Electromagnetic Fields
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Lecture – 13
Displacement Vector

The last time, we would almost reach to the definition of the displacement vector and today we will finally reach there. Let me review what we done.

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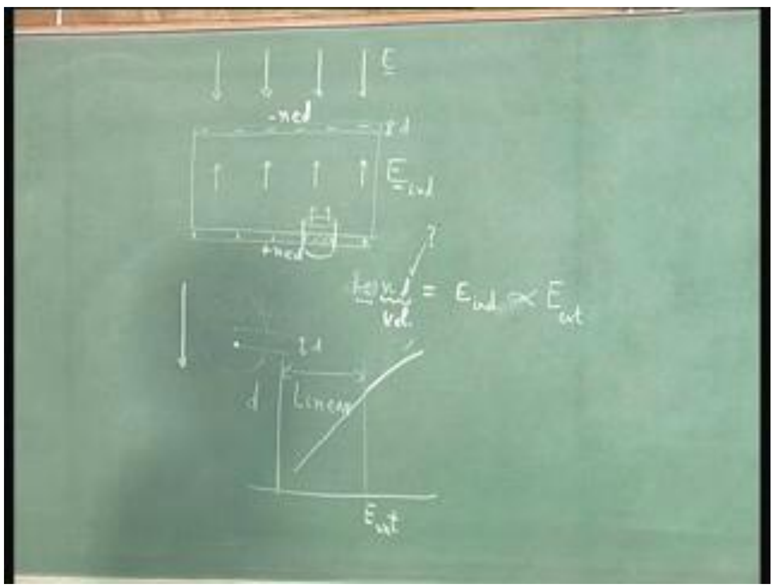

We had a material, some non-conducting material; a uniform electric field is applied. And because this material is made of atoms, the negative charge in the cloud of the electrons moves on the direction of the electric field. And so there is a layer of negative charge, at the top; layer of positive charge, at the bottom. And we already worked out that if you have equal amounts of negative and positive charge, uniform electric field appears that is in the opposite direction.

This is what I called an induced electric field. Now how much was the induced electric field? We worked that out? Besides that, supposing the slab of electrons got shifted upwards due to this electric field, by how much will get shifted? It gets shifted upwards by an amount d, where d is here is the nucleus. You have net electron cloud and they are displaced with respective each other that distance is d. So you take this distance is d multiplied d by the charge of the electron minus sign and then you multiplied by the number of electrons per unit volume. That is the number density that gives you the amount of charge per unit area there is present.

Why because, minus e is the charge and n into d is the volume. So this is the charge density. So there is charge density equal to minus ned and then the charge density equal to plus ned at the bottom. And it is this charge density that causes an electric field. We can apply Gauss's law. Supposing we had only this charge, no applied electric field, then Gauss's law tells us there is no electric field outside; there is an electric field inside. And electric field inside is directly related to the amount of charge enclosed.

So this is equal to E induced or E ind. So E ind is negative in sign. That is, this pointing upwards. This is not very useful because we do not know d, we know the charge of the electron, we know the number density of electrons in the material, but we do not know d. However from a model of what happens to an electron cloud we know that this stronger the applied electric field, the more this cloud shifts. So we can plot saying applied electric field E external and we can plot the amount of d and we find there is approximately linear.

It is a very linear for small values are applied on the electric field. But larger values it deviates from being leaving. So in this region where it is infact truly linear, we can treat this d and therefore this induced electric field as propositional to the applied an electric field. Now there are materials where you can apply an electric field this way and induced electric field will be in some other direction. These are called anisotropic material.

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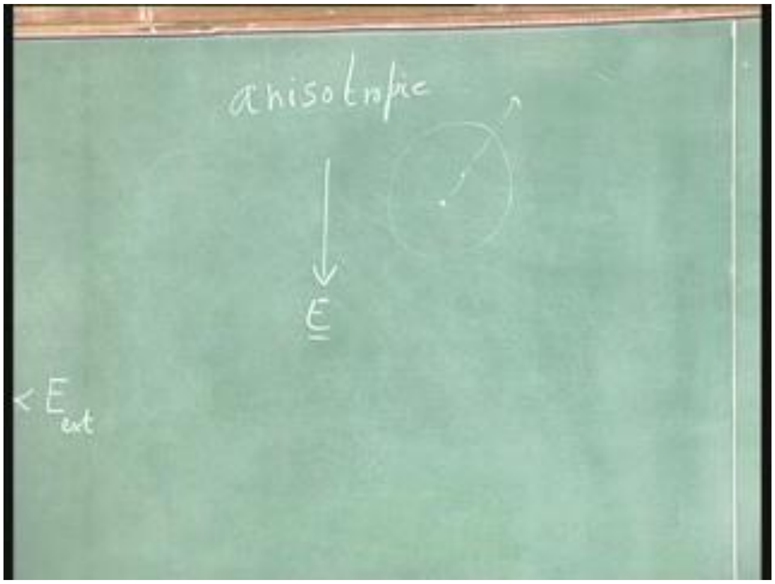

We would not touch anisotropic materials right now. We will assume that if you apply an electric field the electron cloud shifts in the direction opposite direction to the electric field. And anisotropic material would be one per have an electric field this way my nucleus is here. My electron cloud shifted in this direction even though the electric field is this way. The electron cloud will shift directly backwards with shifted side-ways. This can happen because of a crystal structure of the material. And these are more complicated materials but very, very important from an engineer point of view.

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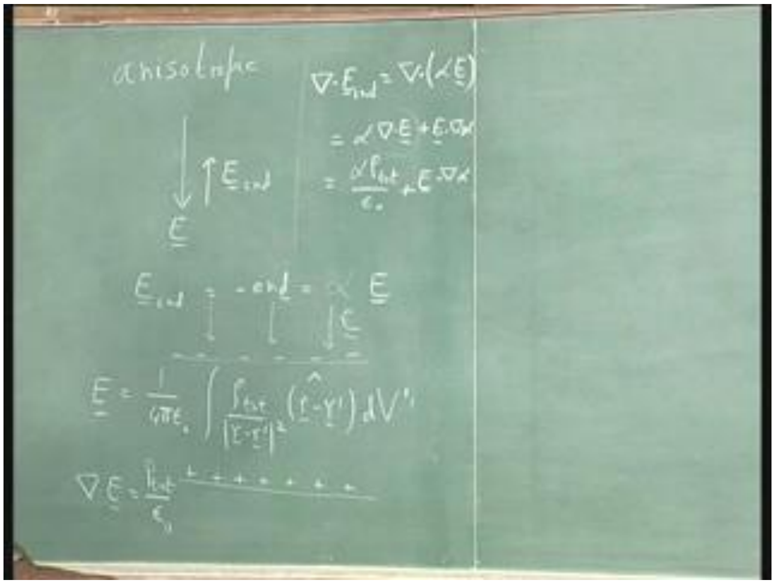

For now we will decide to study only systems where induced electric field is exactly opposite direction to the applied electric field. So we can write E induced is equal to minus en and d and d is a vector. Now is equal to some coefficient I called it alpha last Time: times the external applied electric field. This alpha is clearly negative because the induced electric field tries to cancel the external electric field. Let us get back to the original problem. I have a material and apply an electric field. Due to this apply electric field some kind of charge has developed.

Actually charge has developed everywhere, but the only place where charge does not cancelled out it is the top on the bottom. Now supposing we knew how much this charge was, if you knew exactly what the charge was, we can actually solve for the electric field. We can say that the electric field E is equal to 1 over 4 pi epsilon naught times the total charge integral of row total divided by r minus half rank squared times r, minus half rank unit vector dV. What I mean by this is, if I actually knew how much bound charge was developed by the applied electric field, I can go back to coulomb's law. And coulomb's law is always correct if I knew all the charge.

Only problem is I do not know all the charge, alright. Now that is one other thing, if I take this E induced which propositional to the electric field, so I can take the divergence of both. Now this expression tells me divergence of E is equal to total charge density divided by epsilon not. No matter whether the material or not if I could some more calculate the total charge naught just the charge I put there. But including the charge that induced then this is true.

Let me take divergence of this if I take divergence of this equation, I will get divergence of E induced is equal to divergence of alpha E which can be broken is not very useful to do it alpha times divergence of E plus E dot grad alpha. Only reason for doing it is we can see the divergence of E is already related to something the row total over epsilon naught sp. You could write it as alpha row total over epsilon naught plus E dot gradient alpha. Now what is more useful is to look at this picture. Picture may be drawing from, while now.

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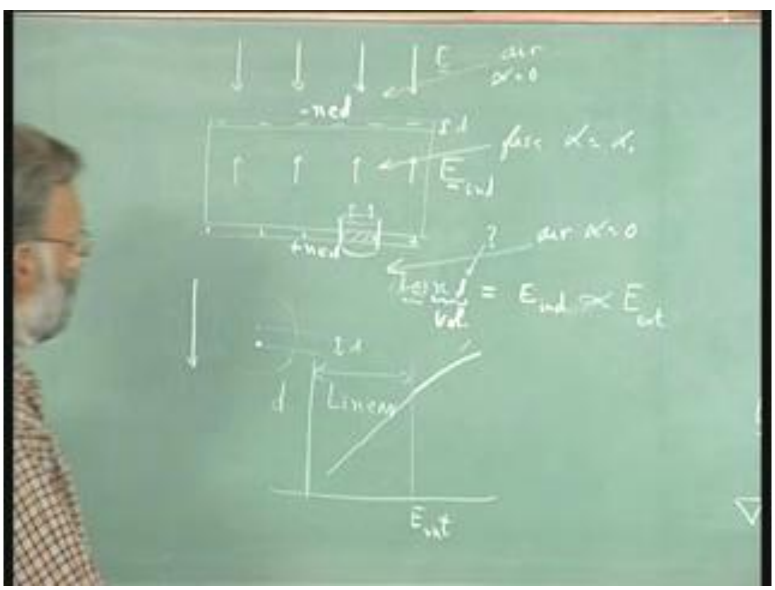

For example, if you look at this and say that the only place charge develops is wave, this factor alpha changes abruptly. Because we look out here this is air alpha equals zero there is no induced charge. This is my glass alpha equal something. This is air again alpha equal 0. So if you look wherever alpha is constant nothing is happened. But where alpha is 0, changing from constant to 0 that is where there is charge developing. So we could thing of this as representing a charge itself wherever E induced changed abruptly.

That is the place where there may be charge. And this picture agrees there, because if I apply Gauss's law, E induced when from 0 to positive the charge enclosed. E induced remain constant, there is no charge enclosed. E induced when from positive to zero this is negative charge enclosed. So it looks like divergence of E induced represents some sort of charge. In fact it represents the induced charge.

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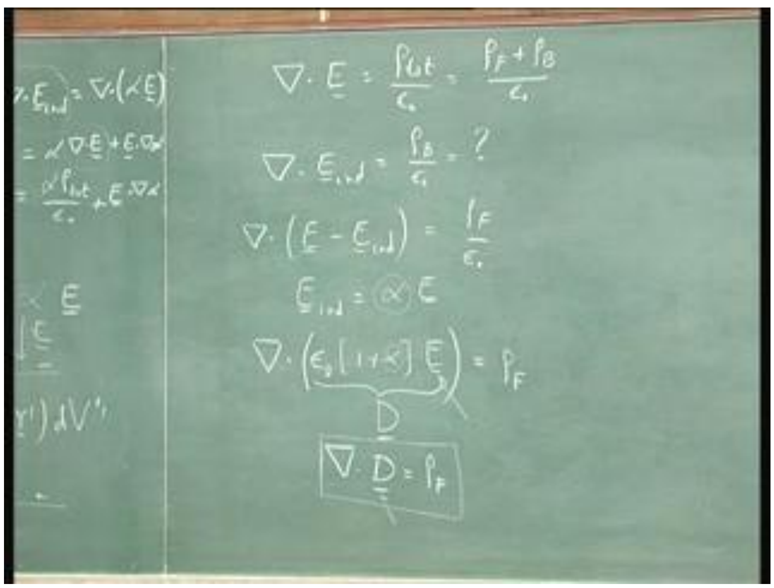

So I will write these two equations, divergence of E is equal to total charge density over epsilon naught, divergence of E induced is equal to row bound over epsilon naught. The B means this is the induced charge. So it is bound to the individual atoms of the material. It is not something we place that it is something appears only in the presence out and applied field. The problem in engineering the problem in all of electromagnetic is we do not know it.

This row bound is simply not known to us. So what we are going to do is instead of working with row total which is the sum of the charges, we have placed which I am calling row free and the induced charges row bound or working with E induced which deals with row bound. I would like to work with some kind of electric field that only relates to row free. That is the only row I know about. The remaining rows are self consistently to be determined. So I am going to take the difference. Because I can see, I can cancel out row bound.

So I will take divergence of E minus E induced is equal to row free divided by epsilon naught. That is more you already written down the E induced is equal to some alpha times E. Now alpha is not this symbol that si used. But I am just show keeping, they for all place holder. So I can write this as divergence of and I am going to take this epsilon naught to the other side. It is harmless, it is a constant times one plus alpha times the electric field is equal to row free.

Now this probably looks like magic. But what it is? It is actually kind of a fancy sort of cheating. And I will go over it again so that you feel more comfortable with what you have done. But we have what we have achieved as a result of all this, is we have an identified a new kind of field. It is related to the electric field. But it has some material properties in it this alpha represents the relation between the applied field and the induced field. In other words it is talking about this slope of this line. Now it is an electric field like quantity, because it depends on the electric field.

But it is multiplied through by some material properties and it is divergence is equal to row F only. This quantity is what we called the displacement vector. The name is purely historical. But its usefulness is extremely high because if you write this equation down you know the right hand side completely you place that charge there. So you know exactly what charge of you placed. Therefore you know mostly what this vector field is going to look like. The problem of course is then how from this displacement vector do I obtain the electric field. Now this all comes down finding this alpha.

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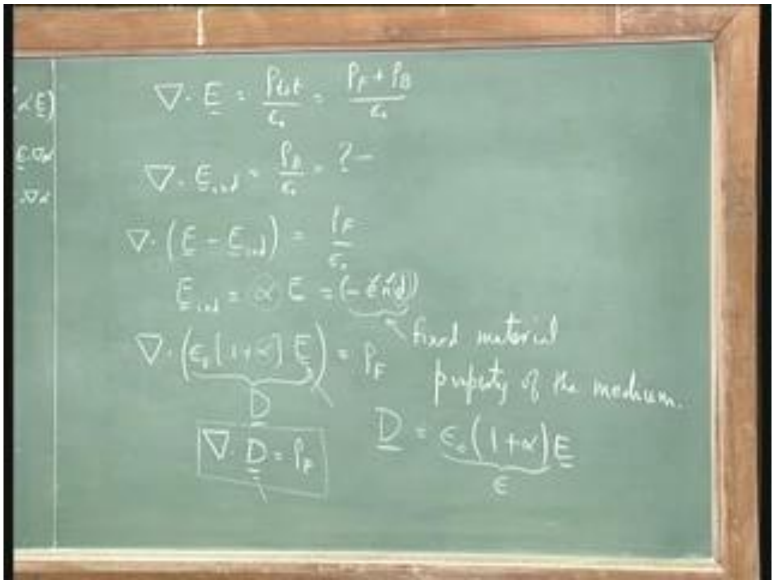

And if you look back to see what is this alpha, E induced is nothing but minus E induced is nothing but minus e n d. This is the Q bound and from Gauss's law. We found that E is equal to minus e n d, e and n are material, e is a constant, n is the just the number of atoms per unit volume. D has to do with the materials tendency to deform in the presence of electric field. So all of this is can you put down as fixed material property of the medium. So what does that mean?

It means that there relation between D and E is something we can put down in a handbook and just used. It does not something to be solved for if I make a plastic then the manufacturer of the plastic will subject the plastic to test. He will find out if I apply 1, volt how much the induced charge. If I put two volts, how much this induced charge and he will give me a plot of E induced to verses E. And if the material is a linear material that is E induced propositional to E he will give me this whole number. And this number is called the dielectric constant. Now let us take go back go back and review.

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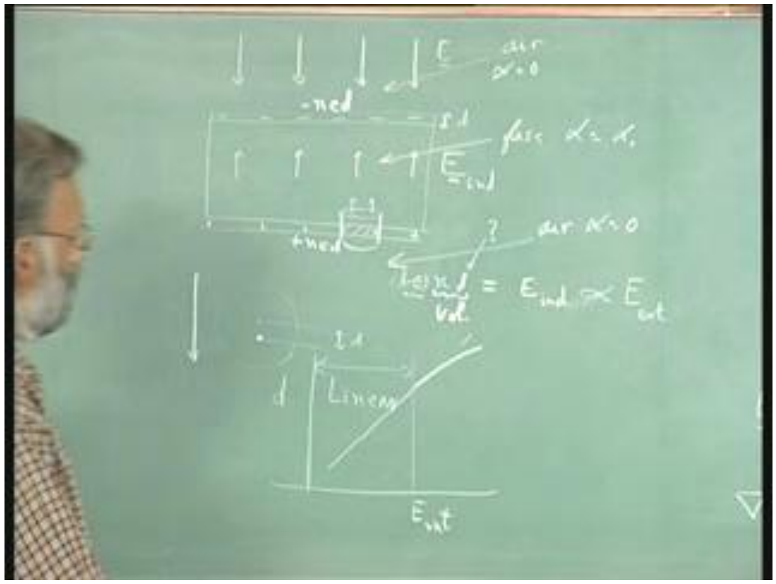

If we apply an electric field to a material induced charges appear. These induced charges for a slab like this form of the top and bottom of the material. In a middle that tends to cancel out, if you ask how much is formed it has to do with the thickness of this layer of charge. Thickness has to do with the amount the electron cloud shifted and the amount of electron cloud shifted experimentally has been seen to be linear. There is no proof that is always has to be linear. In fact it does not always have to be linear there are non linear portion to this curve. But where most engineering applications happen it is a linear curve.

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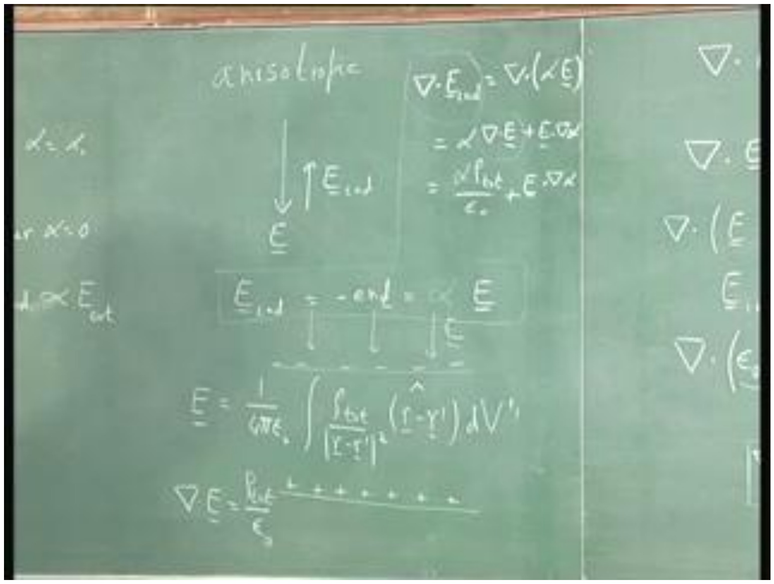

So I can write the induced electric field as something times the applied electric field, start with this this is the first equation. Now the second thing is if I knew every charge present in the entire universe in including the charges I deliberately place and the charge is that are induced on surface of the materials in the bulk of the materials. When you all those charges then coulomb's law till true because coulomb's law says, when you all the charges just write down row over r squared with appropriate direction and integrated over all space which from coulomb's law. We can always derived the divergence theorem divergence equals the same charge density divided by epsilon naught.

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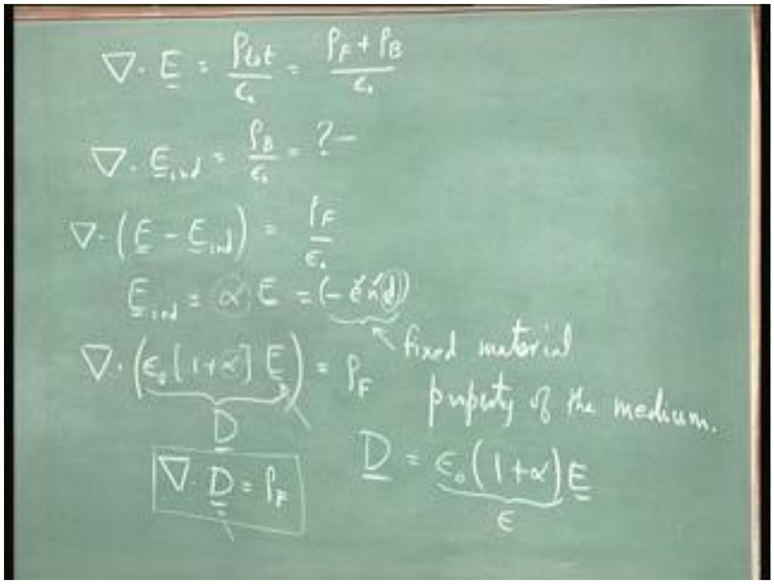

So now we have this equation where this total charge is row free plus row bound that is row the charge replaced plus the charge that was induced. Now the induced electric field is clearly related to the induced charge because that is why it came from. So divergence of E induced is nothing but the bound charge divided by epsilon naught. But we do not know what it is. But I take these two equations then I say I can eliminate row bound. Just subtract the two equations have got it row bound.

That is what I do. I take divergence of E minus E induced and it is only related to row free. And when I write out E induced interms of the applied electric field I get this equation divergence of some other vector is only equal to row free. So crucial equation and it is a bit over tricky equation. The same that you just write it down and use that without realizing what it represents. Let us take an immediate example and see how this applies.

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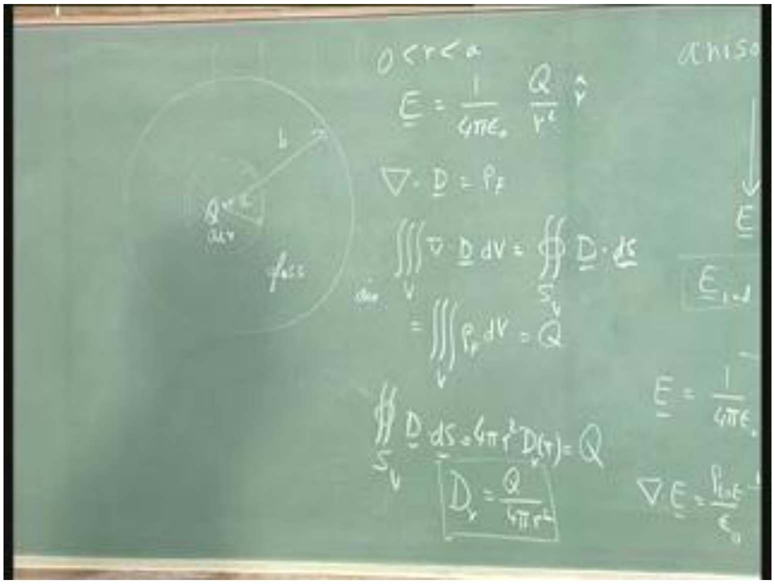

It is a trivial example but it work understand I have a charge Q and a place it inside a sphere this at a center. Here it is air, here it is glass and here it is air. The inner radius of the glass is a, outer radius is b. I want to know what is the electric field and the potential as a function of r alright. Now there are two ways of going about it. One way of going about it is to say well up to this point I am completely in air. So I know the electric field from zero to a nothing but coulomb's law.

Now this electric field when it enters this material is going to cross induced charges and I can try and write down a relation between E induced and the applied electric field and I can try to work out what is going to happen inside the material. However there is an easier way of going about it, easier way is to simply say divergence D is equal to row free. So if I take if I applied Gauss's law not to the electric field but to D I can take any radius sphere.

The integral of divergence D is nothing but surface integral over that V surface of that volume D dot d s. But this equal to volume integral of row free. Now I do not know what is happening inside this material. But whatever is happening inside this material is induced response. That is the charges are all induced only free charge I know about this

Q. So it is equal to Q. It is a spherically symmetric problem so surface integral D dot ds is nothing but 4 pi r squared D of r D in the r direction which is the function of r its equal to Q.

So I know D. D has only a r coordinate and the Dr is Q over 4 pi r squared. I have naught solved anything else but regardless what the dielectric constant of glasses. Regardless of whether I have one glass or many glasses as long as the problem is symmetric in all directions spherically symmetric. I have already solved the problem I have found out D sub r. How does it help me? Well, I will look up my handbook for the glass and I will find out.

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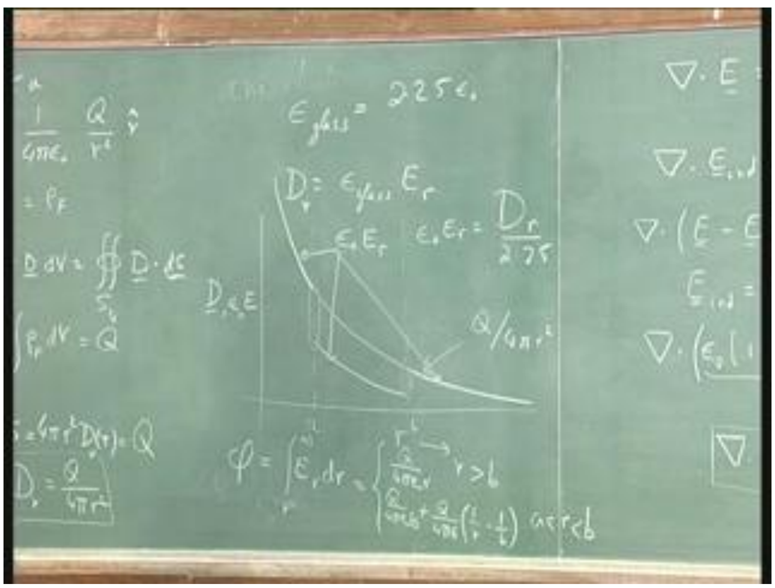

This handbook tells me epsilon glass is equal to “2.25” or it could be “two point five” or it could be 3. It is some number it is written by the manufacturer of the glass. So in that case I know that D sub r is equal to epsilon glass E sub r sp. Let us just look at the plot and make it clearly this is the radial direction. This is a, this is b. If I plot D, D is nothing but Q over 4 pi epsilon sorry Q over 4 pi r squared, so it is going to be one over r squared. This is D. What is E? Well, instead of talking about what is E I am going to talk about epsilon naught E. Epsilon not E is going to be identical to D in these two regions.

But in this region whether is a glass I know that epsilon naught Er is equal to Dr divided by "2.25". So this part of the curve alone is going to be a smaller the well. So this is epsilon naught. Yeah. What is happened is surface charge is developed here and surface charge is developed here so that the electric field has been reduced the applied electric field has been reduced by some induced electric field. So the net electric field is smaller inside the material than what do you expected to be. Once I have got this electric field and getting the potential is easy. Potential pie is equal to integral from minus infinity from infinity.

I should say from r to infinity of Er dr which is going to be equal to Q over 4 pi epsilon naught r for r greater than b. Because it is nothing but coulomb's law up to here. For r less than b is going to be Q over 4 pi epsilon naught b. That is the value at this point plus the integral of this curve. This is what the electric field is so there will be Q over 4 pi epsilon naught epsilon not times one over r minus one over b. This is for a less than r less than b. Similarly when you go inside or you can see. The point of this example was to show you the usefulness of displacement vector.

It is not just an arbitrary field that we invented. We created the displacement vector because when we worked out what divergence E is if find divergence E contains an unknown charge. So we constructed field which depends only on known charge. Now this field therefore is complicated. It is related not to coulomb's law, but to something else. Luckily for linear materials the relation between D and E is very straight forward. D and E are related to local material properties and that is very important. You can see here in this picture the material properties are changing.

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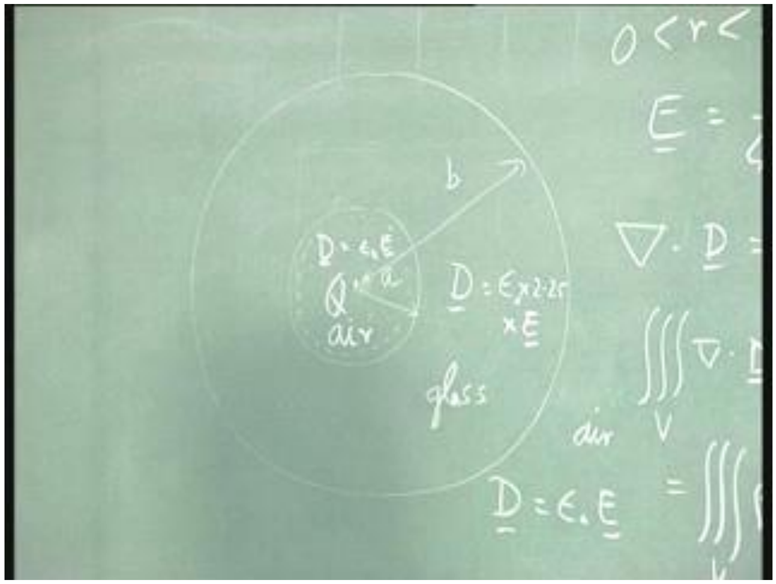

I have air, glass air. However D in here D is equal to epsilon naught E out here, D is equal to epsilon naught into "2.25" into E and D is equal to epsilon naught E. So point by point the relationship between D and E is only through a different constant. So the advantage of this is I have got a new field which is essentially the electric field, if I knew that field I know the electric field. But this new field satisfies the very simple equation it is satisfied the equation whose right hand side I know completely and that is an enormous advantage and you can see it in this example because I know Q. I do not know the induced term, but even show have found the answer alright.

Now I want to go on to one more problem and that problem is supposing I had my slab and I have the electric field coming at it. We worked out there is a induced charge and we know that there is a reduced electric field. Now if I apply Gauss's law we have an incoming electric field and an electric field inside the material. So I call this E 2, I call this E 1. So E2 minus E1 is normal. So I do not have to keep the vector notation is equal to row bound divided by epsilon naught. Is that clear?

That is the amount of the induced charge this is amount by which the electric field reduced. But I know that this E1, this is E2 is related to the displacement vector. Because if I look at the displacement vector nothing changed displacement vector did not even noticed this material. Because there is no free charges only bound charge. So we know that if we scaled up this second electric field by the dielectric constant epsilon of the material. And we scaled up E 1 by the dielectric constant of air okay.

These are nothing but D1, this is nothing but D1. These two cannot really be any different because this same Gauss's law I can applied to divergence D. I can applied divergence D, is equal to row free. And I know the row free is 0, I have naught placed any charge there. Whatever charges developed as developed because it was induced. So I know that these two must be equal. Because, if there are naught equal I would not get zero when I worked out with total amount of in enclosed charge.

Let me repeat because I think this is a shuttle point the amount of electric field entering the amount of electric field leaving a Gaussian cylinder an imaginary cylinder is not equal which means the charge enclosed. How much charge is there? That is nothing but induced charge but if I apply the same thinking to D, then I have that D2 minus D1 is equal to row free divided by epsilon naught. Row free is 0. It is equal to 0. So we get this special condition. If you want to be more accurate and we do what we should say is, let supposing the electric field is coming at some angle.

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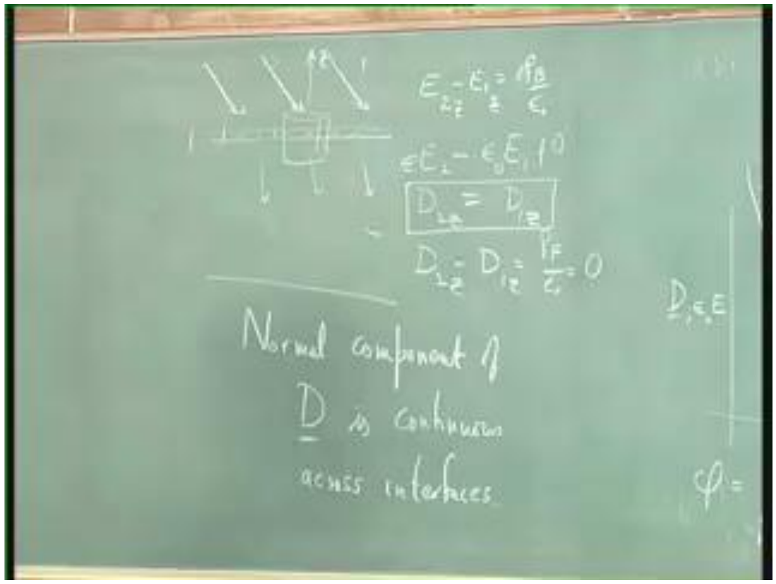

We do not exactly know what angle, the electric field will be inside we know there will be less. We can draw a cylinder and when we draw the cylinder it gives direction z E2 z minus E1 z would be row bound over epsilon naught. So only the normal component that matters you look at the sideways component. The flux entering but the same flux will leave so only the top and bottom contribute. Similarly when you do the argument here we will get D2 z minus D1 z is equal to 0 or D2 z is equal to D1 z. The way this support is normal component of D is continuous across interfaces. I said this continues across interfaces but it is really continuous everywhere. It is continuous here also it is continuous there also.

But that is assume to be true, but in a place where things a changing abruptly. You can see the electric field is not continuous E2 is not equal to E1. Why? Because there is this row bound if E2 are equal to E1, E2 z minus E1 z would be 0. But in state E2 z minus E1 z is equal to the induced charge divided by epsilon naught. However even in such places displacement vector is continuous. For the displacement vector to bE naught continuous you should have a placed charge on the surface. So let us just emphasize that and go on to next part.

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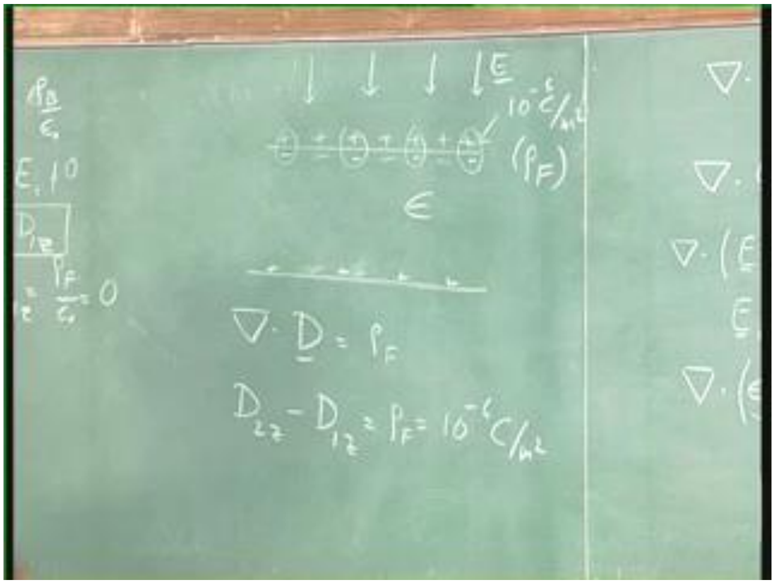

Supposing I had a slab with an epsilon say glass epsilon equals “2.25” and supposing I deliberately put charge. How much charge? Let us say you know ten to the minus six coulombs per meter squared some charge have put. So I know this charge therefore this charge is row free. Because it is it is charge I can account for it is present, even if there is no applied electric field.

Now apply an electric field this applied electric field will create an induced charge, so there will be a small amount of negative charge and small amount of positive charge. So on the top after these this negative charge and this positive charge combine these two will cancel. So what you will be left with a little bit of negative charge? When you try to work out the this problem, you will have divergence D is equal to row free which means D2 z minus D1 z is equal to row free.

Row free is no longer zero, row free is 10 to the minus 6 column per meter squared. So you will find that now you are displacement vector is not continuous is not continuous because we deliberately put charge there. If we add and put charge there then we would had a continuous displacement vector. There are quite a few problems where working with displacement vector simplifies your analysis. And that is why it is an extremely important formula. Now let me remind you other condition.

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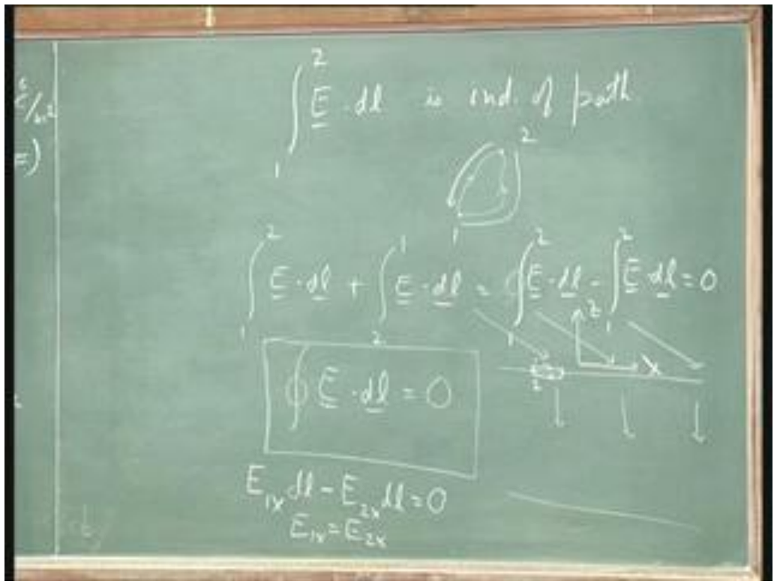

Let be always have been using which is that integral E dot dl form any point any point is independent of path. And I mentioned in a class earlier it have a 0.2 and 0.1, and I take two different paths going from one to two is like minus of going from 2 to 1. This is nothing but integral 1 to 2 E dot dl minus integral 2 to 1 E dot dl is equal to 0. Sorry plus integral 2 to 1 is equal to E dot is equal to 0 or if I take any loop, what is so ever? And integrate E around that loop I get zero. This statement which be made in the very beginning to obtain potential and this statement are identical saying this same thing. What I am saying is I will go this way and then I will come back this way.

Since both gave me the same answer adding of the two answers, this is the negative. So this is nothing but integral 1 to 2E dot dl minus integral 1 to 2E dot dl equals 0. This is the very interesting equation because supposing I had my slab and let us say my electric field I have got sloping electric field and I want to know what is the electric field is going to do. It is going to do some thing what all do is I will take a loop like this. And I will try to integrate E dot dl on this loop, this piece.

The vertical pieces, I will keep very, very thin. So I only large horizontal pieces and if I take this large horizontal piece what I get I called this side one this side two I will get E one this direction is z this direction is x. So E one x times dl minus E two x times dl is equal to zero that is the vertical parts do not contribute much. Because very thin E1 the x component multiplied by the length of this minus E two x component multiplied by the length is equal to 0. That is what this equation says or it says E1 x is equal to E2x. So, I have two equations and let me put them prominently.

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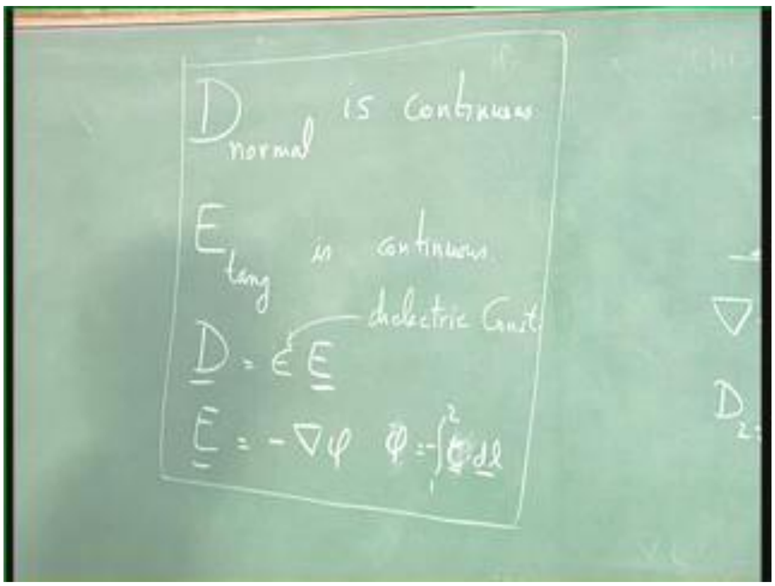

I have that D normal is continuous E tangential is continuous. These are the equations we have which describe every thing we know from the start of this course. From the start you obtain that if you had a charge is you can derive from the electric field potential and the relation is electric field is minus the gradient of potential. From whatever we have done now we know that there is a new field very useful field called displacement vector which is the electric field times the local dielectric constant epsilon is dielectric constant.

For the more looking at the displacement vector and applying Gauss's law we know that the normal component at an interface of D is continuous. And looking at the fact that electric field can be derived from potential through E equals integral minus integral 1 to 2 pie dl. Pie is equal to E dot dl using this I also obtained electric field tangential component is continuous. Let me finish the class by trying and using these equations.

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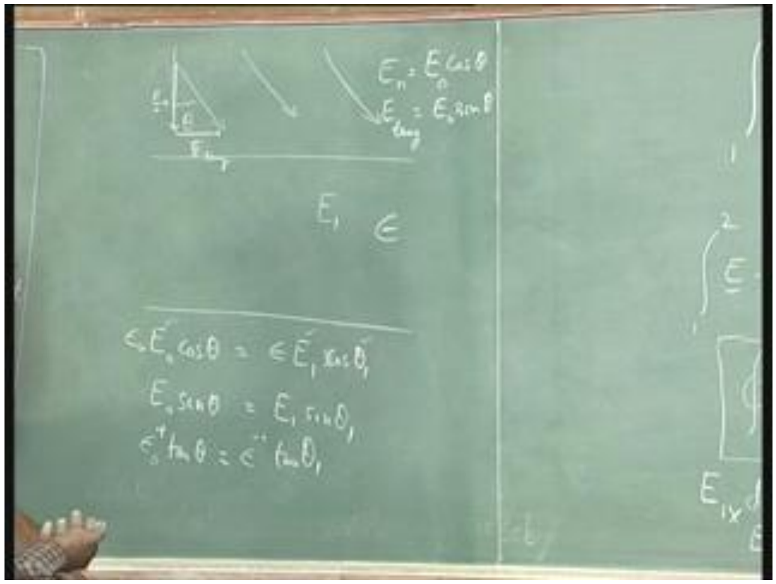

Supposing I have a slab with epsilon and I have an electric field that is arriving at a slope this angle is theta. So the normal electric field E normal is equal to the magnitude of E say the magnitude of E naught cos theta and E tangential is equal to E naught sin theta. So this is E normal this is E tangential. I know that if this is E naught and this is E one I know that E naught cos theta times epsilon naught. That is the normal component of D is equal to epsilon times E1 times cos theta. Whereas I also know that E naught times sin theta is equal to E1 times, sin theta sin theta 1.

So I have two unknowns E naught E1. Sorry E naught is known E1 and cos theta 1. I have two equations. If I divide these two equations I will get epsilon naught tan theta is equal to epsilon naught inverse epsilon inverse tan theta one. All of done is divided this equation by this equation. So sin theta over cos theta is tan theta. Sin theta over cos theta one is tan theta 1. And epsilon naught and epsilon going to the denominators. So I put epsilon naught to the power of minus one and epsilon to the power of minus 1. This is not quite this same thing but it likes Snell's law essentially saying this same thing. Now we can also work out what happens to E1 itself. For that all we need to do is we need to eliminate the theta one.

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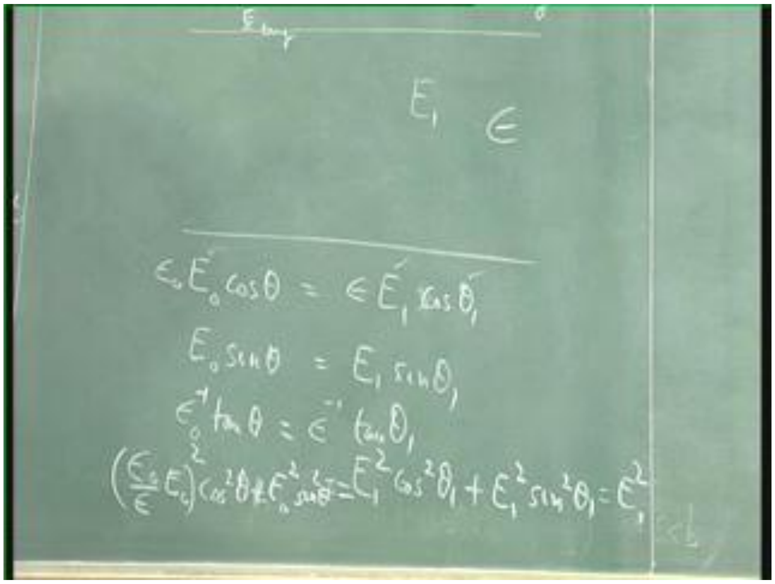

So supposing we divide the first equation by epsilon squared it square the second equation and add them up. What do you get? The right hand side was be E1 squared cos squared theta 1 plus E1 squared sin squared theta 1 cos squared theta 1 plus sin squared theta 1 is 1, that is equal to E1 squared. But what happens to the left hand side. I divided this equation by epsilon so I get epsilon naught over epsilon times E naught whole squared cos squared theta plus E naught squared sin squared theta.

So it is a mess here. But the mess does not matter I know what theta is I know what eps E naught is. So left hand side is a known thing is equal to E1 squared. So essentially if look at what this is saying it is saying that the portion that is tangential portion that along x does not change. It contributes fully to E1 the portion that is pointing in to the slab that part gets reduced in the proposition epsilon naught over epsilon.

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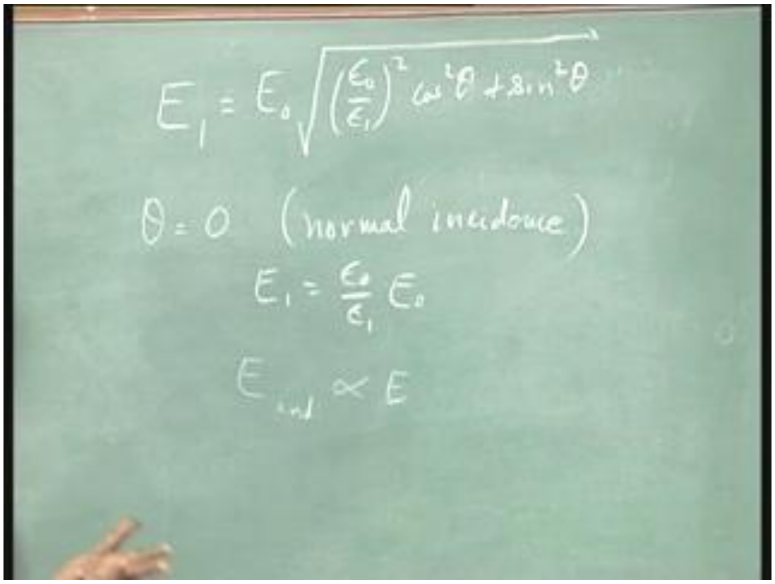

So we finally solve this, you get E1 is equal to E naught square root of epsilon naught over epsilon one squared cos squared theta plus sin squared theta. See if I had normal incident which is theta equals 0. Sin theta goes half cos theta is one. Let it say E1 equals epsilon naught over epsilon 1 E naught. So the electric field inside the material reduces in proportion to the dielectric constant which is exactly what we yet come up with the earlier.

Because we found that E induced was propositional to the applied electric field, and the quantity we came out with this displacement vector was E minus E induced. So obviously D is continuous then you must have reduction in E wherever the dielectric constant is large. Whereas if theta is large then what will start happening is this term will dominate this term. And you will have that E 1 is equal to E naught.

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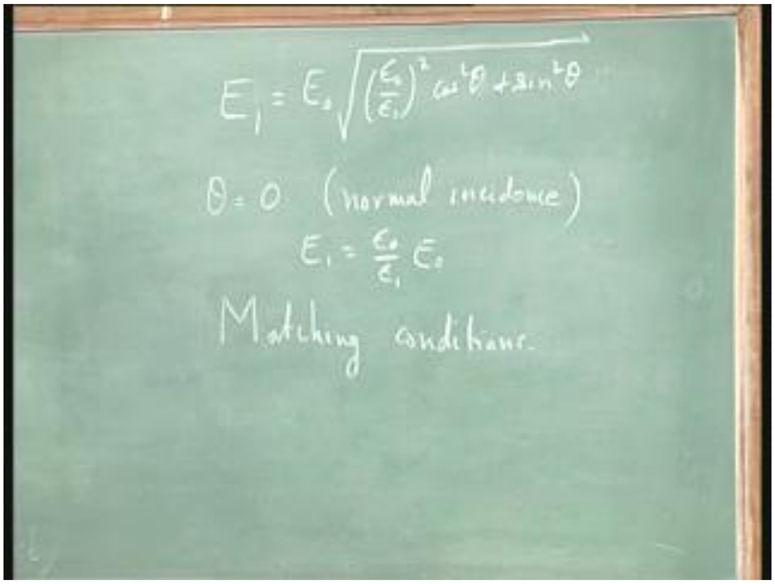

This example should tell you the, a sort of the introduction to the idea of what we called matching conditions. And the equations here these two are matching conditions for electrostatics. Namely the normal component of D is continuous. Tangential component of E is continuous. This more or less establishes all the building blocks we need to solve the problems in electrostatics.