Electromagnetic Fields

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Lecture – 14
Capacitors

Good morning. Up to now, we have been doing a lot of theory. So what I am going to is apply that theory and make it a little useful. We have done of your problems. But this lecture will be almost completely looking at calculating capacitance in different situations. Let me start out with the simplest capacitor that we have already knows about.

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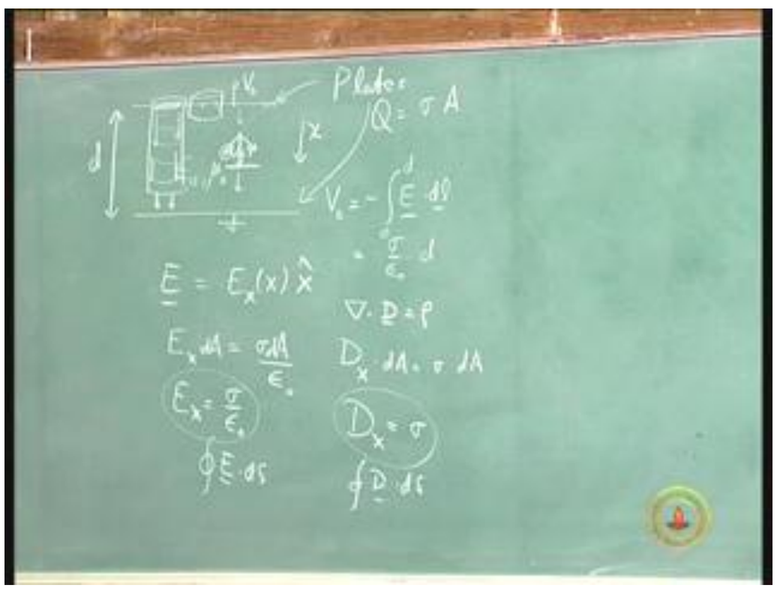

You have two plates infinitely large a separated by distance d. I ground the bottom plate and I am applied a voltage V naught to the top plate. In between the plates I am going to have air. So that means my dielectric constant is epsilon naught permeability is mew not. Now we already done the solution to this, but let us do it again. Idea is that the electric field is going to point away from the plate. Because, there is no reason for it point in any other direction there are no side walls that are trying to cause forces to come side-ways.

If I take any point here, the every bit of charge that is here that is trying to make me move this way. There are corresponding bit of charge here that makes me move this way, so the net electric field is always downwards. So if this direction is x the electric field is Ex. Once I know that I can apply Gauss's law and Gauss's law tells me if I put the top surface inside the metal bottom surface is inside the air, it tells me that Ex times area is equal to charge enclosed charge density times again the area I call it dA to indicate there is small area divided by epsilon naught.

And this is how we get the electric field is equal to sigma over epsilon naught. Now this same idea can now be brought done using displacement vector. We have divergence D is equal to row. So the same Gaussian cylinder I can draw and I now have Dx times dA is equal to sigma times dA. There is no 1 over epsilon naught, alright? Otherwise it is a trivial extension of what we have already written. So I will get Dx is equal to sigma. Now what is important about either of these is that, I can draw a cylinder that extends into the region between the two plates. It does not have to be a very thin cylinder.

It can be very long cylinder because the electric field is always pointing downwards so that the slope sides cannot contribute anything. There is no electric field entering there is no electric field leaving. Similarly there is no displacement vector entering there is no displacement vector leaving. So the only contribution to surface integral E dot dS or surface integral D dot dS is at the bottom plate. And the charge enclosed is coming from where it into set surface of the metallic plate. So we find that the electric field and the displacement vector of constants ion this problem and from that we can write that the voltage drop V naught is equal to minus integral 0 to d of E dot dl.

E is downwards dl is upwards, so the minus sign goes away and you get it is equal to sigma over epsilon naught times vector total height. Because the constant vector it can pull it out of the integral. So only the total height comes up and finally the amount of charge Q on the plate is equal to sigma times the total area of the plate. These two pieces of information tell me something very interesting. The total charge is propositional to sigma; the voltage on the plate is also propositional to sigma. So if I divide these two expressions sigma should go away. Now if you look at this derivation even though I got Ex and Dx interms of sigma, I do not know what sigma is. I know what epsilon naught is. I know what dS. I know what area is. But I do not know what sigma is. By dividing these two equations, I can eliminate this unknown sigma so what do I get well.

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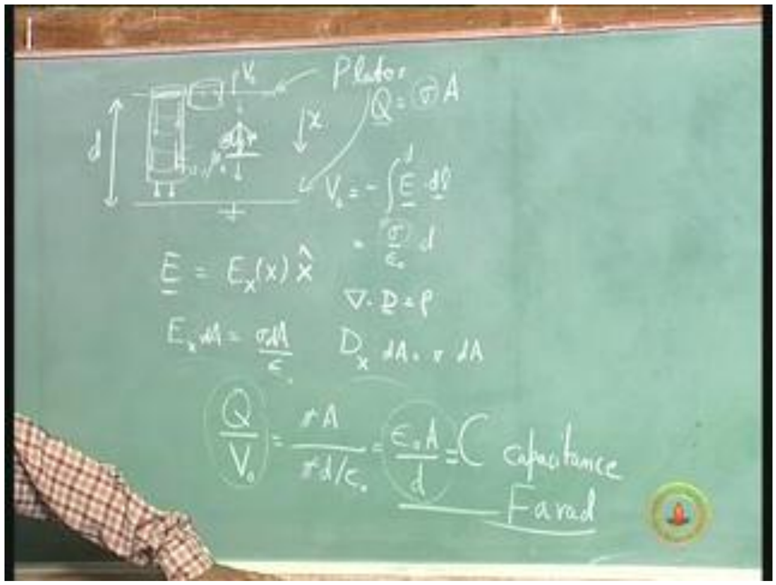

What I get is Q divided by V naught is equal to sigma A divided by sigma d divided by epsilon naught. Something there is in denominator of the denominator goes to the numerator and the sigma cancels. So epsilon naught A over d. Now what is important about this is not the derivation. If done this, I do not know how many times in school and college, what is important is starting with two equations that both dependent on unknown sigma have not got a relation between Q and V naught, that depends only on the device, depends on epsilon naught, it depends on the size of the plates, it depends on the distance between the plates.

It does not depend on the fields themselves, does not depend on the charge. It does not depend on anything which is dynamic which depends on circuit conditions and this quantity that is only a function of device parameters is called capacitance. So this is the very important concept and it does not have to repeat this as many times of possible.

Capacitance is a major of how much charge is induced on a plate when you put that plate at one volt with respect to the ground.

If I said V naught to 1, then the amount of charge I will induce in coulombs will be the capacitance itself. So the unit of capacitance should be coulombs per volt directly. But it is given a new unit; the unit is called farad. Now, the farad is such a large number because, the column is such a large number. One column per charge is huge and you will never see it in a normal device. So typical values of capacitance you will see will be ranging from the peek of farads to the micro farads. You will never see a milli farad and a farad you can forget. This was the simplest kind of experiment. It is a thought experiment we can do.

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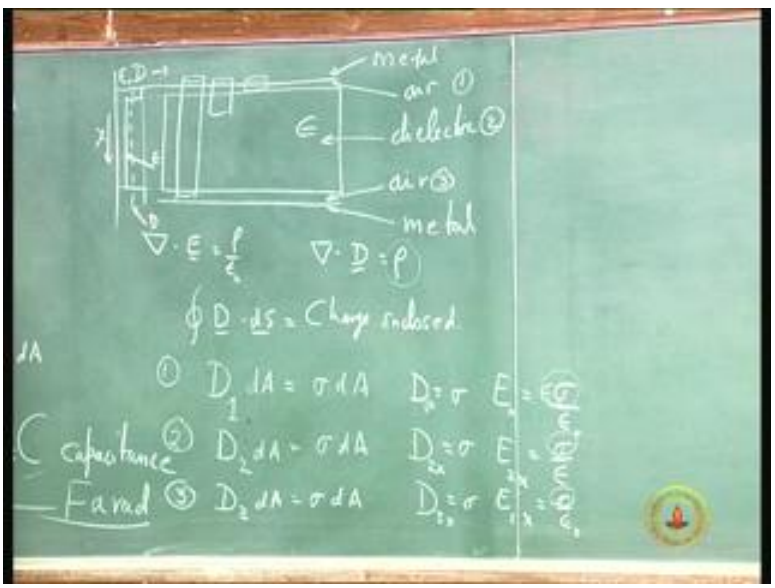

But let us do a slightly more complicated. I have got a parallel plate capacitor. But inside this parallel plate have put a dielectric material. So I have metal, I have air; I have dielectric, air and metal. See you can see it is a sandwich kind of construction and forget the side walls. Let us assume there is infinitely large. I would like to solve this problem. Now if I apply Gauss's law. I can again get the fields in this region, in this region, in this region and there by solved the whole problems. Unfortunately there are induced charges on the dielectric. So how do I handle this problem?

The answer is, instead of using divergence E equals row over epsilon naught, I will instead use divergence D equals row. This is free charge. It is the charge I deliberately put anywhere. This row includes the charge on the surface of the plates. But it does not include the charge induced on the dielectric. So if I apply this formula apply the divergence theorem, it tells me surface integral D dot dS is equal to charge enclosed. Let me call this region 1, this region 2 and this region 3. So in region 1, that is by cylinder as it top side on the metal the bottom side in region 1.

Once again you can argue that the fields will be directly down. So the side walls will not contribute top wall is in a metal so no field. So the bottom wall is only thing that matters. So I get D1 times area is going to be equal to induced charge on the wall times, area. So, D is equal to sigma. Which means, electric field Dx Ex is equal to epsilon naught sorry sigma over epsilon naught. Is that clear? Now in region two I draw my Gaussian cylinder and I again look at this, the slope parts do not contribute. Top part is in the metal. Bottom part is important. So D2 times, area is equal to the charge enclosed.

This is the true charge enclosed and there is only one place whether true charge which is the plate. So it is sigma dA which means this is D1 x, D2 x is equal to sigma or E2 x is equal to sigma over epsilon. Here there is a change. The displacement vector was the same. It is a same here and it is a same here or the electric field is changed earlier the electric field was sigma over epsilon naught. Now the electric field is sigma over epsilon. The third region D3 dA is equal to sigma dA same argument which gives me D3x equals sigma E3x equals sigma over epsilon naught.

Because, you see that third region, I am back into air and therefore I am back into epsilon naught. See if I plot this I will plot it here this this is D or E and this is x going down, the displacement vector is a constant. So it is a same at all values are positioned. But you look at the electric field; it is some value up to this point. And it is that the same value here. But in the region where there is dielectric the electric field is small, alright? Now drawn these two lines is overlapping I soon do that D is something. So this is D this is E. So the displacement vector is uniform in x, the electric field vector is not uniform in x. But I know what the electric field is. But again I know it only in terms of an unknown charge density okay. Can we make use of that?

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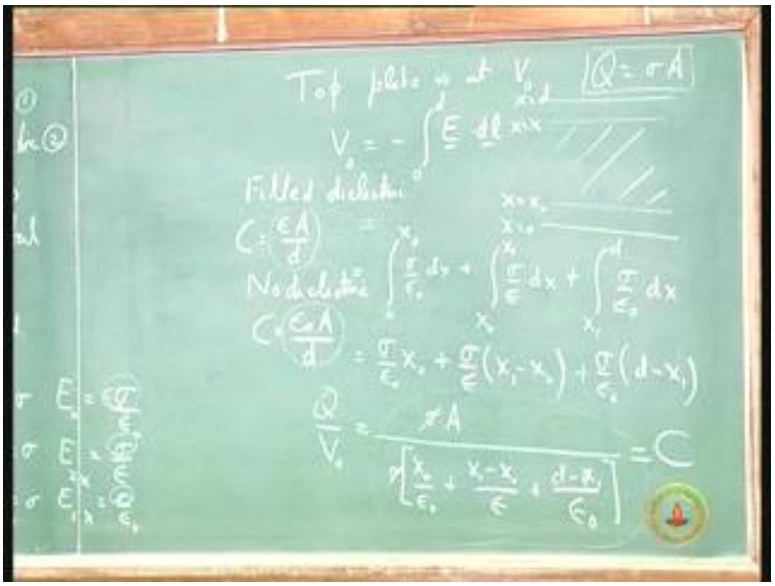

Well the top plate is at V naught and I know that V naught is equal to minus integral 0 to d of E dot dl. That is to say this relation is still true potential is related to electric field, it is not related to displacement vector. But I know what the electric field is interms of this unknown sigma. So I can write it as I will draw the picture again. So this is x equals 0, this is x equals say x naught, x equals x 1, x equals d. So, it is equal to integral 0 to x naught sigma over epsilon naught dx. Let two minus signs gets cancelled out. Plus, integral x naught to x1, sigma over epsilon dx plus integral x1 to d sigma over epsilon naught dx.

So I am just taking the solutions that we already worked out the electric field is strong here strong here weak here. But I do not know it is actual strength. Because I do not know what is sigma is. But I know it is form as a function of sigma, so I put that in. Well, sigma is a constant and I can pull it out. Epsilon naught and epsilon are again constant. So the integrals are very trivial. So what do I get? It is sigma over epsilon naught times x naught plus sigma over epsilon times x1 minus x naught plus sigma over epsilon naught times d minus x 1. So, we now have an expression for total voltage and the expression for charge Q is nothing but sigma times, area.

So, once again I have an expression for voltage. I have an expression for charge. Both of them are interms even unknown sigma, but I can divide and what happens if I divide? Q divided by V naught is equal to sigma A divided by sigma times x naught over epsilon naught plus x1 minus x naught over epsilon plus d minus x1 over epsilon naught. Again I can cut out the sigma and you can see let the answer does not depend on sigma anymore. It depends on device parameters because, it depends on A; it depends on epsilon which is the material of this dielectric which dielectric I put in. It depends on the spacing on the gap size and another thickness of the dielectric.

All of these are based on how I made this capacitor. It does not depend on what is happening in the rest of the circuit. So this is the quantity that depends only on the device physics and therefore it can be defined independent of anything else and so again I can define it as a capacitance. Now it is worth looking at this capacitance and trying to understand it. Supposing I let this x naught go to 0 and x1 go to d. That is this gap goes to 0. So the dielectric fields the entire region. Then x1 minus x naught is nothing but d because x1 is d and x naught is 0. What will I get? I will get epsilon A; over d, that is the case filled dielectric. So then I have capacitance is equal to epsilon A; over d.

Now you can compare this to the other case which is the dielectric strings to nothing and you left only the air gap. So then x1 is equal to x naught. So this term goes away this x naught is equal to x1. So this two all cancels. So you get back no dielectric C is equal to epsilon naught A; over d. So you can see this epsilon naught will go to the top. So I get epsilon naught A; over d. So as you put a dielectric and as you increase the fraction of this gap over which the dielectric is present, it goes from this value to this value. The capacitance increases and what is capacitance? Capacitance is the amount of charge that is the induced on a plate if you applied one volt between the plate and ground. So, as you put a dielectric and as you increase the dielectric constant the amount of stored charge keeps on increasing okay. Let us try and plot some of the fields.

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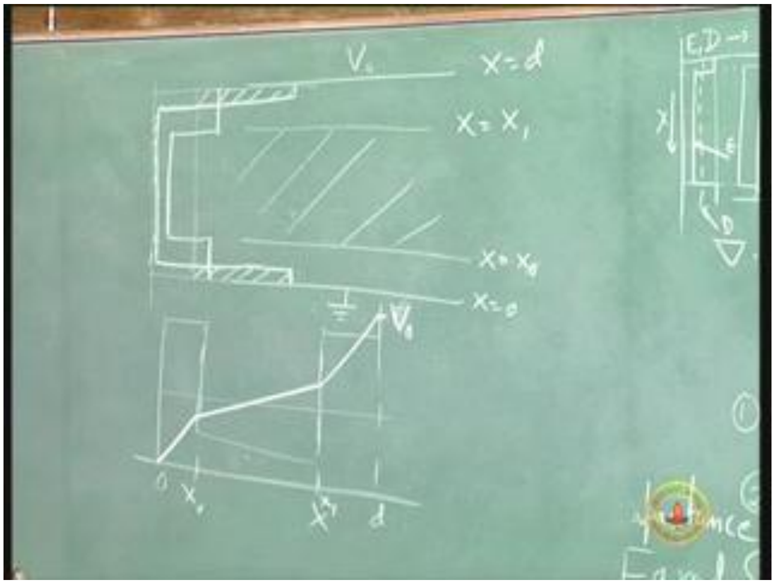

So this is again the same plot this is at voltage V naught this is at ground. This is x equals 0, x equals x1 x naught, x equals x1, x equals d. Now we already plotted it. We have shown that the electric field looks like this. So it is low inside it is high outside. Now what is actually happening? What is happening as follows: Overall, you need an average field. So that we integrate the fields from 0 to d you build up V naught. So I will put this average field like this. This is the average field you need. Now when you put a dielectric in the region whether the dielectric is present the field is actually lower than it is average value.

That is compensated for by having a very high average value high value outside. So that is these two regions. But having a high electric field means having a lot of induced charge. So I am putting a high large dielectric constant material inside this capacitor and making it fill most of the gap corresponds to something like this. That is I have got an average that this much but that average is made of very large electric field here and very small electric field in the rest of the region. And I am putting this gap as a very small quantity. I am actually creating a very large electric field here. And that is what is giving me a large amount of induced charge. That is what gives given me large capacitance.

If you look at the voltage, however I will plot that this way 0, this is d, this is x naught, this is x1. So the electric field is doing this. What is the voltage doing? The voltage this slope of the voltage is related to the value of the electric fields. So the voltage grows strongly. Then here the electric field is low so the slope is less and then it should sub again and this is what has to be V naught. And finally what is the displacement vector doing? Well, the displacement vector is not on the same scale just constant.

Now at the advantage of what we have done is that, we did not really have to do any hard work trying to understand what is happening at the surface of this dielectric. This surface of the dielectric, we used displacement vector is does not see any bound charge, solve the problem as it was a vacuum capacitor and came back and calculated all the pieces it was that simple. If you did not have the concept of displacement vector this would I mean a much harder problem. Now let us take slightly different problem.

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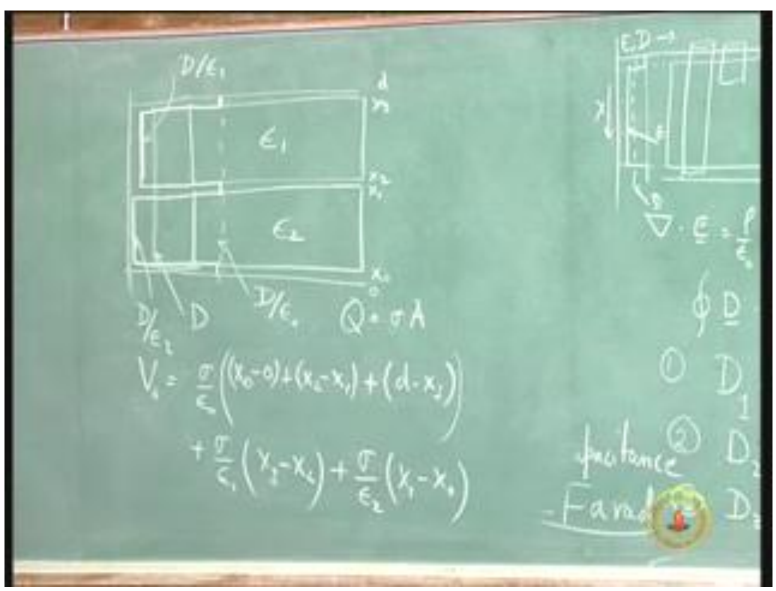

I have a parallel plate capacitor but I have two dielectrics. Problem is essentially is same is that is that approach is almost trivial to solve. Let me just plot. What we will get? We know that the displacement vector is going to be constant right through because only free charge is at the top plate and at the bottom plate. Whatever induced charge there is in this surface, this surface, this surface and this surface does not show up in D, it will only show up when you look at electric field, so that displacement vector is constant. We do not know the value of it. But we know it is constant.

The electric field well the electric field is displacement vector multiplied divided by a constant. That constant depends on the material. So the electric field is going to look like. So it is going to have some value this value is D over epsilon naught. This value is D over epsilon one and this value is D over epsilon 2. So here, in this diagram I am assuming that epsilon one is smaller than epsilon 2 and both of them are much larger than epsilon naught. That is why you see a very large voltage, very large field in the air gaps relatively small field in dielectric 1 and an extremely small field in dielectric 2. So then how do I calculate this D or how do I finally solve the problem?

Well I again use V is equal to minus integral E dot df. So I have V naught is equal to minus integral E dot dl which means I have sigma over epsilon naught times all the air gaps. I am just going to add them up. So this is 0, this is x naught. This is x1 this is x 2, x 3 and d. This air gap is x naught minus 0 plus x 2 minus x1 plus d minus x 3 plus sigma over epsilon 1. Epsilon is in this region so is x 3 minus x 2 plus sigma over epsilon 2 x1 minus x0. As usual Q is sigma A.

So I can divide Q by V naught and eliminate sigma and I can get an answer for capacitance. It is identical to what we have already done by adding the second dielectric. This approach does not make it any more difficult. They just some extra terms and not even that meaning, the one extra term. So this problem of solving for capacitance in one dimensional geometry is made very simple the moment you start using displacement vector. Now earlier we talked about electric field and stored energy in the electric field.

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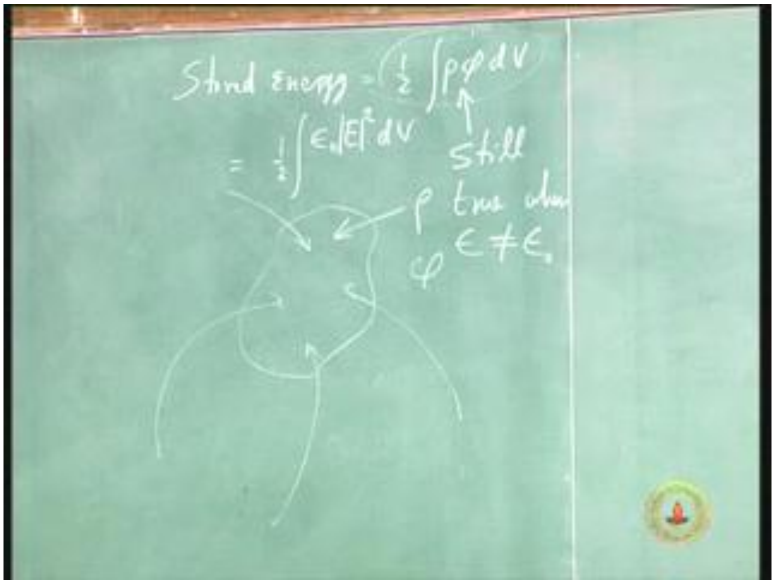

So we have got an expression that stored energy which is worked done set up the field is equal to one half integral row pie dV that is integrated over the volume. The charge density times, the potential at that point and divide the whole thing by two. I gave you a derivation, of that and we found that is was equal to one half integral epsilon naught E squared dV. Now let us look at what happens if you have dielectrics. Now this formula came from the idea that I have some charge density row and this charge density created the potential pie.

And if I bring charges from infinity in all directions and build up this charge slowly. The amount of work I have to do at any point is to bring in a small charge 0 is to work against some fraction of this potential pie and if I integrate all that work we get one half row pie. That argument does not depend on whether there is dielectric or not. The presence of dielectric will make calculation of coulomb's law more difficult. But it does not change this argument. So this is still true. What changes is this expression? This expression is only correct in vacuum. So let us see what happens when you have a material.

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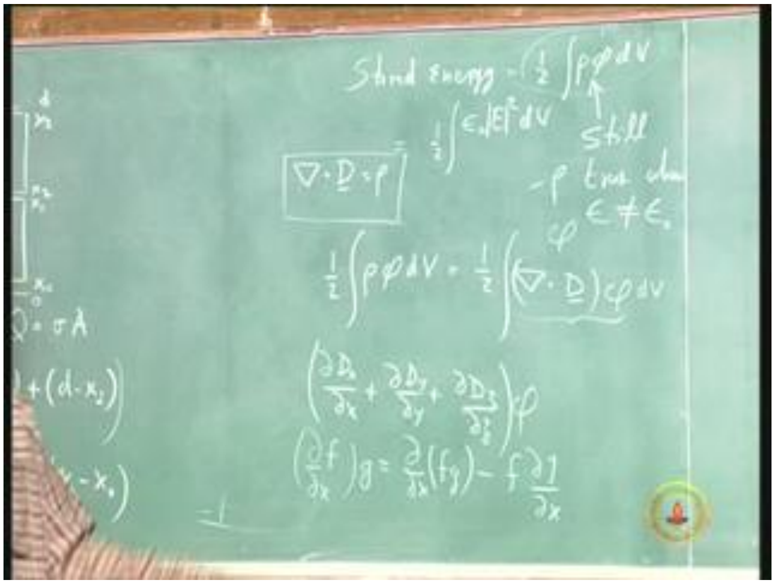

I go back to this expression and then I say one half volume integral row pie dV. Now what is row? We have an equation. Let says divergence D is equal to row. We worked out very hard to get this and should sort of dream it when you study electromagnetics is one of the foundation equations of Maxwell. So put that in this becomes one half volume integral divergence of D times, pie dV. Now you would like to do something to this equation it looks little too complicated. So let me just look at this piece. It says del Dx del x plus del Dy del y plus del Dz del z multiplied by pie.

Now this is just as a derivative times of function and I know that del del x of f times g is equal to del del x of fg minus f times del g del x. That is, if I take the derivative of a product it breaks up into being the derivative of the first term multiplied by the second term and the derivative of the second term multiplied by the first term. I took one of them to other sides. So there is a minus sign. So I can use that here and what do I get?

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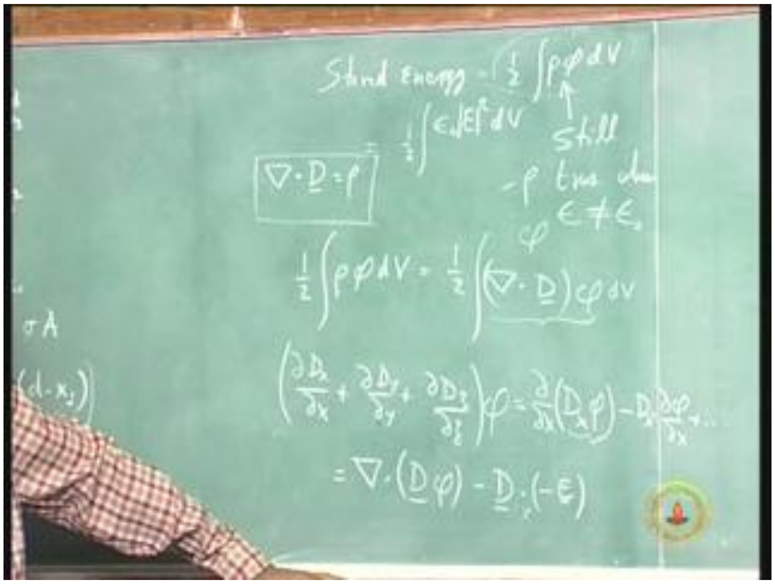

I get del del x of Dx pie minus Dx del pie del x plus similar terms y and z. Now this is a vector Dx times pie Dy times, pie D pie is a vector and I am taking it is divergence. So this is nothing but divergence of D times, pie. Whereas this is D dot minus E gradient of pie is minus of the electric field. Now let us see whether they take us. It is exactly the same derivation as we did for the previous case except now, we have a material present.

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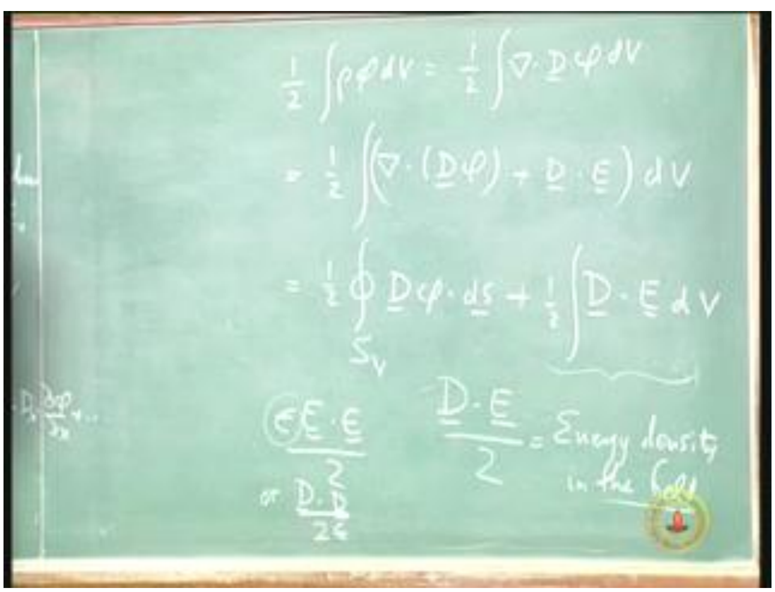

So I have one half integral row pie dV is equal to one half integral divergence D pie dV which is now equal to one half volume integral divergence of D pie minus sorry plus D dot E whole into dV. Actually I had minus D dot dot minus E so the two minuses make it as plus. Now this is an integral over a divergence for volume any integral of divergence over a volume is a surface integral. So it becomes half surface integral over the volume D pie dot dS plus one half integral over the volume D dot E dV.

Now what we do with this? Surface integral depends on what device? We have if we have a capacitor this surfaces are infact in the metallic plates and it is not true that pie goes to 0 there. But if we are talking about charges and free space with some materials present then pie goes to 0. So we cannot actually throw away this surface term minimum. This surface term can be important. But this piece is the piece we commonly associated with energy in the field. It is quantity D dot E over two is like energy density in the field. So when you have a dielectric material.

You cannot use epsilon E squared or you cannot use epsilon naught E squared. You have to use D dot E and what is D dot E? It is epsilon E dot E over 2. So, this stored energy increases due to this factor epsilon. Of course the problem actually is, that you can also write this as or D dot D over 2 epsilon. You can say it increases for fixed E or it decreases for fixed D and this is an very important thing if you manage to keep electric field constant. Let me write it down the statement is so important needs to be written down.

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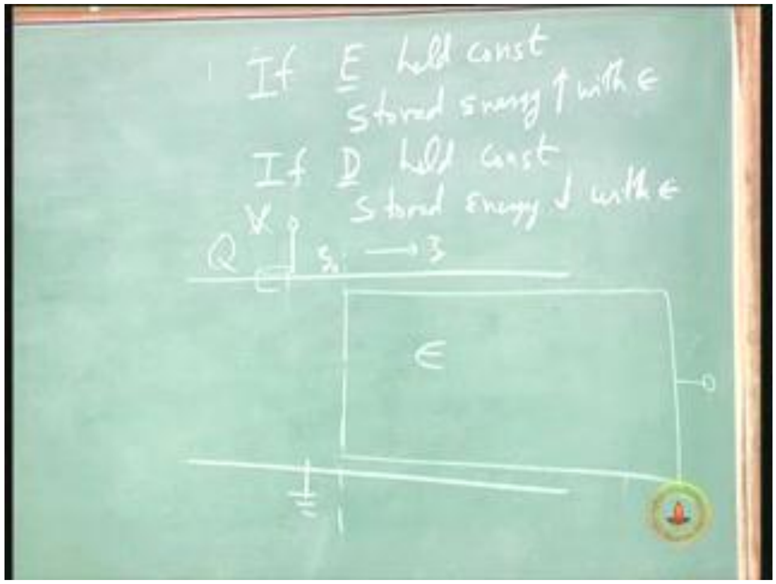

If E held constant stored energy up with epsilon. If D held constant stored energy down with epsilon. Where does this make a difference? It is actually extremely important because it is used in realize, it is used in many applications. Let me give your examples. Supposing I have a capacitor plate system and I put a block of dielectric long block which has the dielectric epsilon. This coordinate let us call it z and at this particular point it is at some position z not.

Now the question is the capacitor exerting a force on this dielectric and can we controllably exert this force. For example, this dielectric block could be connected to a relay and by giving an electrical signal I could be exerting a mechanical force. Now if I want to do that I need to know what force I am applying here. Now the way to do that is first of all to understand why they should be force and then to calculate. Let us look at the problem. Supposing I held my capacitor to some voltage V, now I know that if I hold the voltage to some constant then effectively I am pinning down E.

Because the integral E dl is the voltage, so I am essentially holding E constant. If I hold E constant, I find stored energy increases epsilon which means that by pulling this in I am going to increase my stored energy. Supposing instead of applying a voltage V I just deposited a certain amount of charge and then disconnected the circuit. So instead of V I have a charge Q on the plate. Now what happens? Moment I talked about charge on the plate, I have divergence D equals Q equal sigma. Therefore Q directly implies displacement vector and if Q is constant displacement vector is constant. Displacement vector is constant stored energy drops with epsilon. So what it means is that the system will try to move in different direction exactly different direction because it is trying to minimize energy.

So depending on whether I am holding the voltage constant or the charge constant, it will do exactly opposite things. It is very important concept. I will work out the detailed solution using method of virtual work, but I will do that later. The main point to understand is if I apply a voltage condition on the plate I am saying electric field is constant and when electric field is constant epsilon E squared over two is constant. Epsilon E squared over two describes the stored energy. So epsilon increases stored energy increases. If on the other hand I have stored charge, charge directly talks about displacement vector. It does not care about dielectric constant.

And displacement vector is held constant and stored energy drops with epsilon. And that tells me that if I push it in the amount of stored energy is dropping. So these automatically give you an idea of what D Alembert's method is going to tell you will come back to the actual solution later. In fact we will be having good idea for you to go and revise your applied mechanics course right now. Because of method of virtual work you would have learned in an earlier course. That is important that you understand that derivation. Okay now I am going to go to a new idea and it is an important new idea. I am not going to prove it. I am going to only stated. The proof is not very important. The idea is as follows.

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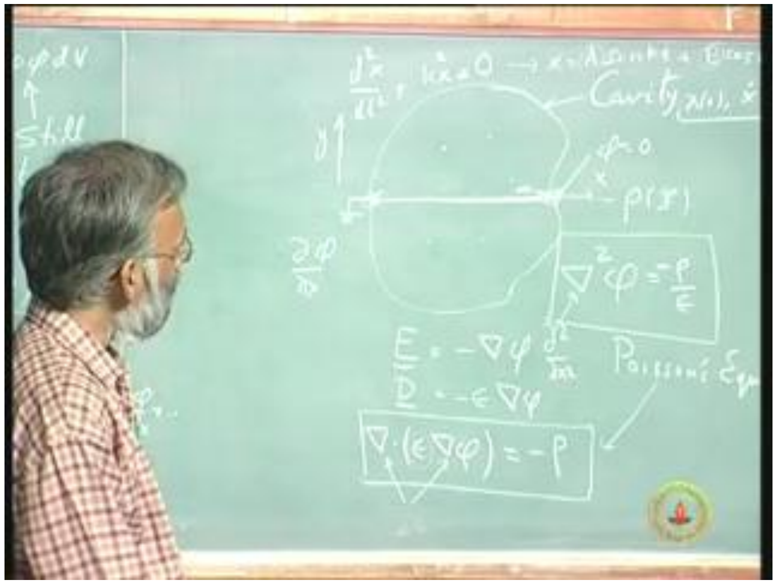

Supposing I have a cavity, this cavity could be a capacitor it could be insulating a box. It could be anything it could be a motor. I have a cavity and inside this cavity I have some charges. So I have row of x y and z call it r. Now I know that if I have charges that the problem, I need to solve it dell squared pie is equal to row over epsilon. Now if I have material properties that are varying inside this cavity, then I need a better equation than this. The equation now have is the electric field is equal to minus grad pie.

The displacement vector D is equal to minus epsilon grad pie and the divergence of that displacement vector. So divergence of epsilon grad pie is equal to minus row. So this is what is commonly called Poisson's equation. But really this is the more correct equation? But both are essentially the same equation. Unless epsilon has some very non-uniform properties from properties, but either way this is an equation I want to solve. Now if you look at this geometry of this problem, it is a second order operator. For example it has del squared by del x squared del squared by del y squared del squared by del z squared.

So it is second order differential equation to solve for five in terms of charge. Now if you know anything about your second order equation. You know that the second order equation. For example d squared x dt squared minus kx is equal to 0. That is the second order equation and we know that the answer for that let me put plus and put k squared to make a look familiar. You know that the answer to that is x is equal to A sin kt plus B cos kt. Now what is important or these A and B. This system, this differential equation actually ahs two kinds of solutions. It has sin kt and cos kt and the amount of sin kt or the amount of cos kt does not get decided by this equation. It gets decided by what we called initial conditions.

That is unless we know what x of 0 is and what x dot of 0 is we cannot pin down A and B. So this is what is called an initial value problem? You have to first solve the differential equation then you have to put in your knowledge of the starting values. There is a problem here. Supposing I have put this as the x direction and this as a y direction and out of the board is z. Well this is the region over which I want to solve the problem beyond it there is a wall. At on this wall there are unknown induced charges it is a metallic wall.

So I do not know what charges there are. However I could say that I know there are this point have grounded the wall. So I know x of 0, but I do not know x dot of 0 or I do not know del pie del x. Knowing del pie del x at this point means I know the amount of charge. That is sitting on the wall at that point. But I do not know that. What I happened to know is that if I go one and half meters down I will give the wall again I am at that point pie is again 0. So the kind of problem I am trying to solve is not this kind of problem.

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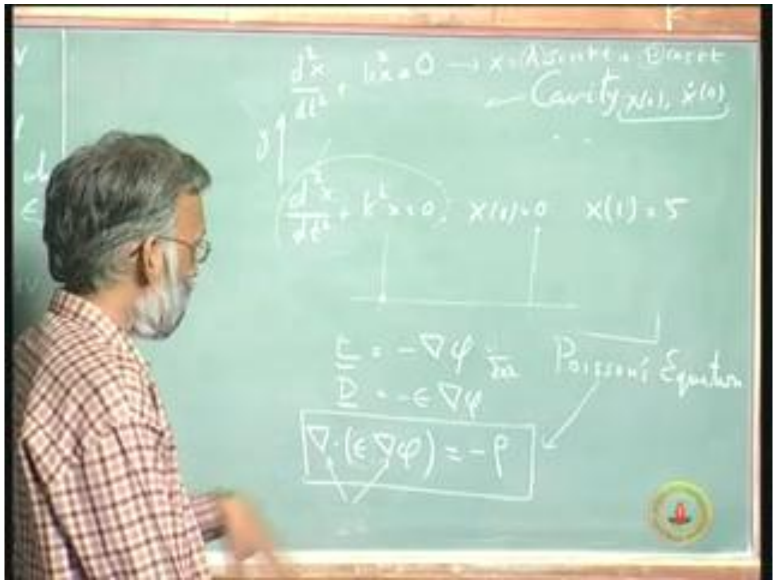

It is in fact this kind of problem which is d squared x dt squared plus k squared x is equal to 0, x of 0 equal 0, x of 1 equal say 5. That is I am saying I have this equation governing my behavior at 0. I am 0 at pie I am some value. Now find that proper combination of A and B so that I go from here to the, this is the very standard problem and I expect to be able to solve it already. Cannot able to solve it, go back to maths book. Because this is not the course in which to teach you solution of ordinary differential equations.

The solution of partial differential equation will go in to, but only in a little bit because this is not meant to handle this course is not meant to handle very detailed problem. But ordinary differential equation you should know. You are covered it in a first your curriculum. So all you have to do is you have to find A times sin kt plus B times cos kt evaluated at t equal to 0 and evaluated at t equals 1. And so you will get a matrix equation trivial matrix one which one you can solved by inspection. Now exactly the same kind of problem has to be done for Poisson’s equation.

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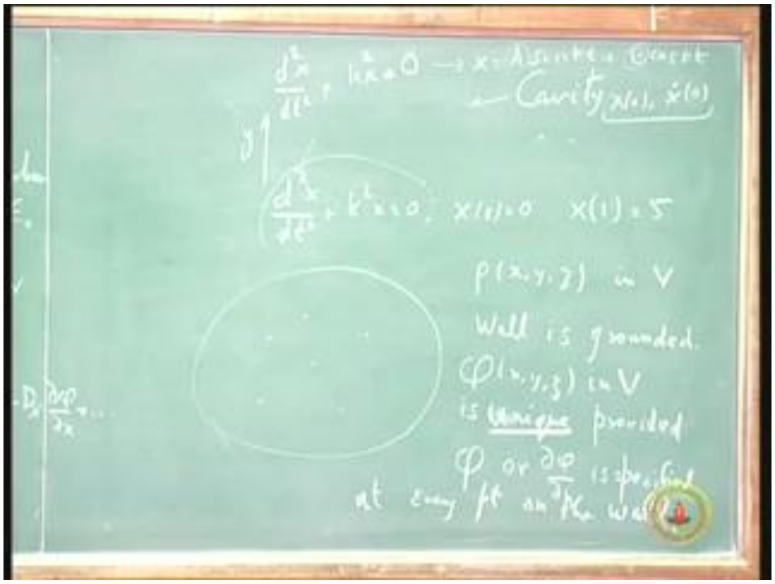

But there is a question here. The question is supposing I have this cavity I have some charges so I say I have row of x y z in V. And I say wall is grounded. Does this problem have a solution? Does it a one solution or does it a many solutions? After all I am not saying how the charge is on the wall or distributed. Could it be that by distributing the charges on the wall differently or different kinds of solutions? The theory is actually quite easy, but I will rather state the answer.

Answer is that, pie of x y z in V is unique. Unique meaning there is only one solution provided pie or del pie del n is specified at every point on wall. See it is very powerful statement it says that I can say the top order this wall is a two volts. I can say the remaining part one third of it is at 0 volts and the last part by some method del pie del n equal 0. The theorem says that if you give me such a wall there is one and only solution you can get out of it.

Now this is interesting as a theorem but what is much more interesting is by using this theorem. We can find ways of solving quite a few problems and I want to show you one of the problems. It is again one of those problems that you would have done in school. But you would have done it without knowing why it was.

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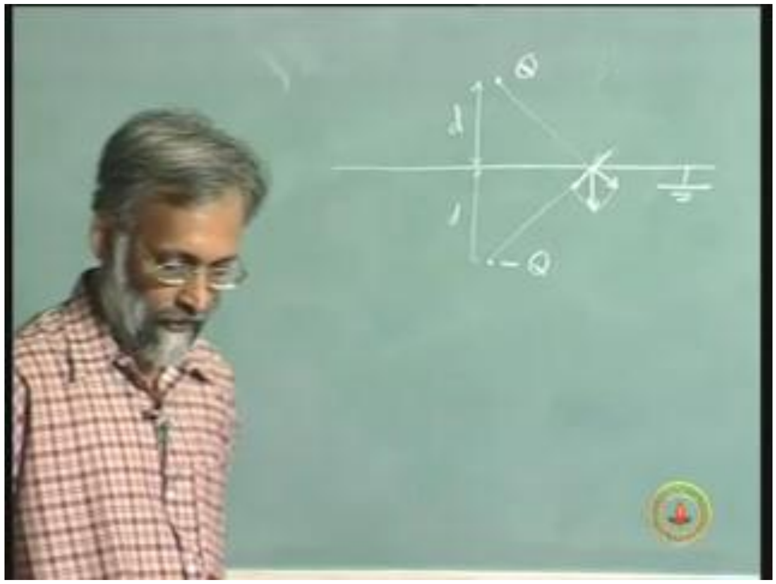

Supposing I have a large plate which I have grounded, and I am put a charge Q above that plate at height d. Now this is the problem that where I specified the potential on this surface and potential anyway goes to 0 at infinity. So I know all the charge inside the cavity and I know the potential on the cavity wall. So I should be able to solve this. There is there is exactly one solution that satisfies all this conditions but which solution is it. But what I do is I do a trick. I say this nothing shared over what is here below this plate. It could be anything. So I can put anything I like and if I put it and get the correct answer.

That answer must be correct because there is only one correct answer. How? Say I where get the correct answer is okay. So what I do I put a charge minus Q this same distance below the plate as the original charges above the plate and why do I do this? I do this because I know that if I take any point on the plate it is equate distance to both this these locations. So there will be an electric field that will point towards this charge there will be an electric field pointing away from this charge. And if I combine these two I get a field pointing straight down which means it is a constant potential surface. I will complete this example next time. But, read up your textbook and understand method of images.