate. So I get integral 0 to Lx V naught sin m pi over Lx x dx must be equal to sum n equals 1 to infinity Cn integral 0 to Lx sin m pi x over Lx sin n pi x over Lx dx. Well, this side we just worked out this is equal to sum n equals one to infinity Cn multiplied by 0 if n not equal to m and Lx over two if n equals m. So I am summing over an infinite number of terms, but only one of those terms is not 0 all the rest are 0.

So, this becomes Cm Lx over 2 because only this term at use, all the others are 0. What about the left hand side? The left hand side I can pull the V naught out integral of sin I know how to do that its minus cos. So it is minus V naught times Lx over m pi cos of m pi over Lx x between 0 and Lx which can be written out is V naught Lx over m pi times the minus 1 will make it 1 minus cos of m pi Lx over Lx. So, that cancels out. When m is 0, 2, 4, etcetera. This becomes cos of 2 pi cos of 4 pi cos of 6 pi. That is nothing but 1. So, this will go to 0. This is equal to V naught Lx over m pi with the factor of 2 if m odd, 0 if m even. So now, let us combine these two statements and get a solution. I will rewrite the equation. You can see where I am coming from.

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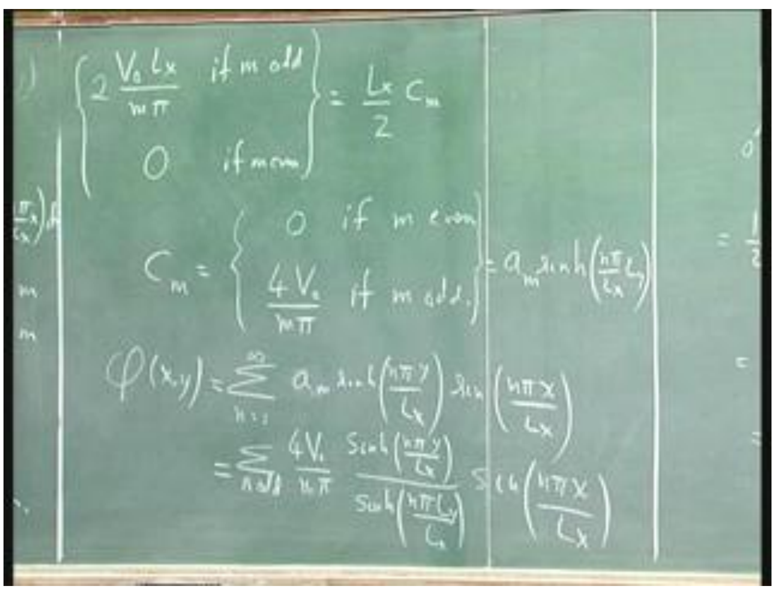

So the left hand side which was the integral of V naught sin n pi x over Lx has become V naught Lx. So let m pi with the factor of 2, okay? And I will put, if m odd, 0 if m even. This is the left hand side is equal to Lx over 2 Cm. So what it means is if m is even Lx over 2 Cm is equal to 0 if m is odd Lx over 2 Cm is twice V naught Lx over m pi. So we can solve Cm is equal to well if m is even Cm is 0. If m is odd the Lx cancels out this factor of two goes that side becomes 4 V naught over m pie if m odd.

So my answer becomes since I have found Cm I can now write down what the potential is. Potential pie of x y was equal to sum on n equals 1 to infinity Cm. Well, it was not Cm it is am sin h n pi y over Lx sin n pi x over Lx. What we have found Cm what is Cm. Cm is equal to am sin hypopolic m pi over Lx times Ly. Since I know Cm, so therefore I know am so I can substitute in here and I get the final answers for potential exist only if it is odd.

So, m odd there is n equals 1, 3, 5, etcetera and it becomes 4 V naught over m n pi times sin hypopolic n pi y over Lx divided by sin hypopolic n pi Ly over Lx times sin n pi x over Lx. So complicated looking expression, but what is important about it is that, now we have no nothing in this expression. That is left to be determined everything is known. So, summation over n, it depends on V naught, depends on Lx and depends on Ly. But it does not any longer depend on three parameters. Everything has been pin down. Now, there are something. That is, we can immediately learn from this which are very important.

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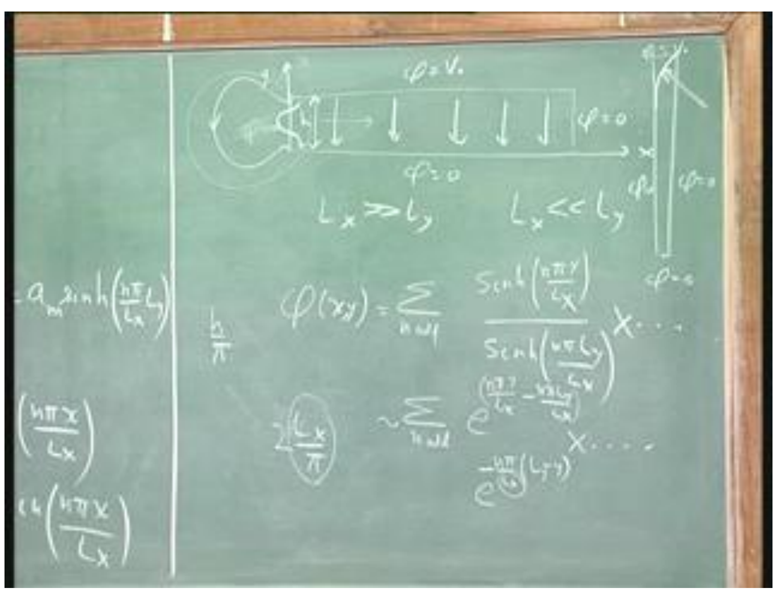

Supposing I took this box and tilted it. So my x is this way now or this way and y is this way. So I have got pie equals 0 here. No I am sorry. Let me correct that. I have just made it very flat box and I still have x and y correctly pie equals 0 here, pie equals V naught here and pie equals 0 at the ends. So now supposing Lx is much greater than Ly I have drawn it that way. Supposing Lx is 1000 times Ly, then we can ask what kind of solution comes out of this equation.

We know that we should get back to the parallel plate capacitor and supposing on the other hand Lx is much less than Ly we like to know what happens, namely this kind of problems. Let pie equals 0 pie equals 0 pie equals 0 but pie equals V naught. Now, the particular problem is solved gives us the answer to the second problem immediately. Because of if you look at the answer; answer says pie of x y is equal to some summation on n odd. But it depends on sin hypopolic of n pi y over Ly Lx. Sorry divided by sin hypopolic n pi Ly over Lx.

Now, when you go away from the bottom plate Ly over Lx is very large number. Similarly y over Lx will be a large number and both of this hypopolic functions can be approximated. Each of them is like an exponential. So this looks like sum over n odd various other things times e to the power of n pi y over Lx minus n pi Ly over Lx times other things. So what it tells us is something very important. It says that, as you go away from the top plate the potential V naught that drew a plot, let us say this is the base and I am drawing a plot of potential.

The potential will decade exponentially. It decades as if you write this out its e to the power of n pi over Lx times Ly minus y here minus n. So it is equal to V naught as you go away it becomes less not slowly becomes less exponentially fast. The exponential is the fastest varying function we know. I mean functions are exponential faster but it of all the common function, we have this is the one the decades quickly. See you can see that if you have any kind of field if you have any kind of voltage at one end of a long pipe because this is looking like a long pipe that voltage does not penetrate the pipe at all in fact, it decades on a distance that is of the order of Lx.

In this case the decade distance is Lx over pi. As soon as Ly minus y is of order Lx over pi, you already got e to the minus 1 as the worst case e to the minus n for in general. So the function is become quite small. And if you it is twice this, it is e to the minus 2 and so on also. The same idea can be applied to the parallel plate capacitor I would not do it here. But you can show that if you look at brunching fields. That is you draw the voltage you draw the field lines of a parallel plate capacitor. You know that out here that field lines will bulge out.

Similarly on the other side, the question is to what extend does this behavior penetrate into the middle. How form from the age must be b before we can say it is an ideal capacitor. The answer is again the same you have this quantity. If this is the height h, then if you move from the end at distance is equal to h over pi you are already in very good shape. The few times h over pi and you can forget about the edge of the capacitor and we treated as a one dimensional capacitor.

In this course we do not really go further in to multidimensional Poisson’s equation. However I think for your own interest you should go in to it because there are many, many interesting results that come out of doing electrostatics in multiple dimensions. I will do one more problem however because that is very important from the point of view of power applications.

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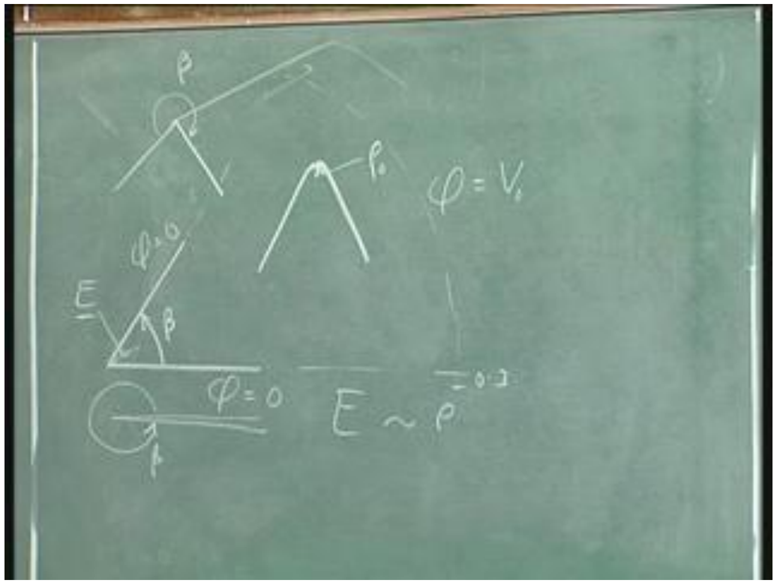

The particular problem I am going to take up is the following. Supposing I have a blade edge a knife edge, so there is some angle I call it beta typically greater than 180 degrees nearly 360 degrees. And there is a long straight edge and I have this charged and I like to know what are the potentials near this sharp edge. Even more interesting would be a cone. Both of these problems are easily solved I am going to solve this one and leave this answer for this. But in both cases it turns out that very high fields developed near the sharp edge and making beta nearly 360 degrees make this a needle.

And this is nothing but lightning arrested and sharp edges again appear quite often in our designs. You have to worry about discharges happening in cracks for example. So, let me look at this particular problem. It is a two dimensional problem because the third dimension nothing is happening. So I am going to use rn theta you can see there is a angle is important here. So, Cartesian coordinates are not good. So I will assume that I have these two edges I will assume that there is some angle beta.

Now I am really interested in the case where beta is, this but I will solve the general problem. I will assume that potential is 0 here potential is 0 here. But for a way there is some potential say V naught and I am interested in knowing what is the nature of the potential near here. What is the nature of electric field rather near the point. The answer I will give you immediately as long as this angle is less than 180 degrees.

This electric field is vanishing it becomes very timing. But the moment you cross 180 degrees and the moment you start having a point there the electric field becomes infinity and you need to know how is the electric field grows as you come closer to this point. The reason is in any real design this point is not a true point. So you would actually if you look at very closely it would be rounded.

So they would be a curvature at the tip it may be one micron on it or it may be ten microns. So if you knew this general solution and you knew that the electric field went as row to the power of minus “0.3”. Then you can substitute row not and find out the maximum electric field and that is extremely important for break down for discharges and for other kinds of affix. So now let us try and solve this problem.

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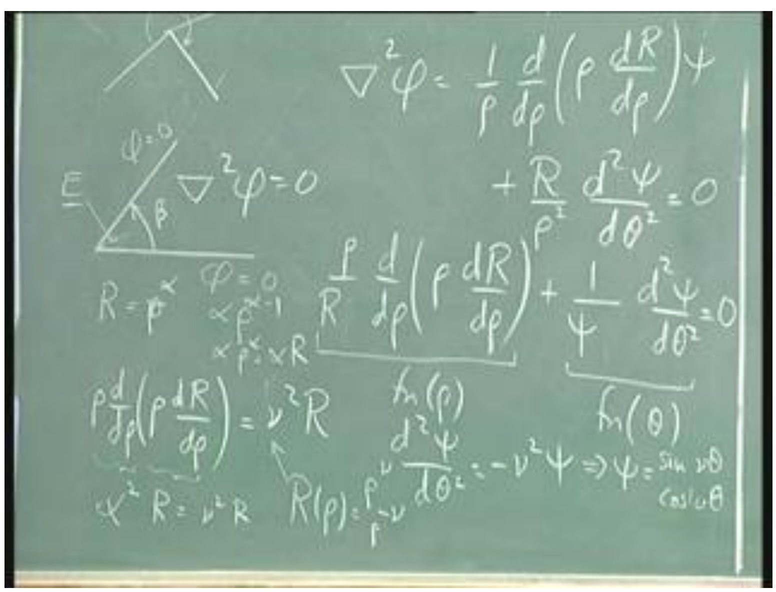

This is r theta coordinates. So I need my cylindrical coordinates Laplace’s equation one over. Well I am going to use row and theta and I am going to assume pie is equal to R of row and psi of theta I am not very good at writing capital row and capital theta. And so I am using different symbols plus times psi plus R over row squared d squared psi d theta squared the whole thing is equal to 0 Laplace's equation. Once again I divide through by R psi and I get an equation that is, but I still have this row here but I can easily get rid of that way of multiplying through by row squared. So I will do that if I multiply through by a row squared I will get a row in the numeric.

Now this function this part of the expression is a function of row only and this part is a function of theta only because you can see that psi is a function of theta R is a function of row. So I can equate each of these two a constant the same argument I have used several times in the last two lectures. So when I do that I get row d d row of row d capital R d row is equal to some I would not use k new squared R and on this side I have d squared psi d theta squared is equal to minus new squared psi. This is again the pendulum equation. So this implies psi is equal to sin new theta and cos new theta. Here, this is the rather simple equation even though it does not look simple.

The reason is when I differentiate and R has any form if R looks like row to the power of alpha I take the derivative will become alpha row to the power of alpha minus one. But after differentiating multiplied by row, so it will become alpha row to the power of alpha it becomes alpha capital R because R is row to the power of alpha. So this operator all it does it multiplies R by alpha and then this operator again multiplies R by alpha. So I will get alpha squared R is equal to new squared R. So alpha squared is equal to new squared or answer is for this equation R of row is equal to either row to the power of new or row to the power of minus new.

Let me repeat if I guess that R goes like some power of row say row to the power of alpha taking the derivative with respect to row gives me alpha times row to the power of alpha minus one. But I then multiply by row so the alpha minus one there becomes row to the power of alpha. But row to the power of alpha is the original function. So, this whole operation acting on R gives me alpha times R. So, if I operative twice I will get alpha squared times R and right hand side is some constant new squared times R. So it must be whatever power I need to use there is nothing but plus or minus new so those other two solutions. So, I can now write the answer.

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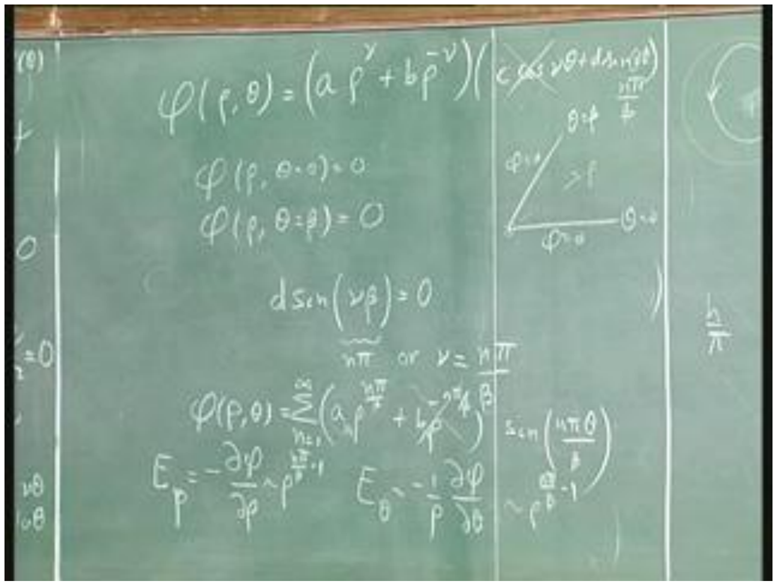

Potential as a function of row and theta is equal to some a row to the power of new plus b row to the power of minus new times c cos new theta plus d sin new theta. But I do not know what new is and I do not know what a, b, c and d are. So, once again I have five unknowns, well, I do have boundary conditions. Boundary condition I have is pie of any row theta equals 0 is equal to 0, pie of any row theta equals this angle beta is also equal to 0 at theta equals.

At theta equals 0 if pie is 0 it must be this is 0. So constant c is gone because cos of 0 is 1. So I cannot have any c, at theta equals beta again if it is 0 I must have d sin of new beta is equal to 0. But I know there is if sin is going to 0 this must be n pi or new is equal to n pi over beta. So this is n pi over beta but I know new I can substitute new here also. So again an answer pie of row theta is equal to a row to the power of n pi over beta plus b row to the power of minus n pi over beta time’s sin n pi theta over beta. I have absorbed the factor d into a and b as usual because I have a new factor n.

So I am going to sum over n and I will put a some subscript n to a and b. So now, I have a general solution for potential its sum over n equals one to infinity. An row to the power of n pi over beta bn row to the power of minus n pi over beta sin n pi over beta. But, this problem where I am trying to solve happens to include the origin. This is theta equals 0, theta equals beta and row is in this direction. So row equals 0 is part of my region and if you look at this function, it is row to the power of minus something. That is, it looks like one over row to the power of something. It is going to go to infinity here.

So I cannot allow this because my conditions are potential is 0. So, this piece must go so I am left with only one coefficient but infinite number of them. All the, an's are still to be determined okay. I have got pie now I need electric field to determine this and I will now have to solve the problem of what is the potential on the wall in the firewall. But that is not what I am interested in. So, I will just leave it this way. If I knew that potential I can solve this an also, but now I am more interested in something else. From this potential I would like to calculate the electric field.

So I can calculate Er or E row is equal to minus del pie del row and E theta equals minus one over row del pie del theta. Now we take the derivative with respect to row I am going to act on row to the power of n pi over beta. So what I will get is something there will go like row to the power of n pi over beta minus one times other things. If I take E theta the derivative with respect to theta will act on this term. But the there is one over row. So it also goes like row to the power of n pi over beta minus 1.

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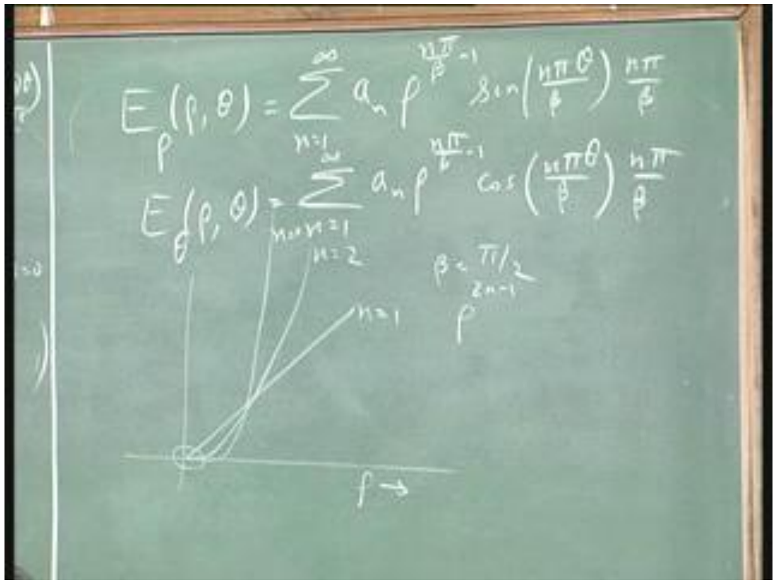

So, if I write down the answer, what I get is that E row of row theta is going to be equal to sum n equals 1 to infinity and I still do not know what an is row to the power of n pi over beta minus one times sin n pi theta over beta. And E theta as a function of row and theta is again sum n equals 1 to infinity I have to multiplied by n pi over beta. It is going to be equal to a row to the power of n pi over beta minus 1 cos n pi theta over beta times n pi over beta. So, these are the solutions and what is important about them is this minus 1. Now, when you look at the electric field and you sketch it for each of these ns, what will I get?

Supposing I take beta equals pie over 2, so n pi over pi over 2, so it is 2 n minus 1. So, the variations are 2 n minus 1. So for n equals 1, the electric fields will grow linearly. For n equals 2, the electric field will grow like a cubic. For n equals 3, the electric field will grow like a fifth order and so on so forth. So, the electric field clearly goes to 0 at row equals 0. So, this is the very control case. Now, the fact that is glowing up as you go for further away it does not matter because we got to be still displayed what the values of anr. The values of an will be suitably small so that these terms do not become very large.

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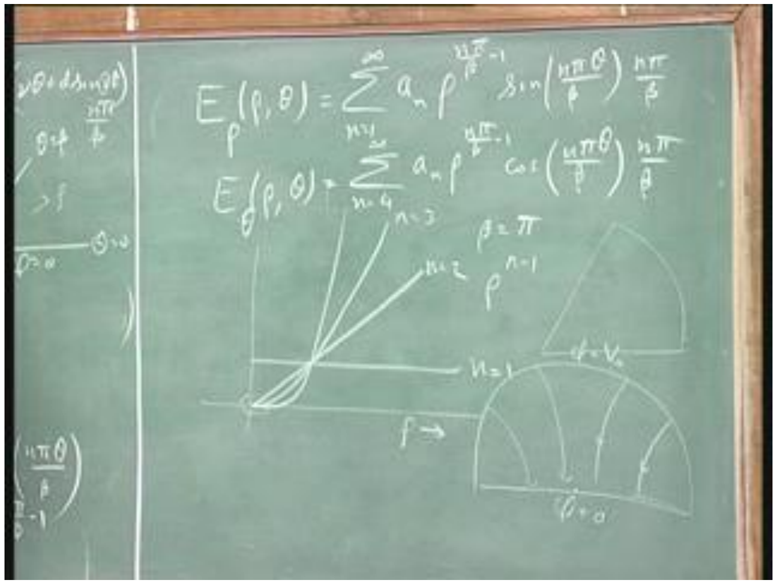

But now what happens? If you try the same analysis but we put beta equals pi, we out beta equals pi then the pi was cancel out will get row of n minus 1. So for n equals 1 it is a constant, for n equals 2 it is linear, for n equals 3 it is a quadratic, for n equals 4 it is a cubic, etcetera. So now for the first you get a non 0 potential a non 0 electric field at the point now does that make sense well think of it. Let us say we have a plate of this type.

Now we let the plate open up completely then we are talking about the semi-circle and we have pie equals 0 here and pie equals let us say V naught here. Obviously you will have electric field coming at the plate which means the electric field at this point is no longer 0. So that is what this pieces saying. It is saying there is infact the uniform piece in the electric field. Electric field does not change with row as far as this component is concerned. All the other components will correct the edge of it is. Now what happens when beta becomes larger than pie? It is the same picture. But I will redraw it.

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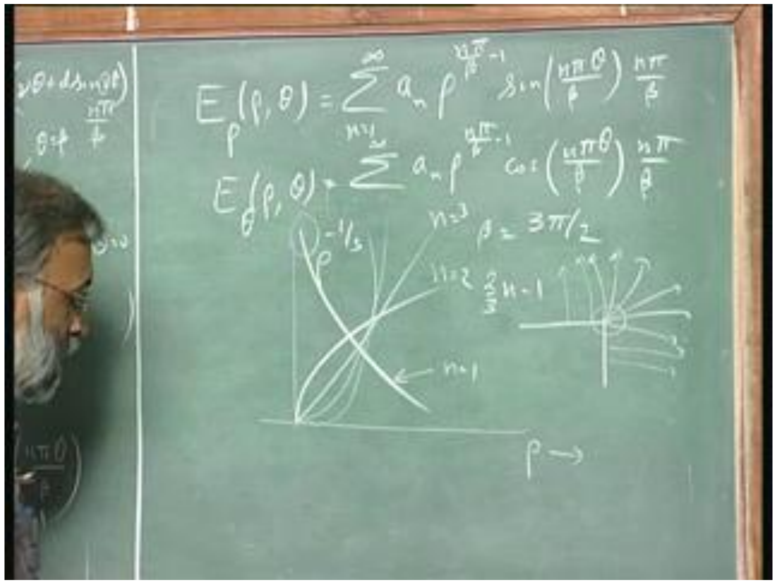

I will try beta equals 3 pi by 2 okay. This row if I substitute beta equals 3 pi by 2 that pi cancels out and I get 2 thirds n minus 1. So, n equals one two thirds minus 1 is minus one thirds. So I will get a curve that it is like this. For n equals 2, it is four thirds minus one so that becomes one third. This is n equals 2 then there will be six thirds minus one which is n equals 1. I do not know whether it will how it will go but it is linear and then after that you get curves that become flatiron plot in your origin. This is the one that is crucial.

This curve actually does not remain bounded as row goes to 0 blows up. It means that if you had region like this, from this point or to this point, you will have large number of electric field lines. So the field lines here there is a large number of field lines landing up that point. Which means the closer you come the hire them electric field and the electric field becomes infinity. For beta equals 3 3 3 pi by 2 this singularity goes like row to the power of minus one thirds.

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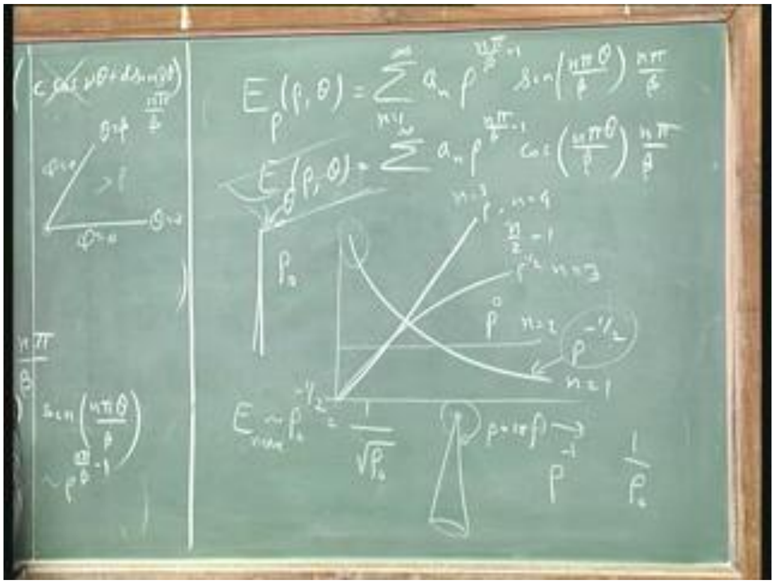

Now, the worst case that you can have is beta equals 2 pi which is the interesting case because play interested in a knife edge. You put a knife edge and you put an electro near at knife edge, then what kind of electric field well this knife edge see. Now we substitute beta equals 2 pi we get n over 2 minus 1 and we put n over 2 minus 1 the first term will go down. This is row to the power of minus one half and corresponds to, n equals 1. The second term will be 0 so its constant row to the power of 0 which is n equals 2, third term will be row to the power of half which is n equals 3 and after that you will have a straight line solution which is row n equals four and so on so forth.

This electric field blows up very strongly it blows up as row to the power of minus half and this is why if you have a knife edge near an electrode a discharge happens almost spontaneous. It is because the electric fields at these points are extremely strong. There all the lines are force on landing on that edge and so the electric field is single. But infact it is a single of any angle greater than 180 degrees, but the amount singularity changes and this is where it is important to know that are real knife edge is not a true knife edge at all. You will have to machine yet you will have to machine yet so that it can take the heat anytime that discharge there will be heat produced so that that knife edge cannot truly be a point.

It will have some some radius at the top. If that radius is row not say one micron then you know that electric fields maximum electric field will go like row not to the power of minus half or one over square root of row not. It is a large row not is 10 to the minus 6 meters. This will be 1000 into normalization factors. So it is going to be a large electric field. But the extern to which it is a large electric field depends very strongly on how how find a point you can may.

Now all of this was for knife edge which is very important. But even more important is the cone because for example if you have a lightning or wrester, lightning or wrester is nothing but a cone with beta equals 2 pi basically a straight line a needle. Now if you do the same analysis, you can in fact all just as easy to solve you will get that the potential. The electric field goes like row to the power of minus one. This goes like one over square root of row. This will go like one over row not itself. See, if that needle has at it strips it ends in a ball whose radius is one micron, you are going to have 10 to the 6 times the normalization of your electric field at that tip.

So, if typically you are having 1 volt, you will have a million volts there per meter whereas you will have a 1000 volts per meter. So that is why these kinds of designs are very important. The electric field here is a million times are showed stronger than it will be anywhere else. So any time a lightning happen the break down happens here first and since it happens here, first a lightning strikes the tip before it strikes any other point. And if you connect a good conductor to earth, well the lightning keeps striking there and carries way the current safely. Now this all are plan to say about electrostatics.

Ours planning do one or two other problems, but I think the course time is limited. So I will stop with this material. But I will say that electrostatics pretty much is a capsule of electromagnetic. All the techniques we used in other parts of electromagnetics or used in electrostatics as well. A variation of Poisson's equation is what you do in magnetic as well and the way equation uses many techniques separation of variables all those techniques that we used here. So if you are comfortable with electrostatics is only one more concept that you have to learn before you can master the whole subject and that is basically the cross product and the curve.

And if you can understand how you add one more vector operator decides the divergence, we have the whole field is covered. There are some minor additional problems that you could have done. For example, I want to do the problem of the solenoid of a capacitor attracting a dielectric or repelling the dielectric. I will do that problem in connection with the solenoid and the ferrite. It is a same problem whether it is in electrostatic or in magnitostatics. And the problem of use solving dielectric problems with Poisson Laplace's equation is fairly straight forward, so there is nothing additional to do.

But the problem of doing Poisson's equation would require a little more effort do read a text book and see if it makes sense to you because the mechanics of the problem solving is the same. You have already done more or less or we have to do to solve Laplace's equation the same we do to solve Poisson's equation. But it always would have been better if I worked out the problem as it is I will close this chapter an electrostatics here. And next lecture onwards I will be looking at magnetostatics.

Thank you.