Electromagnetic Fields
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Lecture – 15
Method of Image

Good morning. Last time I had talked about how to solve for capacitor problems. And I introduced the theorem; a theorem that we call the uniqueness theorem.

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What the theorem says that if I have any volume call it V, if I have some known charges inside that volume and if on the surface of V, I know either the potential or the normal electric field. It is del pie del n is minus En at every point on S. So, if I know what charges are inside this volume and on the surface of this volume, I know either, at every point I know either pie of del pie del n. Then the solution inside is unique. Meaning that cannot be two solutions that satisfies these this information.

Now let us say the important of this comes from looking at ordinary differentially equations. If I have solving d squared x dt squared plus k squared x equals 0, that is like solving Poisson's equation. Well, I get sins and cosines. It does not tell me the function x.

The function x could be any combination of sin and cosine. Supposing, I further knew that x is equal to 0 at t equals 0, that still does not determine what x is, because the general solution per x would be the form A cos kt plus B sin kt.

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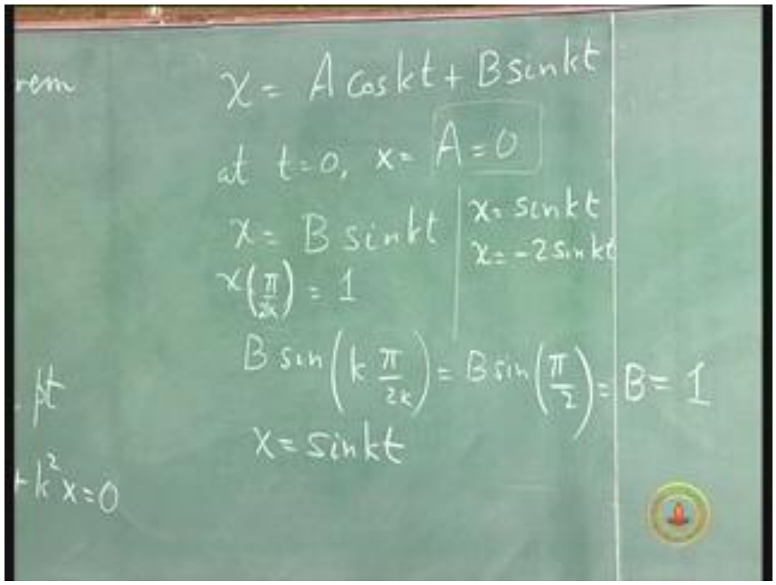

So at t equals 0 x equals A and I am told that at t equals 0, x is equal to 0. So I know that A is 0. So that tells me that x equals B sin kt. If I now know one more piece of information, let us say that x of pie over k is equal to pi over k is back, pi over 2 k is equal to 1. Then I can substitute in here, I will get B sin k times pi over 2 k k cancels out. So there is B sin of pi over 2. But sin of pi over 2 is 1.

So that is nothing but B and that is equal to unit. So given both these of this information I am now able to say x is equal to sin kt which finally fixes x. So it is only after I know both the differential equation and two pieces of information that I am able to solve and get the actual form of x. If I did not have this, then any function of this form namely x equals sin kt, x equals minus 2 sin kt all of them would have been solutions. So when you solve, when you try to propose a problem of solving Poisson’s equation inside the volume.

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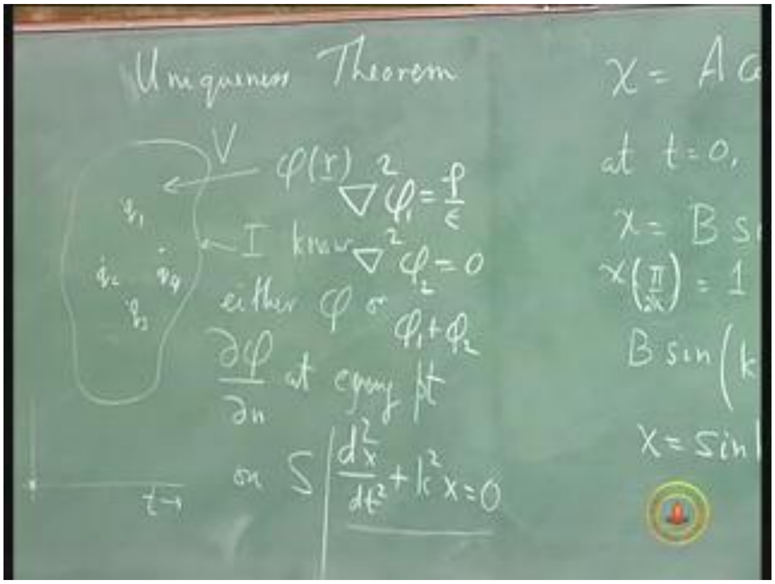

The question becomes, what additional information do I have to say? This besides saying dell squared pie equals minus row over epsilon. This is the equation itself and this equation is exactly like specifying the equation. And just as this equation does not pin down x itself because you need additional information similarly specifying this equation does not pin down pie. We pins down lot of the details of pie but some information is still remaining. Basically if I could solve dell squared pie equals 0 inside, this volume V then if I add this 2. So if I call this pie 1, and call this pie 2, any solution pie 2 that satisfies Laplace’s equation can be added to this to this solution. So pie 1 plus pie 2 is always also a solution.

So that is why this equation by itself does not give me a unique answer. So what this theorem is telling as is, that you could get a unique answer that is to specify enough information all I need to do is on the boundary at every point of the boundary I specify either pie or specify the normal component fact. The proof of this is in your textbook and you can read it. I will leave it as an exercise because it is not very illuminating. What is more useful is to use it. I am going to use it in two ways in today’s class. First I am going to talk about the method of images and secondly I am going to actually use it to solve Poisson’s equation.

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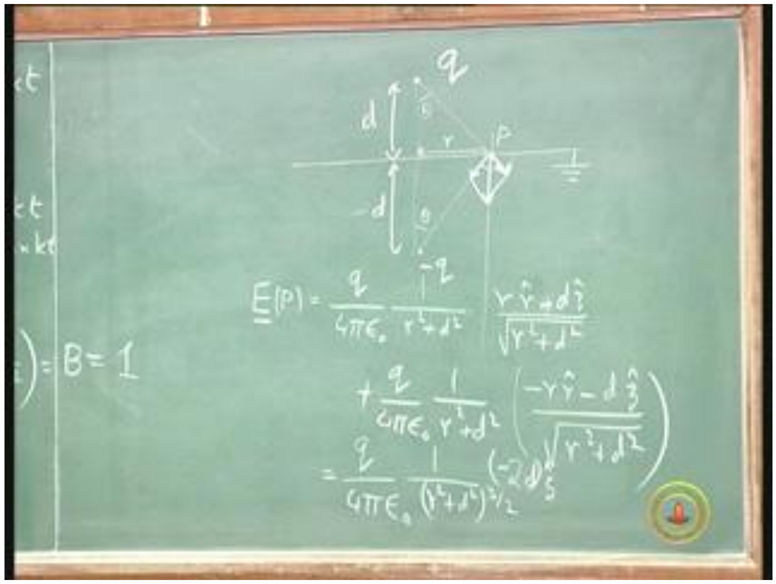

Let us go first for the method of image. You have a ground plane it should be a straight line imagines it is. This plane is kept a 0 voltage and I want to find the solution of pie for this problem. This problem is difficult because I do not know what charge is present on this plate what I know is the potential. Therefore I cannot applied coulomb's law, however as I did last time, supposing you put a charge minus q and you put the charge Minus q as far below this plate as the charge was above.

Then if you take any point on the plate create the triangle because these two distances are equal clearly these two angles are equal, this charge attracts two at itself so the electric field is pointed this. This charge ripples so the electric field is pointed this edge. Both this angles are equal that is both are theta. So if you add these two vectors vectorially you will get net electric field that points straight down. What is the advantage of that? So we got that the electric field at any point p is equal to q over 4 pi epsilon naught times 1 over I am going to call this distance r r squared plus d squared times the direction is I can constructive out of these two components.

So it is equal to r hat plus sorry minus d z hat divided by square root of r squared plus d squared. What I have done is I said that construct this direction I have to go d distance in a minus z direction and r distance in the r direction. So minus d z hat plus r r hat that gives me a vector with length. So I am going to divide by the length of the highpoint in itself which is square root of r squared plus d squared. So the magnitude of this vector is one and the direction is the correct direction.

This is the electric field due to this charge and the electric field due to this charge is minus q over, 4 pi epsilon naught 1 over r squared plus d squared. Now, the vector direction is minus r r hat minus d z hat divided by square root of r squared plus d squared. So I should since I have taken the sign of the arrow inter con this is actually plus. So now you can see what is happened d is minus in both cases so they are going to add up, r is plus r here and minus r here so they are going to cancel.

So the net result is I get an answer which says the electric field is q over 4 pie epsilon naught 1 over by combine this r squared plus d squared to the power of three halves. That is r squared plus d squared to the power of 1 multiplied by r squared plus d squared to the power of half. So r squared plus d squared to the power of three halves and then multiplied by 2 r sorry 2d z that is adding of this term and this term. So as this figure shows the electric field is minus z hat direction. Now what is that mean? If the electric field is minus z hat direction you can conclude something immediately.

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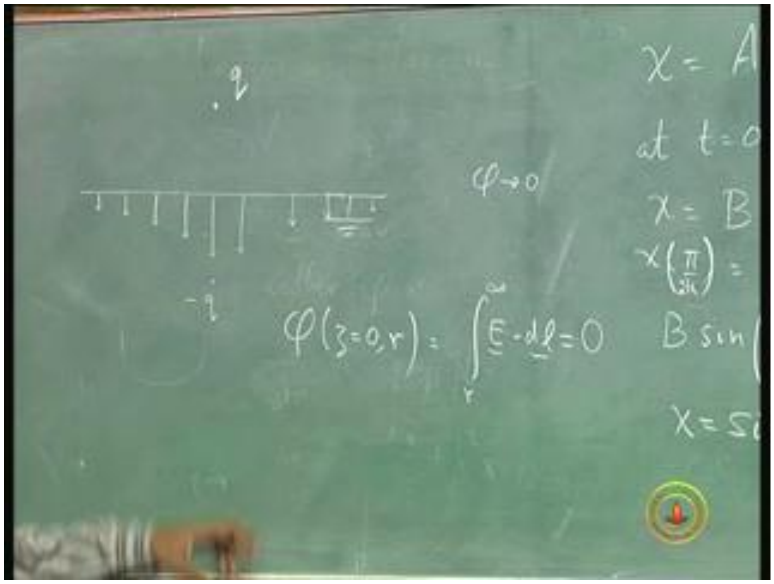

We have a point charge q. We had the ground plane and we had r image of minus q. Now at every point on this ground plane the electric field is in the z direction this is the electric field. Now far away the potential goes to 0. That is because if we are point charges or derivation from coulomb's law tells us potential far away is 0. Now the potential anywhere here pie at any point z equals 0, r is equal to minus integral r to infinity E dot dl. That is, by definition, that is how we define potential. Potential is negative of E dot dl. But E is in this direction and dl is in this direction. Therefore E dot dl is identically 0. See if I move along this ground plate the electric field is always perpendicular to the way I am move. So this is identically 0, okay.

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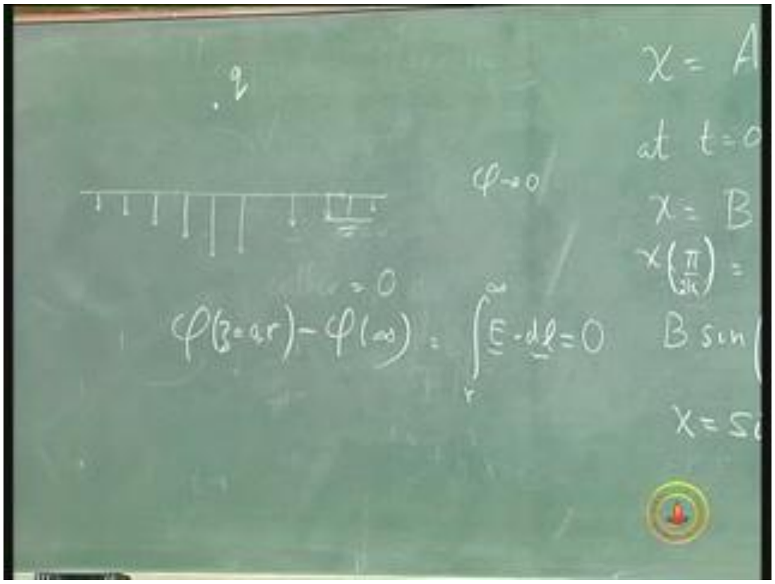

So when I write this as z pie equals z equals 0 or I really mean pie of z equals 0 or minus pie at infinity, but pie at infinity is 0. So I have got pie of z equals 0 or it is always 0. So what does they achieved? I started with this problem. I started with the problem that I wanted a charge and a plate which is kept at ground voltage.

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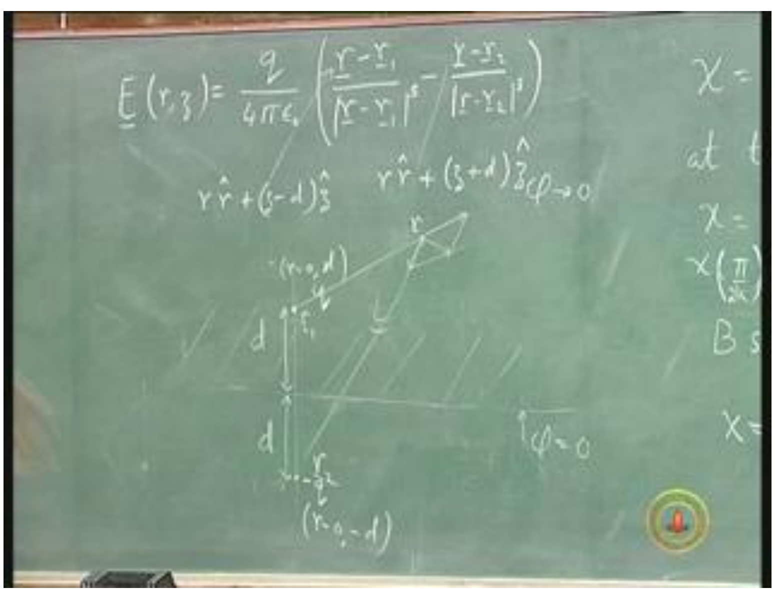

I want to know pie here. I took this problem and I replaced with new problem. I still have a charge q this charge q is a distance d above the origin I put a minus q charge here which is a distance d below the origin. And I remove this ground plate. Ground plate is no longer there. So I have only two charges left and if I have only two charges whose location I know, whose values I know, coulomb's law gives me the answers. So the electric field anywhere for any r any z is going to be equal to let me write it rather place. The electric field anywhere at r z is going to be equal to q over 4 pi epsilon naught this charge is sitting at r equals 0 d.

This charge is sitting at r equals 0 minus d. So the electric field is going to point away due to this charge and going to point towards this charge and its going to be the vector some which is going to be the actual electric field, it is a dipole. So it is going to be the, I will just write it symbolically because otherwise it becomes messy r minus r one divided by r minus r 1 cubed minus r minus r 2 over r minus r 2 cubed.

Where, this is r 1, this is r 2, this is r, I will leave it as exercise for you to work out the actual values. It is actually quite easy because r minus r 1 is nothing but r r hat plus z minus d z hat and r minus r 2 is nothing but r r hat plus z plus d z hat. So just substituting these you can work out what this expression is. Now what is important to understand is give taken one problem where we had a charge and a conducting plate.

You solved and different problem where we have two charges. Now both of them give you solutions. So these problem gives me a solutions, so does the other problem. The uniqueness theorem tells us that since I have the correct charge in the region above that z equals 0 point, because the solution with two charges gave me the correct potential here. This answer is exactly the same answer that I would have got if I solve this problem of the plate and the charge. So let me repeat, I will put down the step we followed.

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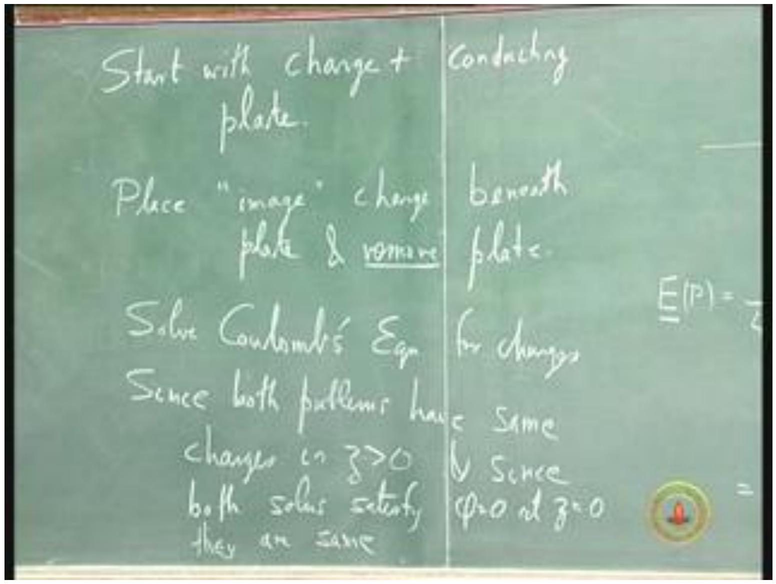

Start with charge plus conducting plate alright, place image charge beneath plate and remove plate. Solve coulomb's equation law for charges. So we do just this step. We start with the problem charge plus conducting plate we replaced the plate by an image charge. Having replaces the plate by an image charge we could use column's law and we get the electric field. The important point about this is a solving a different problem you are not solving the same problem at all. However since both problems have same charges in z greater than 0 and since both problems and both solutions satisfy pie equals 0 at z equals 0.

Since both of these two things there is they have same charges and z is greater than 0 namely the original charge and they both go to 0 at z equals 0 they are the same solutions. It is a remarkable theorem actually. It is a remarkable simplification you get because of this theorem. The method of images when we used in different geometries, it is very specialized tool where you can use it is simplify the lot. But there are only a few geometries where actually you can use it. So in state we go to more general kind of technique which includes the method of images, but that technique also uses the same trick and I am going to illustrate that technique by solving a problem we have already solved.

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Take a capacitor plate we have the bottom plate grounded top plate held at V naught and we have epsilon 1 and epsilon 2, two dielectrics. This is X equals 0, X equals x naught, X equals d. We solved this problem earlier by saying let us take the displacement vector. Displacement vector does not say any charge except top and bottom plates, then apply Gauss's law improve the displacement vector is constant. Thus the displacement vector is constant. Electric field Ex is equal to some constants c divided by epsilon 1 in region x naught less than x less than d and c, over epsilon 2 0 less than x less than x naught and from this we can calculate the solution and we have done that.

Now what I am going to do I am going to do slightly differently. I am going to go back to the original equation and try and solve it two parts. So in order to do that for reference let me write down what I get as the solution using this net, the potential pie is equal to Ex in this region Ex times x and its equal to Ex times x naught plus c, over epsilon 1 x minus x naught with this Ex c, over epsilon 2. So let us see we can recover this solution by another method. What I will do is the following.

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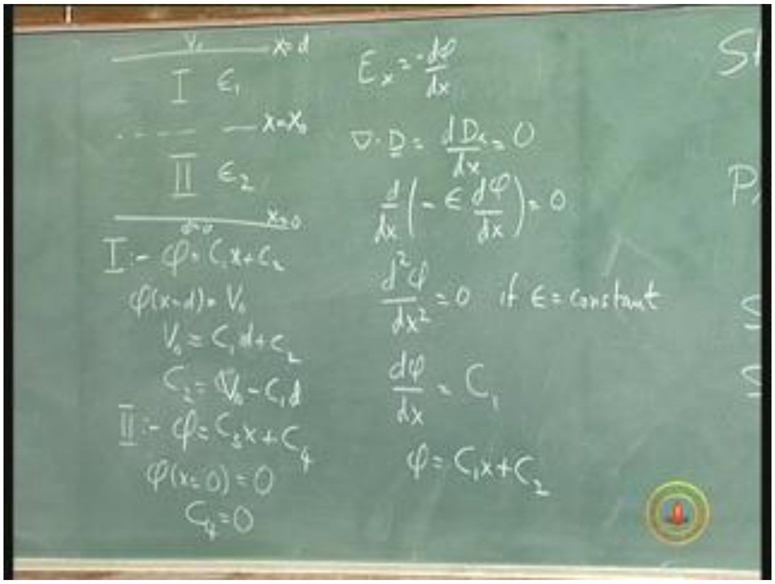

I have my parallel plate capacitor. So X equals d, X equals x naught, X equals 0 that I am going to label this region as region 1 and this region as region two. This region is dielectric epsilon naught. This region is dielectric epsilon 2 and I am going to say I am going to solve Laplace's equations. Because I know that the electric field Ex is equal to minus d pie dx. I know that divergence of D which is equal to d Dx dx is equal to 0; there is no charge no real charge here.

So what that means is d Dx of minus epsilon d pie dx is equal to 0. So if epsilon is a constant I can combine these I can pull that epsilon out. So I get d squared pie dx squared equals 0, if epsilon equals constant, but epsilon is constant in region 1 and it is constant in region 2 where it not constant is at the boundary between region 1 and region two. So let us solve this problem d squared pie dx squared equals 0, the solution I integrate both sides of respect to x I get d pie dx is equal to some constant. Let me call it constant C 1 I integrate it again I get pie equals C 1 x plus C 2. So pie is a straight line.

So now let us apply this two region one. In region 1 pie is equal to C 1 x plus C 2 but I do know that pie at the top pie of x equals d is equal to V not. That is part of the problem specification V naught here and its pie equals 0 at the bottom. So if I substitute d in here I should get V not. V naught is equal to C 1 x C 1 d plus C 2. This allows me to connect up C 1 and C 2. For example I can say C 2 is equal to C 1 is equal to V naught minus C 1 d. In region 2 pie is equal to say C 3 x plus C 4. Now again I have a piece of information which says pie of x equals 0 is equal to 0. Substituting x equals 0, C 3 into 0 goes away. So 0 equals C 4. So I have C 4 equals 0. So now what do I have? Let me summarize.

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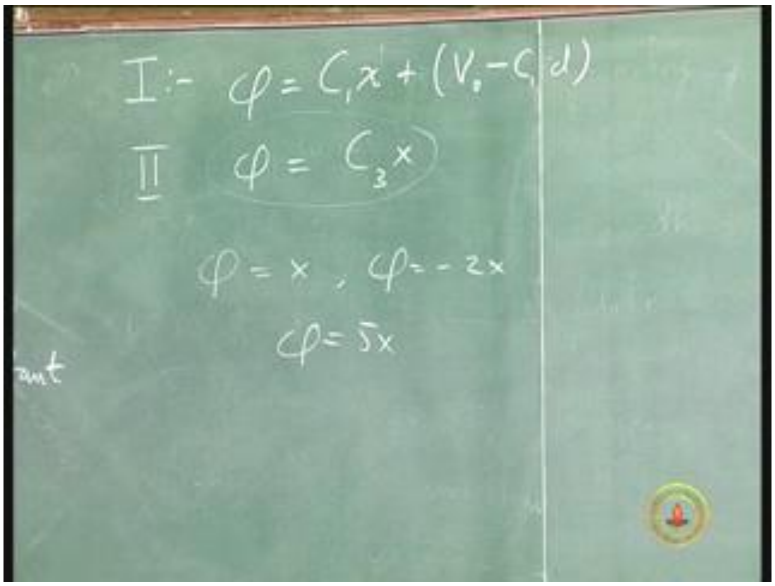

I have in region 1, I have pie is equal to C 1 x plus V naught minus C 1 d, it is a C 2 and in region 2 I have pie is equal to C C 3 into x. Now what I am going to do is I am going to connect a these two solutions and this is where once again the uniqueness theorem comes. Now you can see that this region 2 I have a solution for potential for the solution has a free parameter. So the solution is could be see pie equals x or pie equals minus 2 x or pie equals 5 x, all of them are value. That is because I do not have enough information to decide which coefficient is multiplying x. It is just an unknown coefficient.

But actually the problem knows problem has been fully specified. It is just that in the part that I am looking at I do not have enough information. What I will do is, I will say, supposing I do know, what the potential at x equals x naught is. If I knew that piece of information then I can pin down C 3. So how can I get that value? Well, I can get that value from looking at region 1.

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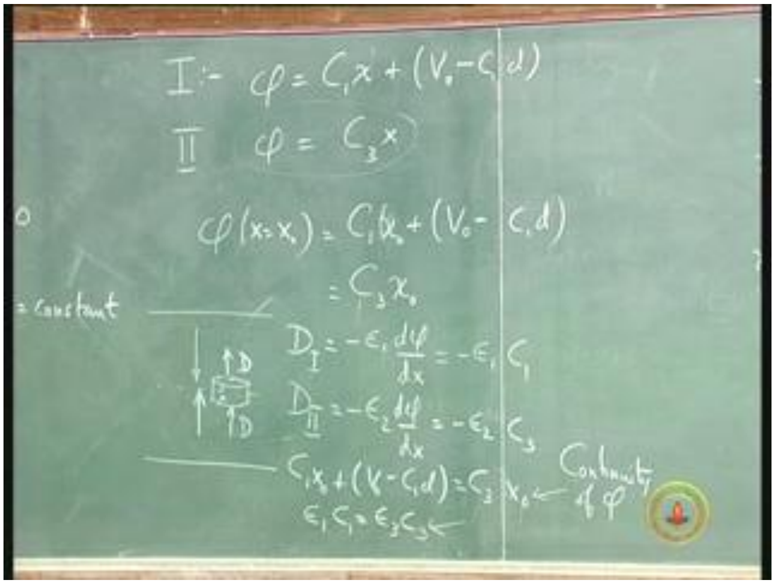

Pie at X equals x naught; according to region 1 C 1 times x naught plus V naught minus C 1 d. And according to region x region 2 is equal to C 3 times x naught. So I have a connection between these two and other words I know these two of both talking about the same potential. In this diagram this piece is talking about the potential as I approach from above towards X equals x naught. This is talking about the same potential as I approach from below. With the potential should be the same because the potential is nothing but the work I did to get to there, whether I got in this way or this way we already shown it does not change. Therefore these two must be equal. So it already gives us one equation.

Now in other equation we have the other equation is that if you make an imaginary cylinder which is very flats so its side sides are narrow, top and bottom are much broader then the amount of flux entering must be equal to the amount of flux leaving because there is no charge at an interface. I can work out the amount of flux entering and leaving. Actually because of the way I have put it should be pointing upwards. So D in region 1 is equal to minus epsilon 1 d pie dx. If I take the derivative of pie, I get that is equal to minus epsilon 1 C 1.

D in region 2 is equal to minus epsilon 2 d pie dx which is equal to minus epsilon 2 C 3. So I therefore I have two equations. One equation says C 1 x naught plus V naught minus C 1 d is equal to C 3 x naught. The second equation says epsilon 1 C 1 equals epsilon three C 3. This equation is what is called continuity of potential and this is nothing but Gauss's law. I have two unknowns C 1 and C 3 and have two equations I can solve them. By looking at this first equation, sorry the second equation, let me write it out epsilon 1 C 1 equals epsilon 2 C 3 should be an epsilon 2. So that tells me how to what C 3 is in terms of C 1.

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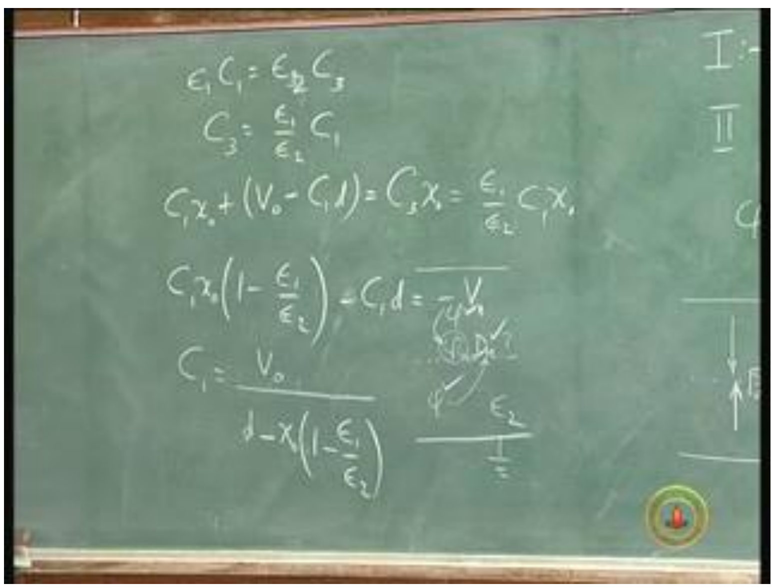

So I can write C 3 is equal to epsilon 1 over epsilon 2 C 1. Now let me substitute into the second equation into other equation. C 1 x naught plus V naught minus C 1 d is equal to C 3 x naught. But I know C 3 is its equal to epsilon 1 over epsilon 2 C 1 x naught. So now I have only one unknown and I have one equation. Let me combine the 2 C 1 x naught and 1 minus epsilon 1 over epsilon 2 is equal to minus C 1 d is equal to minus V not. So if I take on C 1 x naught minus epsilon 1 over epsilon 2 C 1 x naught minus C 1 d is equal to minus V not.

And this gives us C 1, there may be algebra errors along the way. But I do not think so. Look so far now where did I use the uniqueness theorem, it seems like I have done every thing perfectly reasonable. I had enough equation enough unknowns I solved them. For the uniqueness theorem comes you mat ability to look at a problem but I said I have a ground plane I have epsilon 2 and I do not know what I have here and I am able to say okay. I know let at this point I have a potential or I will say I have I know what the normal component of the flux is. By specifying one or the other of these conditions, I have uniqueness here.

Once I have uniqueness here there is only one solution possible here which means that if I specify pie, pie in here is completely known which means intern Dn is completely known. So at this mid-point if I assume pie I have solved that though lower region I have got normal derivative. But in the upper region if I have got normal derivative I know pie I can obtain pie again. This allows me to build up an equation that allows me to pin down what pie is at the mid-point and therefore what the solution is everywhere. This approach which looks very compression is an extremely powerful. In fact this is how most problems in Poisson's involving Poisson's equation I have solved. So, you will have to get used this kind of problem and I am sure you will encounter many more complicated versions of this problem.

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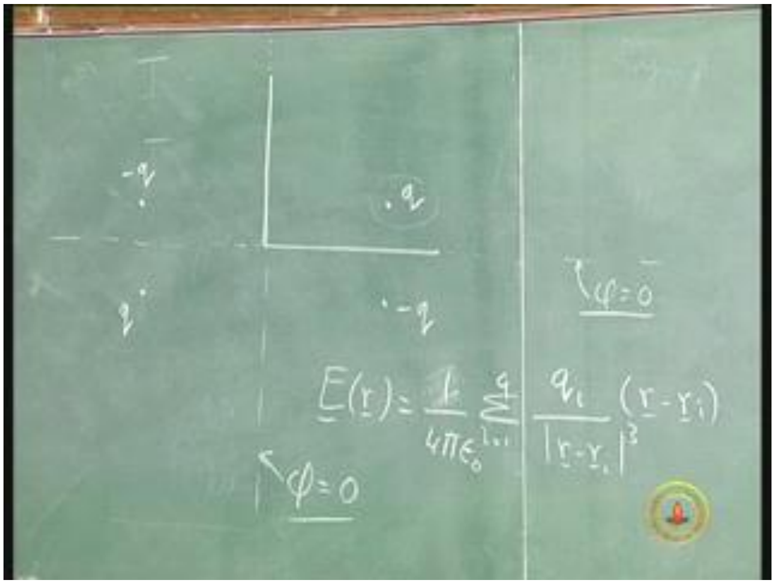

Now the method of images can be used in more somewhat more complex situations. Supposing I have two ground planes which arrived at 90 degrees and I have a charge q I have a charge q here. I can put a charge minus q here and these two charges together will make this entire plane ground. Now for this charge if I put a charge minus q here then this entire plane will become pie equals 0. But if I put this charge here that upsets this boundary condition, if I put this charge here that upsets this boundary condition.

So what would I like to do is I would like to put an image charge for this charge also and luckily it turns out this is not only the image charge of this but also the image charge of this charge. So now if you have all this four charges in place this pair of charges as well as this pair of charges enforce pie equals 0 on the horizontal wall. This pair of charges as well as this pair enforce pie equals zero on the vertical wall, so this set of four charges achieved ground and on both of these plates and I have not introduced any charge in the main chamber.

And therefore by uniqueness I have got the correct solution. So the electric field is again equal to q over 4 pie epsilon naught sum on i equals 1 to 4. Well I keep the q inside qi over r minus ri cubed times r, minus ri. Is it exactly the same as the ground plane except?

Now I have additional charges. Now note the important extra charge of added I have added the extra charge here which is not required by this charge. But it is required by these two charges. But it does not matter as long as the charge have add is not going to fall inside this region it is a perfectly acceptable problem and this as given me an answer.

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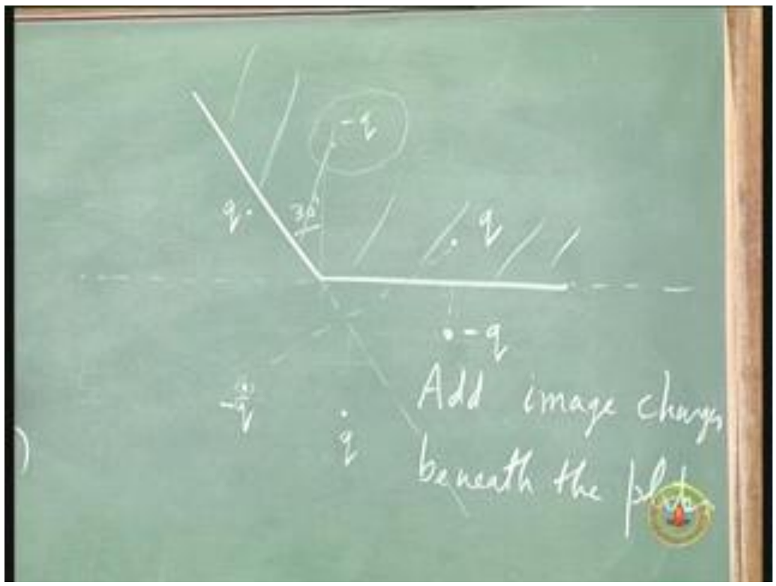

Now we can try to do the same problem for geometry like this. Let us say this is 30 degrees, say it is 120 degrees. Now I put a charge here q I get a charge minus q here. That makes this plate grounded. Now I put a charge here which is minus q to make this plate grounded. I will know want to put a charge reflected against this line so that I will this charge will also be giving me ground and similarly there will be a charge here. Unfortunately when I do this I end of with one more charge here and this is not except to this. After I have done all this reflections I am getting a charge which is inside the region of interests.

The moment that happens the moment I have a charge that is sitting inside my region of interest this whole method fails, the method of images cannot be used. Method of images is based on the idea that we can add image charges beneath the place. And if the image of an image ends up being above the plate, then this method automatically fails. Luckily for us there are other methods. That is all, this particular problem and so we are not really limited by the fact that we could not solve that. But you must keep this in mind, the method of images gives you some very simple answers. But there are lots of time when does not give you a simple answer. And in that case you have to go to other methods.

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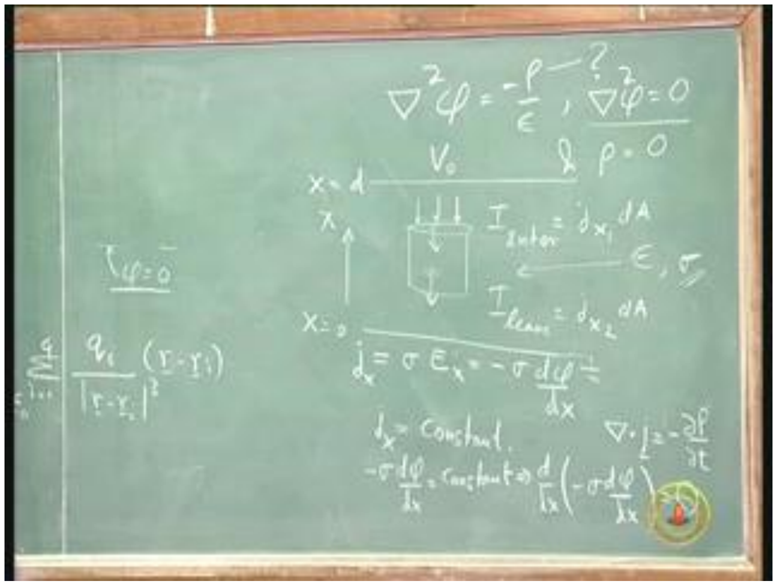

Now let us go back to Laplace's equation and Poisson's equation. I want to go back to one simple problem which needs to be properly understood again this one dimensional capacitor. Again I have a dielectric constant epsilon but now I am going to add conductivity any real capacitor which filled with real material. The real material has a little bit of leakage. If I apply volt voltage one voltage across that dielectric the current through it is not zero, it may it may take year to discharge. But still there is a timing bit of current that is linking through this material. So there is a small amount of sigma.

Now the question is we found a solution using Poisson's equation. But in the presence of sigma does the solution work or does it fail. So let us work out the answers. First of all we have dell squared pie equals minus row over epsilon this is true, but we do not know how. If there is leakage is possible that charge collects here and there. On the other hand we know that if we have conductivity we have that current is equal to sigma jx is equal to sigma Ex, x is the direction x equals 0 is here, x equals d there and since Ex is given by pie is equal to minus sigma d pie dx.

Now jx itself is flowing well we have put a voltage V naught and grounded, current will be flowing down wards. Now this jx itself satisfies another law which is if I draw a cylinder the amount of current entering I entering is equal to jx times dA. The area the current leaving is equal to so I call that jx 1, jx 2, dA. If the current entering and the current leaving are not equal it means charges slowly accumulated, but that cannot happened in a DC problem and therefore I must have jx is equal to constant. More generally the equation I should have is divergence of j is equal to minus del row del t I believed I derived this earlier.

So this is saying the same thing divergence of j which is d jx dx is equal to 0 or jx is constant. So what does that say? It says minus sigma d pie dx equals constant, right? Take the derivative again that tells me ddx of minus sigma d pie dx equals 0. You can assume sigma is constant so we get a second equation. This is one equation, the second equation we get dell squared pie equals 0. So there is one equation coming out of Poisson’s equation, one equation coming out of current. Now as it turns out these are not independent equations. They can both be true if charge density was 0. And so in this particular problem what they are concluding is this equation is true and row is equal to 0.

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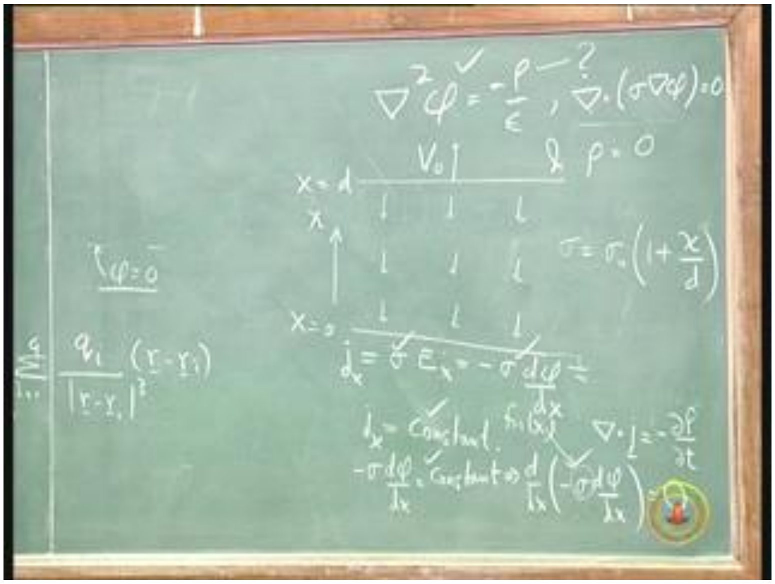

So the way to look at this problem is I apply a voltage V naught 1 volt, may be a current starts up. So, constant current is flowing through. Because constant current is flowing through no charge needs to build up because no charge needs to build up the region of dielectric is charged free and we end up solving Laplace's equation anyway. Now all this would fail if may be the sigma of the system is not constant may be its equal to sigma not times 1 plus x over d. That is the conductivity is weak here. Conductivity is strong here. If you have such a system, then what would happen is this would still be true Poisson's equation is perfectly valid. This would be true j equals sigma E is true.

And E is derived from potential so that is correct, jx is being constant. Well that follows because of these the conditions which means minus sigma d pie dx is constant, that is also correct which means time rate of I mean space rate of change of this quantity is 0. So up to this point the equation is correct. The problem now is sigma is not constant it is a function of x. So when you try to write this equation now that equation would not be true in state that equation becomes divergence of sigma grad pie equals 0. So instead of dell squared pie is equal to minus row over epsilon. Now you got divergence of sigma grad pie equals 0. This problem can be worked out its not very difficult. Let me write it out.

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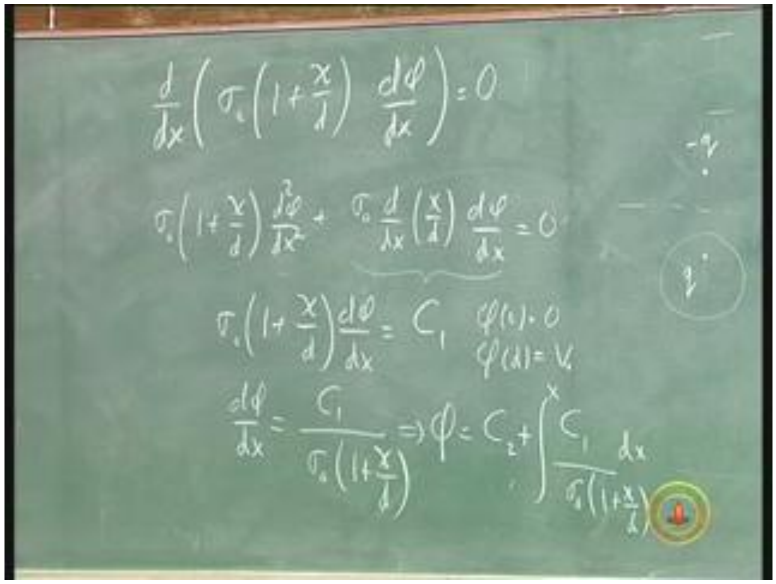

It is ddx of sigma not times one plus x over d times d pie dx is equal to 0. So this ddx either acts on d pie dx or it acts on x over d. So you will get two terms sigma not times one plus x over d d pie dx plus sigma not ddx of x over d d pie dx equals 0. So you can see that there is a new term that as coming which complicates the equation, luckily we can directly solve this problem. So this is equal to 0, it must be true the bracket quantity is constant.

We can write this as sigma not one plus x over d d pie dx equals constant C 1, which means d pie dx is equal to C 1 over sigma not 1 plus x over d which again means, we can integrate this you get pie is equal to C 2 plus integral to x C 1 over sigma not 1 plus x over d dx. Once again the answer is interms of two unknown constants and way of two pieces of information we know that pie at 0 is equal to 0, pie at d equals V naught. Using these two pieces of information we can determine C 1 and C 2.
Now what is this tell us?

I mean we can do algebra. What it tells us is that, if you have certain kinds of capacitors where you have leakage charge and if the properties of the material are not uniform, you can actually have a solution that does not satisfy dell squared pie equals 0 in state it satisfies some thing else and that something else means there is a space charge, a charge of electrons building up inside the dielectric. So the dielectric actually becomes charged and over time that charge settles in every part of the dielectric to make sure that this equation is true.

This is the equation that controls a leaky capacitor because we cannot have current building up. That is not an acceptable, but while it is building up, it is possible to have del row del t being non 0 because charge is building up. Once it is finished building up it must be true the j is constant and j is constant it must be this equation that is true and therefore there is charge present.