Electromagnetic Fields
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Lecture – 16
Poisson's Equation 2Dimensions

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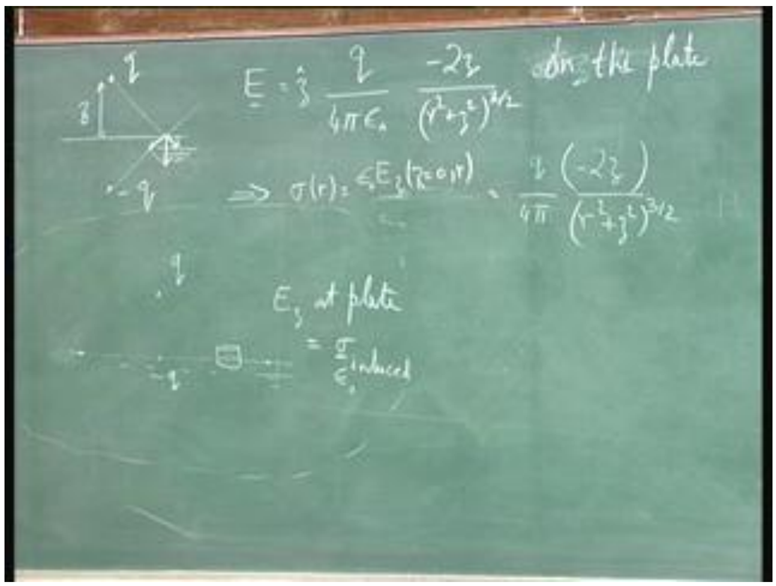

Last time, I had derived using the method of images problem, where I put a charge above a ground plane. Plane that is grounded and worked out that, if you place an image charge with the same magnitude of charge, but opposite sign that the potential is 0 due to these two charges at z equals 0. And so, I removed the ground plane and calculated the electric field everywhere. Answer looks like this, at z equals 0. So basically this is z, so at the observer point on the plate, so what it says is, let the electric field is purely in the z direction.

So, you have attractive force towards this negative charge repulsive force from this positive charge and when we add them up the electric field is a negative z direction. So, that is what you see z hat is minus sign. Now this is fine. But, an interesting question to ask could be if I place a charge q above a grounded plane, what kind of induced charges are created. I know that I am going to have a negative charge because Gauss's law tells me, if I take a sufficiently large surface, so that the potential goes to 0, the field goes to 0. I know the field is going to go to zero very fast. Because, I expect that charges will come in from the ground and I will get a negative charges out here.

So there would not be any net electric field going out. Which means, a charge enclosed to 0, which means I expect minus q on this plate if I have q placed about it? But I do not know exactly how that minus q charge is present. I do not know if all the minus q is in one point here or perhaps the minus q is out here. So, in order to find that out, I take the solution of this problem I know the electric field on the plate. And on the plate if I use Gauss's law the bottom surface is inside the metal there is no electric field the top surface there is an electric field.

And on the surface there is charge. So, I know already that Ez at plate is equal to sigma induced divided by epsilon 0. So sigma can be solved for and sigma is nothing, but epsilon 0 Ez. But already I know Ez the entire electric field is in this z direction. So if I write it down, this is what I get that the charge induced is q over four pi times of constant minus 2z; z is the distance the charge is above the plate. So it is a constant divided by r squared plus z squared to the power of three halves. So, what is that going to say about a charge distribution? Redraw the picture.

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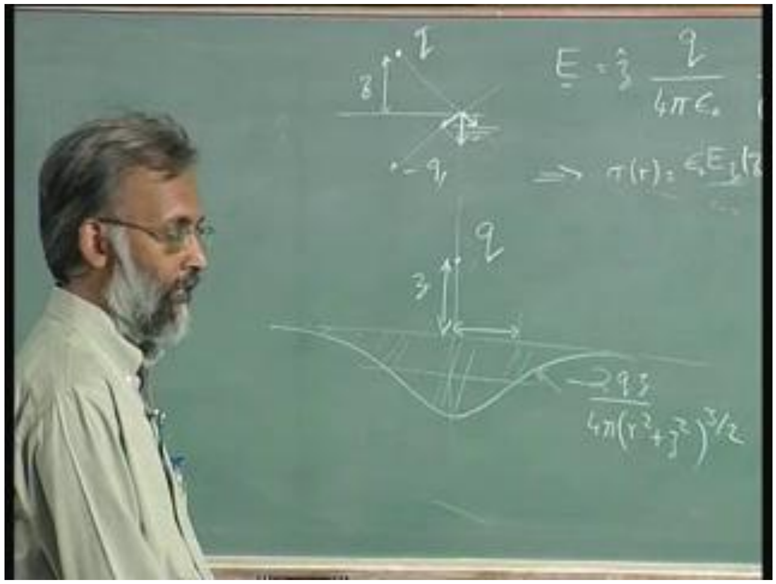

I have the charge q. I have the plate this is the distance z on this plate. I am going to have a negative charge and the negative charge is going to look like this. This is minus 2 q z over 4 pi times r squared plus plus z squared to the power of three halves. So, you can see that the induced charge on a plate is not a simple thing at all which distributed charge. This little charge out there, the more charge out here, the lot of charge out here less charge here.

Well, very little charge there. It distributes itself. It is mere itself out most of the charge being near where q is, if I asked how much distance do I have to go in r before the induced charge is dropped to half its value. This is the maximum value I will take half it is value and draw a line and then I ask how much this distance. So I would like to know this distance which is the distance at which the induced charge becomes half.

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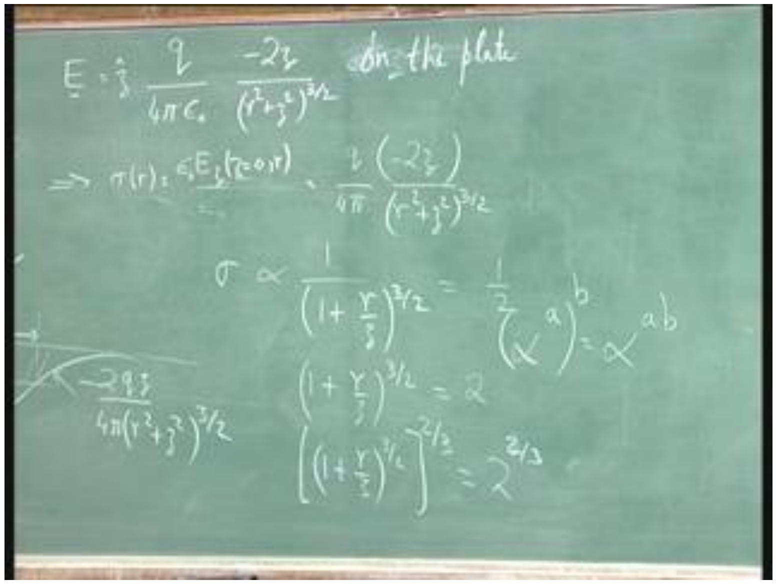

Well, we look at this picture at this expression we know that sigma is propositional to one over r squared plus z squared to the power of three halves. I can take the z below, so I can call this same expression as 1 plus r over z to the power of three halves times some constant. Constant will be q over 4 pi then there will be a minus 2 and 1 over z squared. Now if you look at this expression and say this must be equal to 1 over 2 at z equal at r equals 0 it is 1. So I want to know where it becomes half, then what do I get? I get 1 plus r over z to the power of 3 halves must be equal to 2.

That is, I take both the denominators to the numerator. I have an expression that is to the power of three halves. If I take this whole equation to the power of two thirds, then what do I get? I get 1 plus r over z to the power of three halves, whole to the power of two thirds is equal to 2 to the power of two thirds as you know when I take some quantity alpha to the power of a whole to the power of b, this is nothing but alpha to the power of a times b. So three halves multiplied by two third is one.

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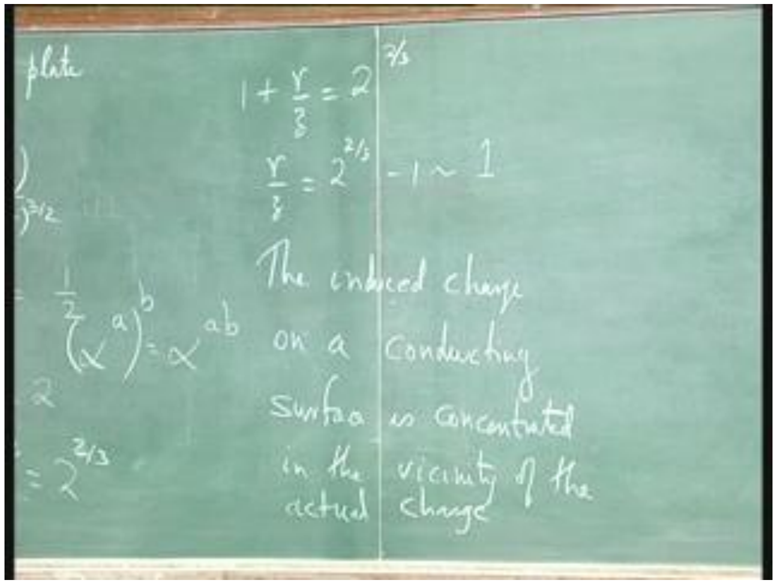

So I get an answer which says 1 plus r over z is equal to 2 to the power of two thirds or r over z is equal to two to the power of two thirds minus 1. Now it is not exactly 1, it is a less than 1 but of the order of 1. So you have to go a distance some what less. But of he same order as the distance the charge is above the plate, before you will find that the induced charge is drop to half its value.

So, most of the charge is in a region that is order in width the height of the charge is above the plate this is very common. So you should know thus the kind of a rule of thumb that the induced charge on a conducting surface is concentrated in the vicinity of the actual charge. See, if I put a charge one centimeter above a plate, you will find there is induced charge, roughly within on a circle one centimeter radius on the plate.

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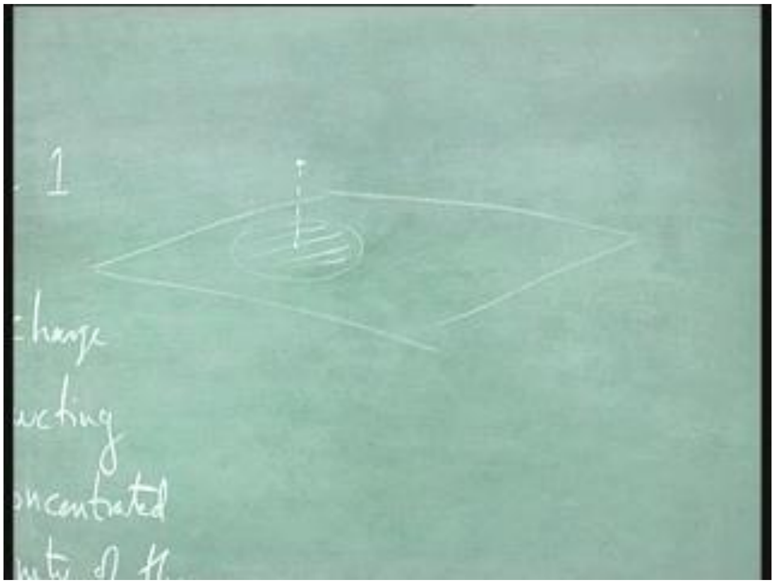

So, if I put a charge up there and I have a plate this charge is nearest to this point on the plate and there will be a circle where most of the induced charges present. And if you go further away thus much less induced charge. So that leaves actually three topics that we still need to cover in electrostatics and I am going to cover one of them today.

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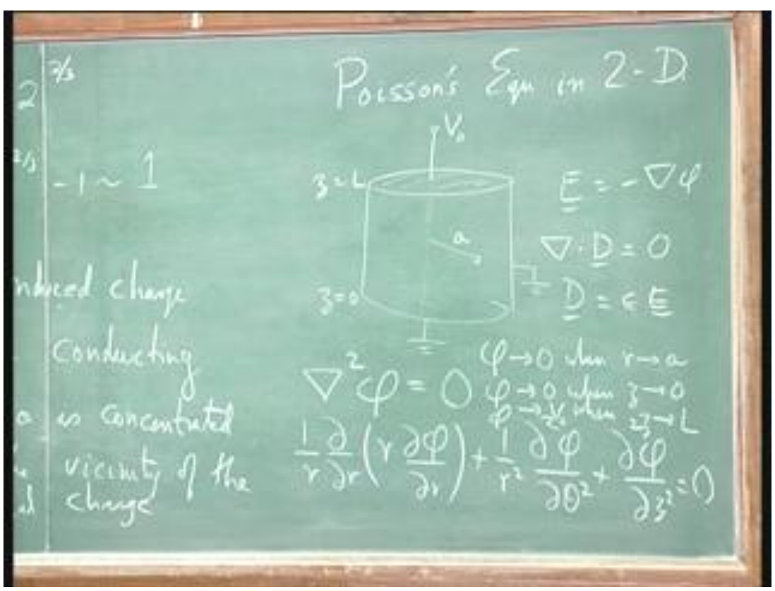

And that topic is Poisson's equation in 2D. And I say Poisson's equation is also Laplace's equation. Now let me motivate the problem. Supposing I have a cylinder and supposing I ground the side walls ground the bottom wall and apply voltage V to the top wall. Now this is actually one of the forms in which capacitors came initially, when initially people were trying to prove that charge could be stored they created laden charge. Laden charge is nothing but bottles where they would put in an electrode. Here instead of putting an electrode it is just making on top or electrode.

I would like to know the capacitance of this object. But how do I do it? If I write down the equation for that has to be solved, the equation is dell squared pie is equal to 0, there is no charge inside. So I had electric field was minus grad pie and divergence of D is equal to 0 and D is equal to epsilon E. So if I combine all three equations I get Laplace's equation. Now, this is to be written in cylindrical coordinates. It is not in Cartesian coordinates. So, the equation looks like 1 over r partial derivative with respect to r r del pie del r plus 1 over r squared del squared pie del theta squared plus del squared pie del z squared is equal to 0.

So, this is the complicated equation and along with the complicated equation I have the boundary conditions that pie goes to 0 when r goes to a. Let a, is the radius of the cylinder, pie goes to 0. When z goes to 0 and pie goes to V naught when z goes to L. So, I have an equation a complicated equation and rather complicated condition on that equation. And I want to know how to solve such a problem. Now this is a very standard way by which we have developed which we have developed to solve such a problem. I will give you the concept first and then I will work out the details.

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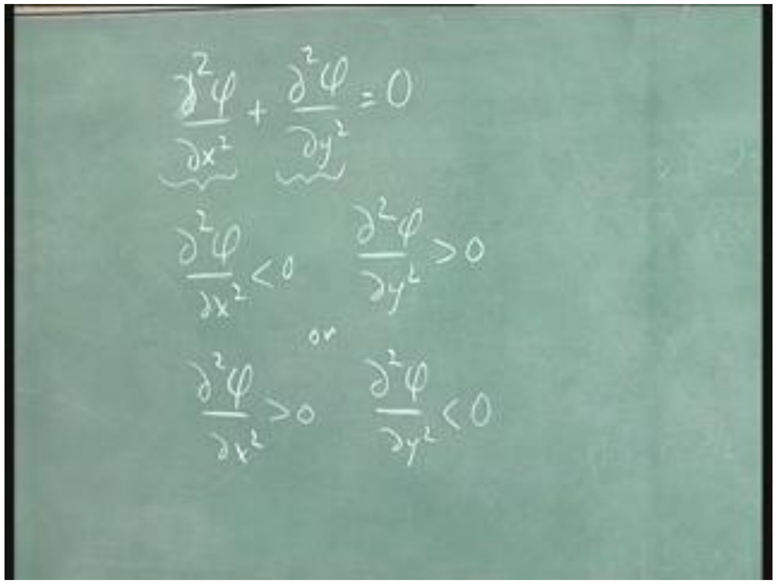

The idea we have is that if you look at Laplace's equation you can find. Let us take the simplest case in Cartesian coordinates d squared del squared pie del x squared plus del squared pie del y squared equals zero. If you look at this you have to second derivatives which added together give you 0. So, you can immediately right down something you can say del squared pie del x squared less than 0 del squared pie del y squared greater than 0 or del squared pie del x squared greater than 0 del squared pie del y squared less than 0.

Why this is important? Well, it is important because we know that if you have spring like behavior oscillatory behavior sins and cosines. The second derivative of the function is negative where as if we have exponential behavior the second derivative is positive. So what it means is that Laplace's equation interestedly contains two kinds of functions. It contains oscillating functions periodic function and it contains non periodic growing functions. What we are going to do? It is going to make a guess. Let us just look at some general solution to this problem. I will I will state that more precisely.

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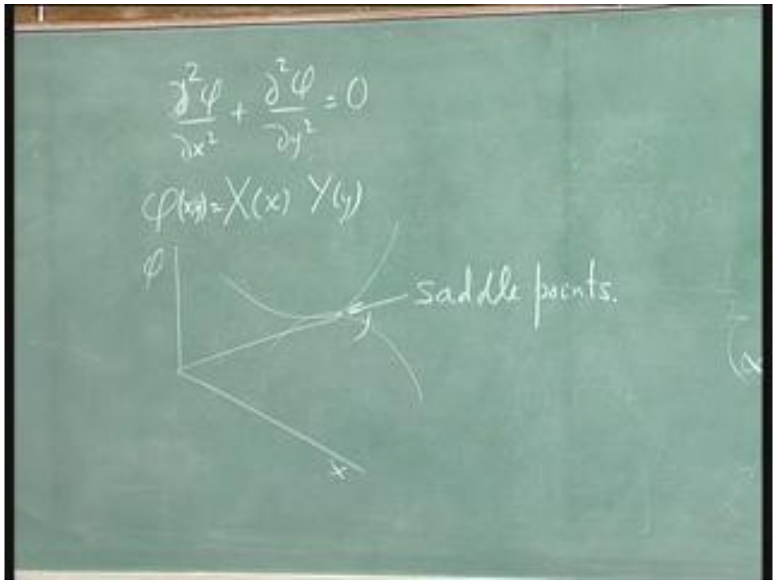

I am going to guess that this potential pie has two different behaviors, one which is oscillatory in x and one which is growing in y. I am going to pertain that each is independent of the other. So I will say that this pie is actually equal to some X of x Y of y. It is a capital X capital Y. This is small x small y. So these are two unknown functions but the important thing about this function is pie itself is a function of both x and y. But, I am going to guess that it is a product of two functions. It is a product of function is that only depending on x and a function is that only depends on y. Why do I assume this? I assume this because of looking at this I can see whether the two different behaviors that seems to be present.

So if I have to sketch this in x and y, supposing in x del squared pie del x squared is negative it means pie is doing this, as I move in x if I reach a maximum and comes direct down. But if I look at del squared pie del y squared, if this is negative this must be positive which means pie must be actual be reaching its minimum and going up. So, at this point there are two very different behaviors in x and y such points are called saddle points and Laplace's equation defines the function which is saddle point at every point where it is defined. So every point for pie is saddle point. May be put that assumption in what do we get? I am going to substitute this into this equation.

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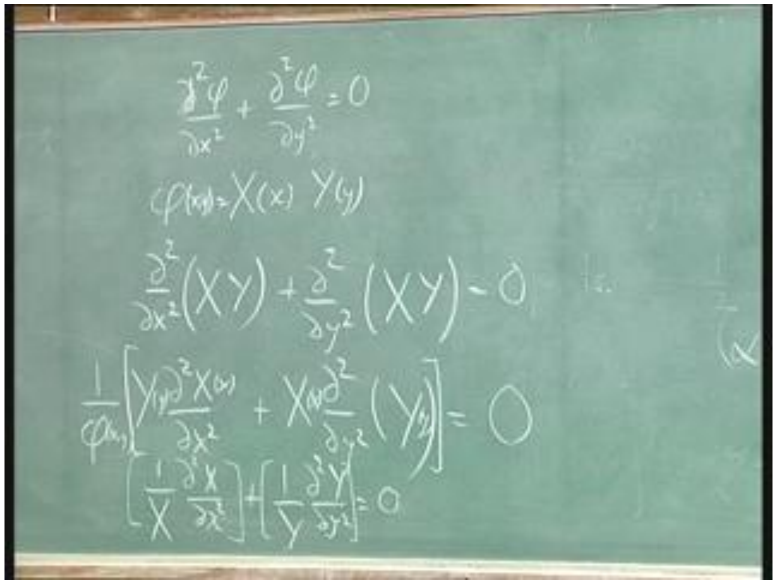

So I get del squared del x squared of capital X capital Y plus del squared del y squared of capital X capital Y is equal to 0. Now capital X is a function of small x, but capital Y does not depend on small x. So, when I take a derivative with respect to small x y can be pulled out. Similarly, capital X is not a function of small y. So, it can be pulled out. So, I can write this as Y del squared capital X del x squared plus X del squared del y squared of capital Y is equal to 0. Now I will do one more thing this is still a function of y and of x and so is all mixed up so I am going to divide this equation by pie itself.

So, I am going to multiply 1 over pi of xy times this, since the bracket is equal to 0, if I divided by pie it is still 0. But pie is equal to X times Y. So if I divide by X times Y 1, Y will cancel out and I will get one over X. So I get one over capital X del squared capital X del x squared plus 1 over capital Y del squared capital Y del y squared is equal to 0. Now there are something very remarkable about this equation. This part of the equation depends on X depends on x depends on x it does not depend on y at all. This part of the equation depends on Y. It does not depend on x at all.

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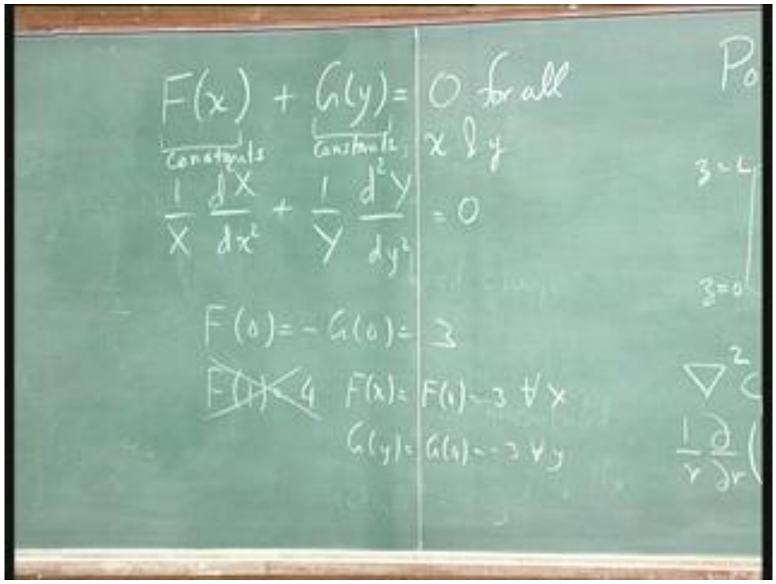

So I have a form that looks like some function of x plus some function of y is equal to 0 for all x and y. Why do I say for all x and y? I say that because Laplace's equation must hold at every point. Therefore the equation I have got which is one over capital X d squared capital X dx squared plus one over capital Y d squared capital Y dy squared is equal to zero must actually hold for every x and every y. It is not at some special x and special y. But this kind of equation cannot really be true. For example, supposing F where three at x equals 0, then G must be equal to minus 3.

So let us say F of 0 is equal to minus G of 0 is equal to 3. Now what is the value of F ay x equals 1, else supposing if F of 1 was equal to 4. Well it is not possible because F of 1 is 4 G of 0 is still three minus 3. So 4 minus 3 is 1 but the point x equals 1 y equals 0. This equation must hold. So I cannot have 4 minus 3 is equal to 0. So this cannot be true. It must be true that F of x is equal to F of 0 equals 3 for all of x.

Similarly, G of y equal to G of 0 equal to minus 3 for all y. What it what you can conclude from such equation says, F of x and G of y or constants. They cannot depend on their variable. Because it F of x dependent on x there is no way G of y can compensate.

We need the sum to be equal to 0. See if F of x varies G does not depend on x, so it cannot vary and it will fail when you apply the condition.

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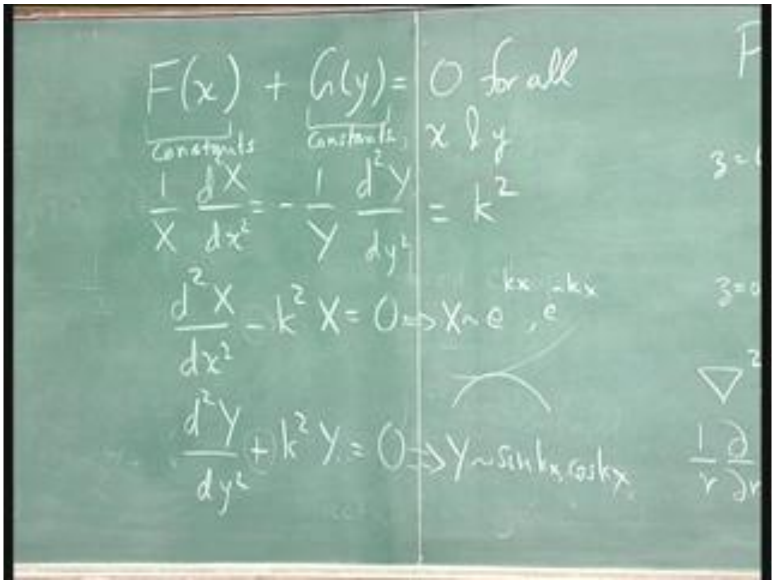

Now when writing this equation have gone over from del X del little x to pull derivatives ddx. The reason is capital X is a function only a one variable and capital Y is a function only one variable. So when I take the derivative of the function of one variable I don't need to put partial derivatives. There is nothing wrong putting derivatives it is just not necessary. So this argument tells us one over X d squared X dx squared it is a constant, 1 over Y d squared Y dy squared is also a constant. So let us give that constant or name that is let us say that this is equal to minus one over Y is equal to some k squared.

Then I would get d squared capital X dx squared minus k squared capital X is equal to 0 d squared capital Y dy squared plus k squared Y equals 0. You can see that important different in sign, because when you add up these quantities. That is when I multiply by Y and multiply by X and add up this equation, this side gives me Laplace's equation whereas these two cancel out. So that is why I must have this difference in sign. So one of them is minus one of them is plus. But the term that has minus sign it implies that x goes like e to the power of kx e to the power of minus kx. Whereas, the Y equation implies Y goes like sin kx cos kx. Once again this is what we have been saying namely the sin kx function looks like this. The e to the kx function looks like this.

So at every point once side is turning up and another direction is turning down. So every point is a saddle point alright. This was the Cartesian geometry case but I really want to solved, this problem. I want to solve the problem of a cylinder which is whose sides and bottom are kept at ground potential whose top is kept at voltage V not. How do I do it? I follow the same procedure.

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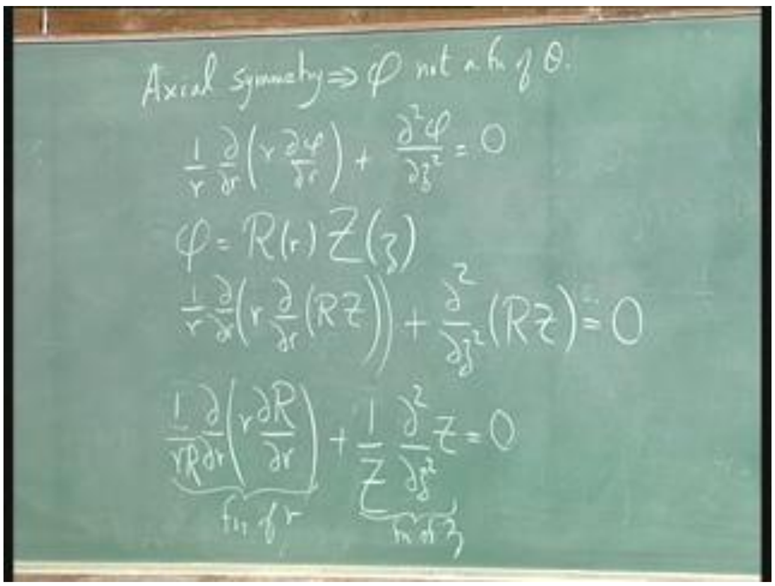

First I say axial symmetry so pie not a function of theta. Since if I rotate this cylinder, it looks like the same problem. Answer cannot depend on the exact value of theta. So, pie may be a function of r and z but it cannot be a function of theta. So my Laplace's equation become one over r del del r r del pie del r plus del squared pie del z squared is equal to 0. Once again I will assume pie is equal to R of r Z of z. Substitute into the equation, so I get one over r del del r r del del r of capital R capital Z plus del squared del z squared capital R capital Z is equal to 0.

But Z is a function of small z and every thing here is taking derivative with respect to r. So this Z can be pulled out. R is a function of capital R is a function of small r only derivative with respect to z. So this capital R can be pulled out. So I will do that and I get capital Z divided by R del del r of r del capital R del r plus capital R del squared del z squared capital Z is equal to 0. Once again we divide through by pie. The reason for dividing by pie is I still have terms which are dependent on both Z and on R. I want to get the answer in the form of something there is a function of R plus something there is a function of Z is equal to 0, then I can separately equate them to constant.

So I will divide by pie if I divide by pie I am dividing by RZ. So this Z cancels out and I get one over capital R and here the R cancels out and I get one over capital Z. Once again this is a function of r and this is function of z. So since each is a function only of one of the two coordinates each is a constant. So what does this give us? We get two equations.

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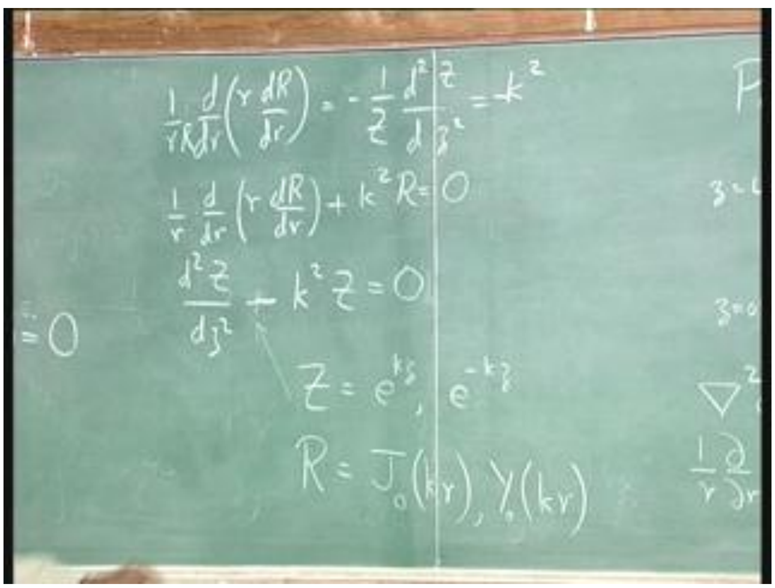

We get 1 over r d by dr 1 over r capital R r d capital R dr is equal to minus one over capital Z d squared capital Z dz squared and I am saying that Z is equal to k squared. So I can make this in to two separate equations one equation says one over r ddr of r d capital R dr minus k squared R is equal to 0. Well, let me make this minus sign so its plus k squared R equals 0 and the other equation is d squared capital Z d small z squared minus k squared capital Z is equal to 0.

Again a plus and a minus you always have to held, that difference in sign because, if I multiply this by z and this by r and add them up, I should get back Laplace's equation. This second solution equation has an immediate solution. It says capital Z is equal to e to the kz or e to the minus kz that is what this says. It is the equation for an exponential and if you solve this problem I am not going to tell you how to solve it. But you can solve it you will find let capital R is equal to what I called Bessel functions J zero of kr and Y zero of kr.

Bessel functions are like sins and cosines except the, occur naturally when you talked about cylindrical geometry. In Cartesian geometry the natural oscillatory function is sin kx and cos kx. In cylindrical geometry the natural oscillatory function is J 0 of kr Y 0 of kr. And similarly there are naturally oscillatory functions in spherical coordinates, alright? So we have got a solution. Now what is of pie?

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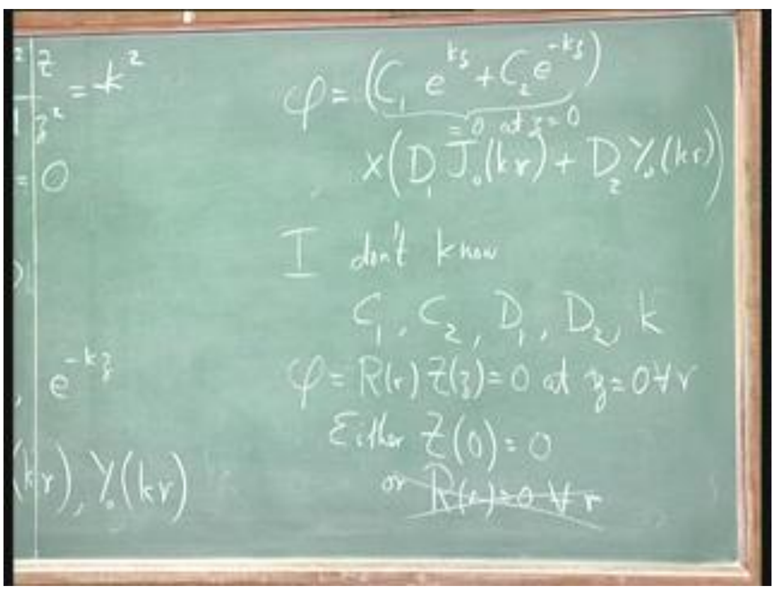

A potential pie is equal to some constant C 1 e to the kz plus another constant C 2 e to the minus kz multiplied by yet another constant J 0 of kr plus fourth constant Y 0 of kr and in these expressions I do not know C 1. I do not know C 2. I do not know D 1. I do not know D 2 and worst of all I do not know k. In fact I know hardly anything, it is very embarrassing situation I am solved the problem but it is of no used to me. If I am going to make some use of this solution I have to put in extra information. But I do have extra information I know that the potential goes to zero at the bottom.

I know that the potential goes to zero at the sides and I know that the potential goes to V naught at the top. So first of all I want to make this potential go to 0 at z equals 0 pie which is equal to R of r Z of z equals 0 at z equals 0. Now how can this happen? Pie can go to 0 at z equals 0 into two ways. Either Z of 0 equals 0 that is one way in which potential can go to 0 at z equals 0 or R of r equals 0 for all r. If r is identically zero then also pie will go to 0. There are only two ways in which this potential can go to 0 at z equals 0 for all r. Now obviously R of r goes to 0 everywhere and I do not have a potential. That is a useless kind of function.

I want something more interesting in that so I through this out I am not going to except such a kind of solution. That says there is no potential anywhere. So translate from a condition on pie to a condition on Z. Z goes to 0 at z equals 0. This function is equal to 0 at z equals 0. Now I am sure you have studied about exponential functions and you know that you can construct trigonometric function from exponential function.

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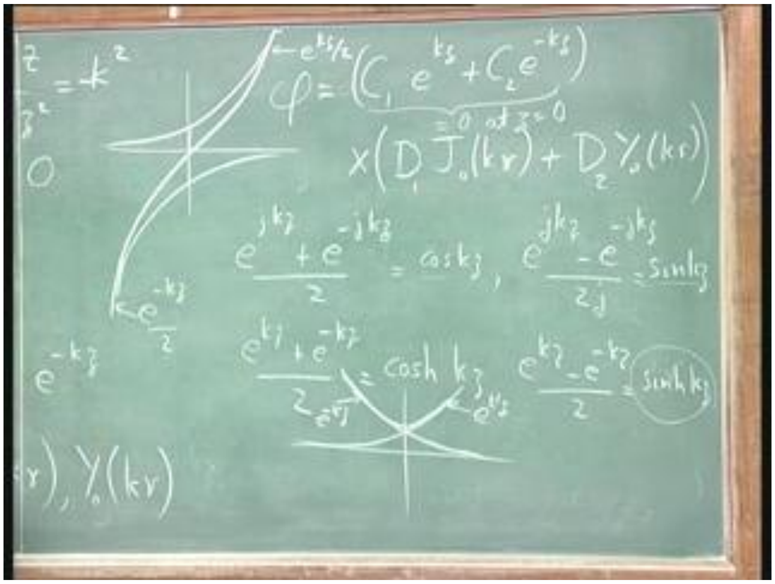

You can construct e to the jkz plus e to the minus jkz over 2. That is cos kz e to the jkz minus e to the minus jkz divided by 2 j that is sin kz. Similarly we can also construct e to the kz plus e to the minus kz over two and that is what is called the hypopolic cosine cos very important function. And e to the kz minus e to the minus kz over 2 is equal to hypopolic sin. So these are all related to exponentials. These two are complex exponentials they give you sins and cosines. These are real exponential they give you sin hypopolic sin hypopolic and cos hypopolic.

Now if you graph, this you know how cos and sin look like. But this is how cos and sin look like cos hypopolic. You can see the exponential looks like this and exponential of minus kz is a mirror reflection of this. So this is e to the kz and this is e to the minus kz. So if I add these two functions together I am get as function that looks like this. That is what the hypopolic cosine looks like at z equals 0 both of these are one. So one plus 1 divided by 2. So the cos hypopolic looks is equal to 1 at z equals 0, the cos hypopolic grows like an exponential both ways.

For large values of z, is like e to the kz over two, because one of the other becomes nearly 0. What does the sin hypopolic look like? Well we plot it again, e to the kz looks like this. Minus e to the minus kz is reflected and changed in sign. So it looks like this. Now if I add these two together, I am not drawn it properly the zero should this point there is equal amount above and below so they add up to 0. For very large values of z is basically this function or this function.

So if I plot the function is going to look like this. It is like e to the power of minus kz over 2 for large negative values of z it is like e to the kz over 2 for large positive values of z and it is 0 at z equals 0. Now it is this thing that is important. When I write it as hypopolic sins and cosines this function goes to 0 at only one point. And that one point is z equals 0 and I want a function of z which is a combinational of exponentials which goes to 0 and z equals 0.

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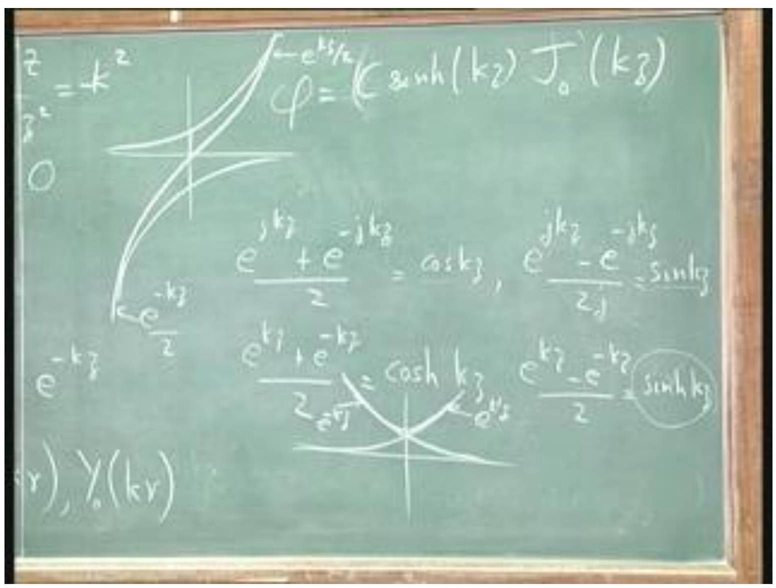

So I can replace this entire function by some constant C sin hypopolic kz. Now there are similar arguments that you can apply which will allow you to eliminate this function. So it terms out that the final form of your express will look like sin hypopolic kz times the Bessel function of kz. I am not going to Bessel functions because that is an advance topic. It is not really suitable for this course. But you do not really need to know what a Bessel function is this is a wiggly function it is a function like a sin or cos, alright? Now, what do we do with this? We still have and solved the problem. Why not because, we had three boundary conditions.

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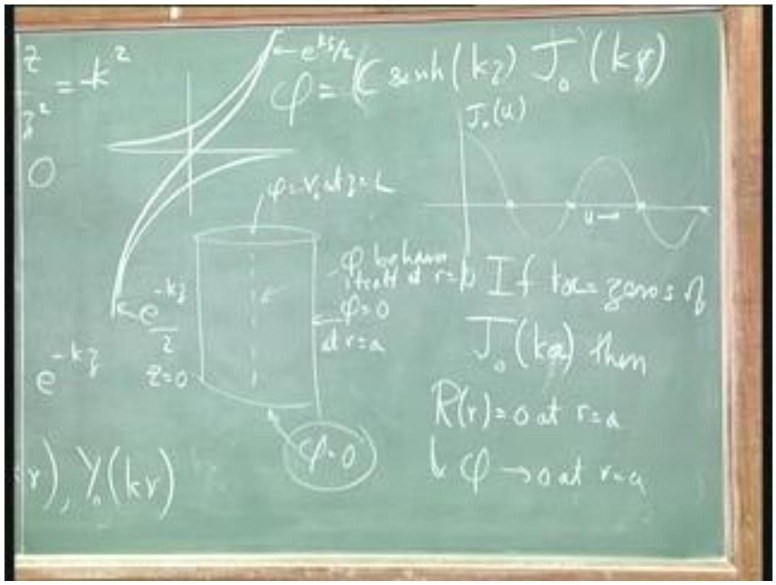

We had a boundary condition that pie was equal to zero at z equals zero that's taken care of sin hypopolic of kz we had a condition that pie equals zero at r equals a, and we had a condition pie equals V naught at z equals capital L. Now we actually had a fourth condition which was that pie behaves itself at r equals 0. That is why I was able to through out the other Bessel function. If you look at Bessel functions this this Bessel function is just a sinusoidal like function.

But the other one glows up at 0 and then interleaves with J 0. So this function you cannot accept because r equals 0, t goes to infinity. So that is still leaves as with three two other boundary conditions, pie must go to zero at r equals a. Pie must go to V naught at z equals L and that else seems to be enough I now have two free variables C and k. What can we do? Well if you look at what J not is. J not is an oscillatory function this is J not of ant argument u verses u.

So if kz were equal to these values if k sorry we have not kr if kr were equal to zeros of J zero of kz. Then R of r equals zero at r equals a and pie goes to 0 at r equals a. So this should be r at r equals a so it is put it as ka. So, let me repeat I have a general form here. This general form satisfies the boundary condition at the bottom. But it does not satisfy anything else. But I am free to choose my k; so far I am not selected yet. If I look at the plot of any oscillatory function, oscillatory function goes to 0 and now and then sin goes to 0 at n pie cos goes to 0 at pie over 2 3 pie over 2 etcetera.

Similarly J zero goes to 0 at a certain values. If I set ka equal to these values okay, ka is equal to zeros of J 0 of u then J 0 of ka will be 0 which means this piece goes to 0 and therefore potential goes to 0. Now there is only one problem with this idea it is a wonderful idea I have taken care of one more boundary condition. But there are many zeros there is not 1 0 there are infinite number of zeros. So instead of reducing my problem to solving one constant, I suddenly find myself within infinite number of constants to solve for. If I call each of the zeros, if I call ka equal to lambda zero n meaning the nth 0 of J 0, then my pie has now become something more peculiar.

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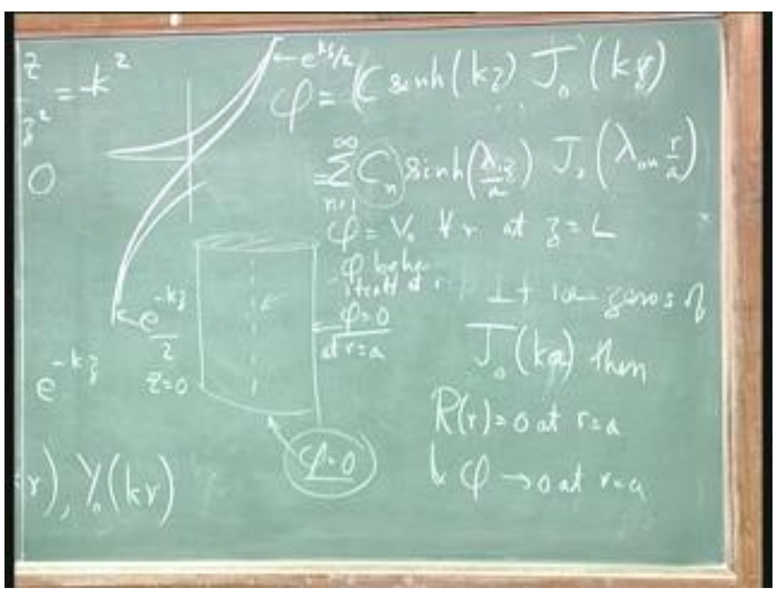

It becomes C sub n sin hypopolic of kz J zero of lambda 0 n r over a and this case is the same case after all must therefore also be lambda 0 n over a, pie does not know anything about n, but suddenly I have introduced a new variable called n and I know way of handling this, unless I summed over all ns. Now the left hand side does not know about n. Right hand side does not know about n needed it is a dummy index.

This now gives me all possible solutions of Laplace's equation which satisfies pie equals zero at the bottom which satisfies pie equals zero at the walls which has pie behaving itself at the origin and pie is equal to something we do not know what at the top. But actually we also require pie equals V naught for all r, at Z equals L and thus the final part of the puzzle. We need to pin down what values of Cn to use such that pie is equal to V naught for every value of r.

I am not going to solve this problem infact the reason for presenting the cylindrical problem before I went back to Cartesian is that the Cartesian problem is too easy because it is so easy, you do not understand what is happening. What is the underlying idea in it? Now I will repeat this whole process for the Cartesian problem and I want you to watch carefully how we are doing this same step we are making the same arguments except that, since they are sins and cosines we know all about that.

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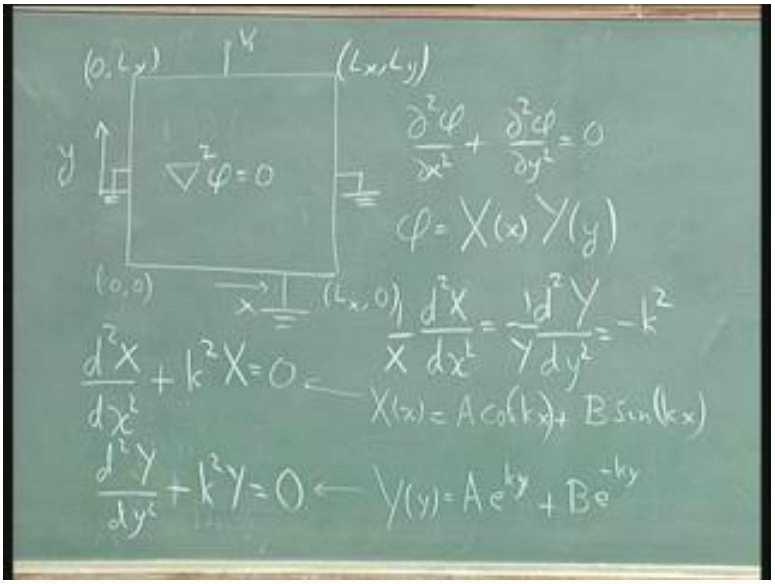

So now, I am going to take up a problem where I have conducting walls. This is the origin, this is x, this is y, this is x equals Lx, y equals zero. This is x equals zero, y equals Ly, this is Lx Ly. Within this region I have dell squared pie equal 0. And I have three walls are grounded and the fourth wall is held at some voltage V not. It is exactly the same problem of the cylindrical problem only change is instead of r z I am talking about x and y. The equations are much simpler I now have del squared pie del x squared plus del squared pie del y squared is equal to zero.

If you go back to your notes you see earlier in the lecture I worked out what you have to do and we found out that we assume pie is equal to some capital X of x capital Y of y. And then we worked through all the details. We find that d squared capital X dx squared is equal to minus d squared capital Y dy squared is equal to minus k squared. I could have put plus k squared but I know before hand I want minus k squared. So I put it there.

So I get two equations the two equations are d squared sorry one over capital X and this is one over capital Y d squared capital X dx squared plus k squared capital X is equal to 0 and d squared capital Y dy squared minus k squared capital Y equals 0. We have reached up to this point even earlier. The solution of this equation is the pendulum equation. So, we know the answers. Answers are capital X of x is equal to A cos kx plus B sin kx. This is the unstable pendulum equation. So I again know the solution it is Y of y equals A e to the power of kx plus B e to the power of minus sorry ky. So I can now write down the answer for pie.

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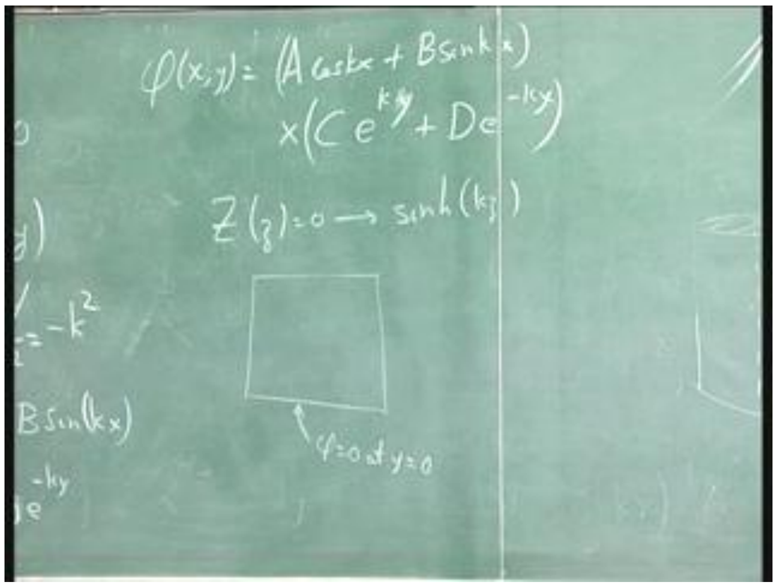

The potential which is a function of x and y is now equal to A cos kx plus B sin kx multiplied by should be some C it should be D C e to the power of kx plus D e ot the power of minus ky. Compare this with what we have already done for the r z case we had A J 0 of kx plus B Y 0 of k kr sorry times C e to the kz plus D e to the minus kz. So, exactly the same structure but now I have to fit my boundary conditions. The same argument that allowed me to say capital Z of z equals zero let to sin hypopolic of kz will now allow me to simplify the y dependents. Because I have a boundary condition where pie equals 0 at y equals 0.

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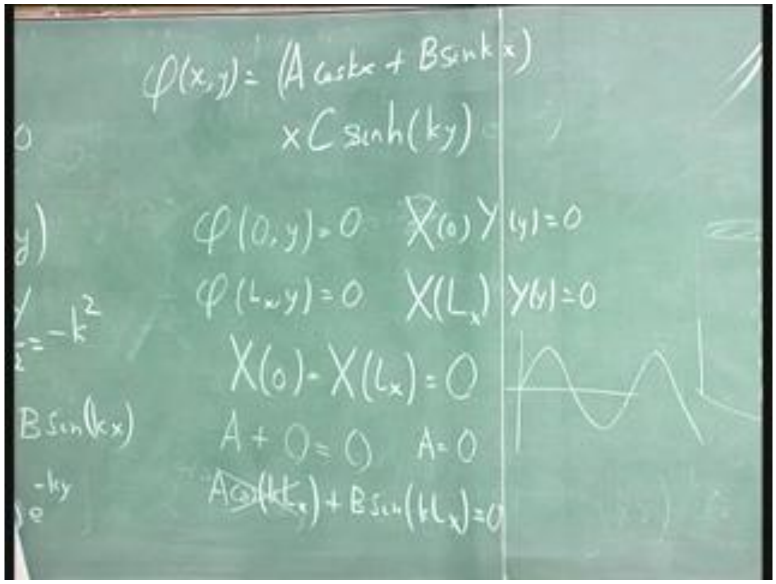

So I am able to answer. Substitute this piece and replace it with some C sin hypopolic of ky. But I still do not know what to do about the first part. Now what are my conditions in x? I have pie of x equals 0 any y is equal to 0 pie of x equals Lx any y equals 0 and if I rewrite this its saying capital X of 0 capital Y of y equal 0. Capital X of Lx capital Y of y equal 0. As before I can choose to make capital Y identically 0, but that is observed I would not have any solution after that.

So the answer must be this capital X of 0 equals, capital X of Lx is equal to 0. Now if you look at this form and substitute zero and Lx. What do we get? We get A plus 0 sin of 0 is 0 is equal to 0 and you have A cos k times Lx plus B sin k times Lx is equal to 0. The first equation tells us A is 0 there is no cosine, since A is zero, this piece goes away. The second term says that B times, sin of something is equal to 0. Now I can said B to 0, but that is very bad because then that says x is identically zero so we want sin kLx to be zero.

How do we make sin kLx 0? I know that if I plot sin, sin goes to zero at periodic values. So if I make k times Lx equals lambda 0 n I am calling it lambda zero n just to make it familiar. Then I have solved my problem. In this particular case, I know what this lambda zero n are. They are just nothing but n pi. Complete the problem next time, but you can see where it is going and we will get a full solution of this problem.