**Electro Magnetic Field**
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Lecture - 10
Example Problems in Electro Statics

Good morning. Last few lectures we have been developing the theory for electrostatics and we have reached up to Poisson's equation and energy in the field; and I did a problem last time. Today I plan to solve several problems which will work out the various parts of the equations we have derived. Last lecture I had started a problem and stopped in the middle.

Let me repeat it. You were given a sphere of charge; the charge density was rho of r. That is to say, the charge did not depend on theta. It did not depend on size it just depended on the radial distance. The sphere has a radius a, and you are told that electric field E of r is a constant. It is a constant inside sphere.

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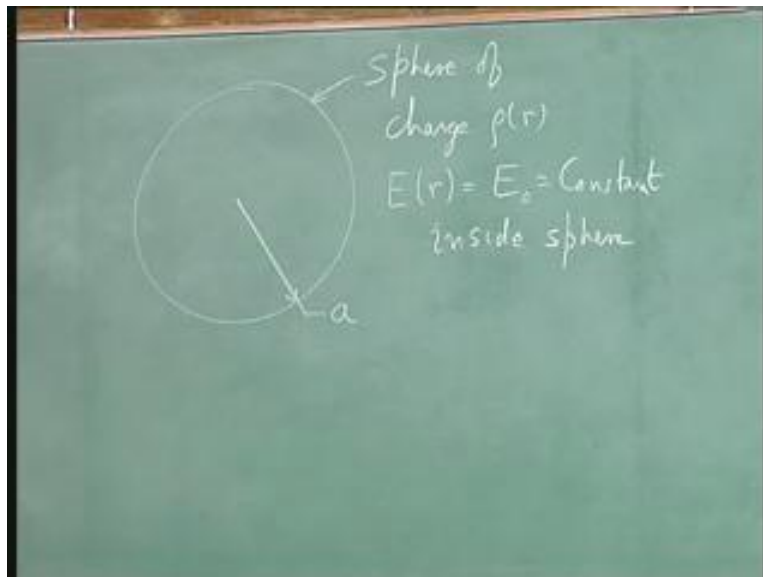
So in other words, if I plot it, the electric field, the electric field is up to this distance constant and then it falls away. This is what I have been given. I do not know the value E

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naught; but in terms of this information, I want to pin down what is rho of r. I am also told, total charge is equal to some Q. So I want to solve this problem. Basically, I want to find E of r and I want to find rho of r and I want to find phi of r.

So in order to do this, you need to use the equations of electrostatics and since the problem is spherically symmetric, the first idea that comes to us is, let us use Gauss's law. Well, if you use Gauss's law, what you get is that integral 0 to some r of 4 pi r square rho of r d r. This 4 pi r square is the surface area of the sphere of radius r. So, 4 pi r square; d r is the volume of a shell multiplied by the charge density rho of r. This is equal to Q enclosed which must be equal to epsilon naught times E naught.

That is what Gauss's law tells us. At least it tells us for 0 less than r less than a. If you took a Gaussian surface that was bigger than the sphere, then you would get a different equation. The epsilon naught...sorry, E of r times...sorry, here I have made a mistake. Let me just repeat charge enclosed is equal to epsilon naught times E of naught times the area and the area is nothing but 4 pi r square.

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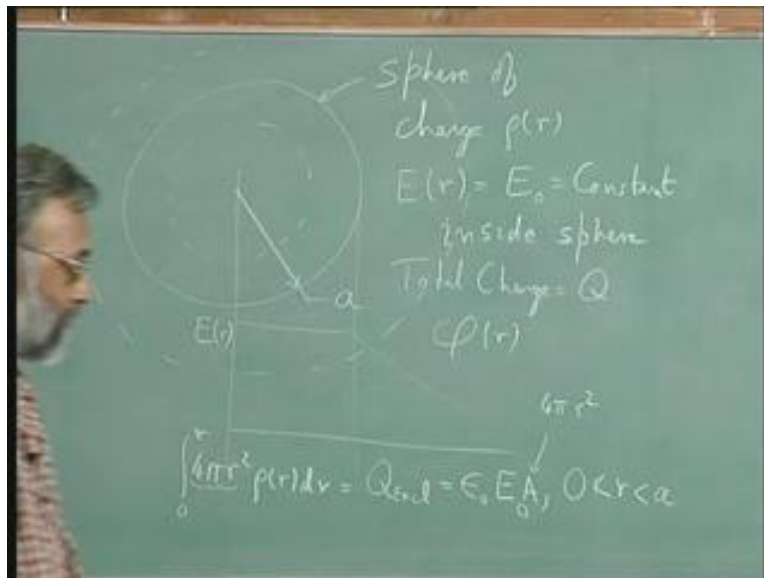
Now, when you go outside the sphere, you have that E of r times 4 pi r square, is equal to charge enclosed divided by epsilon naught. But this charge enclosed is integral 0 to a 4 pi r square r prime square rho of r. What I mean by that is, I have a radius r and I want to know what is the electric field coming out at this radius. The surface area of that radius is 4 pi r square times the electric field. So this is the flux. The flux is equal to charge enclosed divided by epsilon naught. The charge enclosed there is no charge here. The charge is only inside the sphere of charge.

So the...only this portion corresponds to Q enclosed. So that is, this equation integral 0 to a 4 pi r square rho of r d r, luckily we are given this information. We are told that the total charge of the sphere is Q. Then there is 1 over epsilon naught. So outside the sphere, we have E of r is equal to Q over 4 pi epsilon naught r square. It is a very strange result which again I am sure, you are familiar with. The result is following. If I have a spherically symmetric charge distribution, the charge distribution could be anything.

For example, I could have lots of charge out here, relatively little charge in the middle; but as long as it is spherically symmetric, if I measure the electric field further away than this sphere, the electric field, I feel there is the same electric field. I will feel if this entire charge was concentrated at the center. Because you look at this, this is Coulomb's law. So regardless of the detailed shape of rho of r, the answer is Q over r square is the normalization constant.

So, we already know quite a bit. Let me put down what we know. We have figured out what electric field is because up to r equals a, the electric field is constant, E naught. Beyond r equals a, electric field is falling off as Q over 4 pi epsilon naught r square. Now of course we do not know if the electric field is continuous there or not, but in the lack of knowledge, I have just drawn it as continuous.

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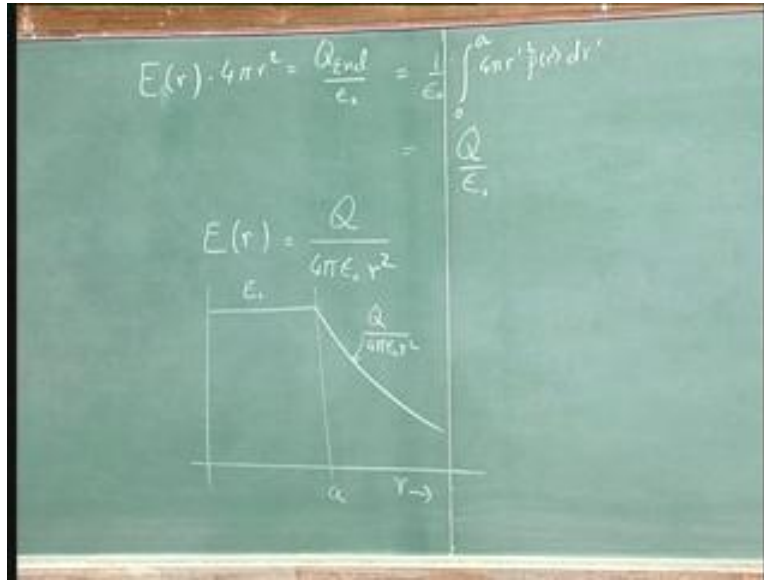
So, I have almost already obtained the electric field. Now, let us go further on and see if we can obtain other quantities. Now going back to this equation here, when I use Gauss's law, inside this sphere, I got an equation that said the enclosed charge up to a radius r. Let me put r primes here to indicate this is dummy variable. The enclosed charge between radius 0 and radius r, this charge alone gave me epsilon naught E naught times the surface area. This is not very useful because I really want to know what rho of r is.

Whereas, this is giving me what is called an integral equation in rho of r and that is not an easy thing to solve. The way out of this is to convert from Gauss's law to the divergence theorem. The divergence theorem told us divergence of E is equal to rho over epsilon naught. Let us apply that and see if we get more information. In spherical coordinates, divergence is 1 over r square del del r of r square E of r and this divergence is equal to rho of r divided by epsilon naught.

Now, this divergence looks rather strange. Let me just remind you where it came from. Divergence in Cartesian says del del x of E x plus del del y of E y plus del del z of E z, that is what divergence says; but when you are talking about non-Cartesian geometries, something happens which is quite different from this. As you move in radius, what is

happening is that the area between these angles, this angle, will be some delta theta. The length of this arc itself is changing. That is because this distance is different from this distance and the arcs are the functions of the distance from the centre of curvature.

So what happens is this area is small and this area is large. So, even as you go in the direction r, the area of your little box is changing. This does not happen in Cartesian. In Cartesian if I drew the... I will draw a cube or a cuboid, so the area here and the area here are the same. So the result is, if I was trying to work out divergence, divergence I got from taking the electric field coming in and the electric field going out; and taking the difference in the amount of flux entering and the amount of flux leaving, since the areas were the same, the difference in flux was equal to the same area times E of x plus delta x minus E of x. And that is why the area drops out. So you get a derivative that acts only on the electric field itself according to the directions.

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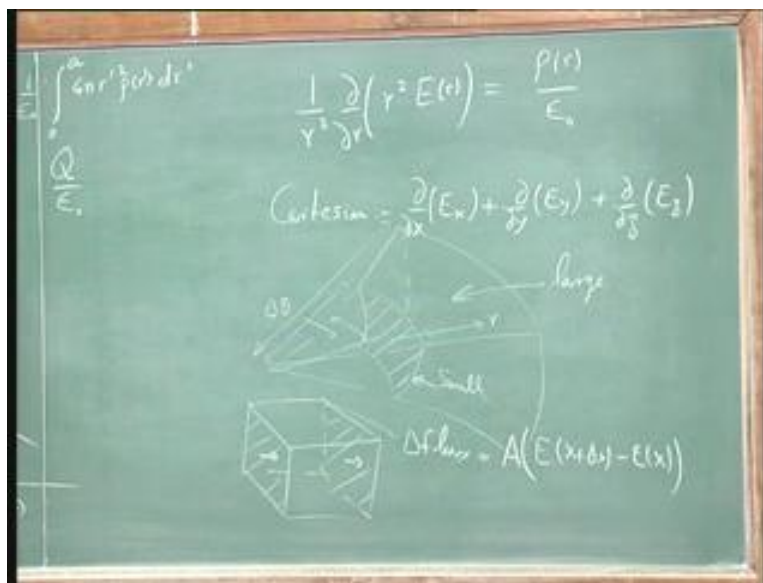
But in a situation like this, what happens is this equation is wrong. Delta flux becomes the area at x plus delta x times electric field at x plus delta x minus area at x E at x. So you see, the area itself depends on x and how does the area depend on x? Well, you know that for a sphere, if you fix the solid angle... So, if you fix delta theta and delta phi and

you just ask what is the area, the area goes as r square and that is what this r square is. It is saying that it is not...you cannot pull the area out. The electric field is scaled by the area over which it is being measured.

So, you have this r square E which is representing nothing else but area times E. So, that is where this equation comes from. You can read the appendix of the textbook where it derives it for you, not very well I must say, but still it derives it; and I do not think the derivation is very important. So we have this equation and we would like to solve it.

Now, solving this equation is easy. Let us take the r square to that side. This E of r is known to be E naught. Differentiate this. What you get is E naught times del r square by del r which is twice E naught r. So what you get is twice E naught divided by r is equal to rho of r divided by epsilon naught; E naught is known, r is known, epsilon naught is known.

So you can calculate rho of r. So it says that the electric field may be constant but charge density is not constant. In fact, if I plotted charge density on this curve, I will draw it as a dashed line. The charge density is going to look like a 1 over r curve and then it drops abruptly to 0 beyond a. This is what rho of r is. We have calculated rho. So now, let us try and use rho to calculate Q. As you can see, we assumed Q to define the electric field outside r equals a, but this Q must satisfy the following equation. Q must be the volume integral of rho.

In other words, it must be rho of r times, 4 pi r square, d r. This is d V and this is rho but we know what rho of r is. Now we have already calculated it. So we can substitute. So it becomes integral 0 to a times twice E naught epsilon naught over r times, 4 pi r square, d r 1. r will cancel and we are left with 8 pi E naught epsilon naught. I have taken all the constants out; integral 0 to a of r d r which is a square over 2.

So the final answer becomes epsilon naught E naught times 4 pi a square and that is not surprising at all. Because we already had from Gauss's law that surface integral E naught

or surface integral E d S was equal to Q over epsilon naught which would have said that E naught times 4 pi a square was equal to Q over epsilon naught. So they are saying the same thing; but in a certain sense, it is saying something useful. You see, I had drawn two curves here. I had drawn E naugh t as constant and I had drawn this curve and I joined them but I had no real proof that they were to be joined. It could easily have been that the curve was here and jumped up and became this.

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But that is not possible because by calculating charge from the rho that we calculated given E naught, the electric field just outside the sphere turns out to be exactly an agreement with what you calculate just inside the sphere. So it means that...in fact, the curve is continuous. Now, from this we can calculate potential. That is easy. All you have to do is integrate phi of r is equal to minus integral E d l up to r and the reference point is infinity. So it becomes integral r to infinity E r d r. The region outside the sphere we already know because it is nothing but Coulomb's law.

So phi of r is equal to Q over 4 pi epsilon naught r, for r greater than a, for r less than a. The potential phi of r minus phi of a must be equal to the amount of work done against a constant electric field. In other words, it is equal to E naught times r minus, sorry, a

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minus r. Since E naught itself is nothing but Q over 4 pi epsilon naught a, this is Q over 4 pi epsilon naught a times a minus r. So phi of r finally becomes Q over 4 pi epsilon naught r, if r is greater than a, and it becomes Q over 4 pi epsilon naught a plus Q over 4 pi epsilon naught a times a minus r, for r less than a.

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So this is the solution we need, if you want to solve the problem of a charge distribution that has constant electric field. The important thing to realize is it is not natural for a sphere to have constant electric field. To achieve this constant electric field, we actually have to have a very singular charge distribution. Only if the charge distribution is very spiky will your electric field be uniform.
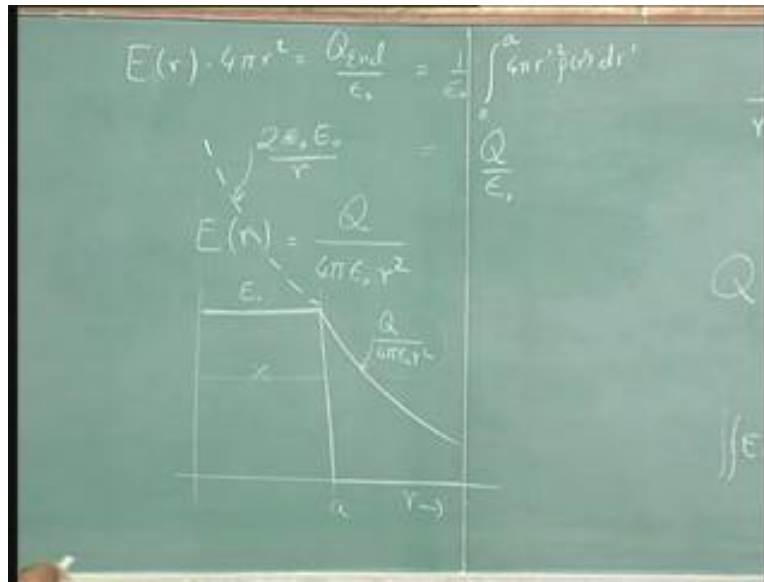
Let us do another problem. This problem is very straight forward but worth understanding. Supposing I put a charge Q in the middle of a hollow sphere and the hollow sphere has an inner radius a, it has an outer radius b. So this region is air, this region is metal; and this region is air. So the metal is in between two regions of air. Furthermore, the metal region is neutral, that is no net charge.

Now, what I want to know is what charge distribution can exist inside this metal that will minimize the stored energy, minimize the energy in the field due to this charge Q. Now, let us back up a bit. If I have only a charge Q, then I knows that the potential is Q over 4 pi epsilon naught r. The electric field is the radial electric field which is Q over 4 pi epsilon naught r square. So the energy in the field would have been integral mod E r square times epsilon naught over 2 d V.

That is, energy density in the field is epsilon E square over 2 and I would have taken that integral from 0 to infinity. There is something sneaky about what I am doing because, if you remember how we calculated this epsilon over 2, we said that the energy to bring the first charge in was 0. Yet I have got a single charge in there and I have calculated energy

in the electric field. So this energy that I am talking about is the energy that exists around a charge.

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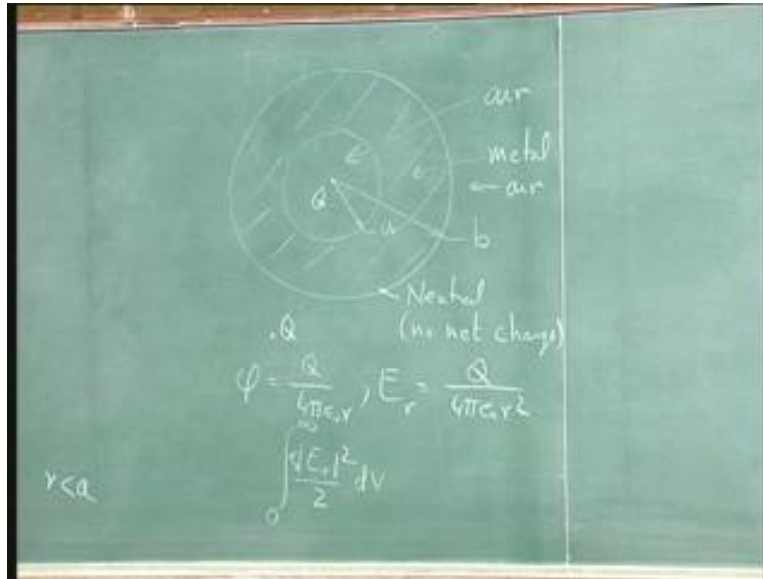
That energy was already there. I did not have to do work to create it. When I put charge together, to make a point charge of charge Q, this is the energy in that field. So now, if I have this energy in a point charge and I now put a spherical conducting shell around it, how is that energy going to change? I am going to do this problem in two ways and it is worth understanding the two different ways.

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First way, I will use Gauss's law. Well, I will...say I have a charge Q and I have an inner radius a, and an outer radius b, and I will put an imaginary Gaussian surface, a sphere of radius r. Now, this is inside the metallic region and we have already discussed that inside any metallic region, I cannot have electric field. So surface integral E dot d S is 0 because the surface is entirely inside the metal.

Metal cannot support any electric field. So that means Q enclosed is equal to 0. Furthermore, there is another calculation you can do which is, I can draw two of these Gaussian surfaces and I can talk about the region between them. In that case, it says surface integral S 1 minus surface integral S 2 is equal to 0 because each of them is separately 0, the difference of them is also 0. So the charge enclosed inside any such sphere is also 0.

This is the problem. There is no charge in the metal but there is no charge enclosed. The charge must be somewhere and the only place the charge can be is on the inner wall of the metal. So there is a charge minus Q on the inner wall and if you draw a Gaussian surface outside the cylindrical sphere, then total charge enclosed...well, the charge on the metal is 0 because metal has no net charge.

Charge at the center is Q. So out here, Q enclosed is equal to Q. So from Gauss's law, what we find is the electric field E r as a function of r is equal to Q over 4 pi epsilon naught r for r less than a. It is equal to 0 for a less than r less than b and it is equal to Q over 4 pi epsilon naught r, for b less than r. It is just coming out of the application of Gauss's law.

If I apply Gauss's law inside the air, I find the first result. If I apply Gauss's law inside the metal, I get the second result. If I apply Gauss's law in the air outside the sphere, I get the third result. So Gauss's law solves the problem for us, but I want to look at this problem slightly differently.

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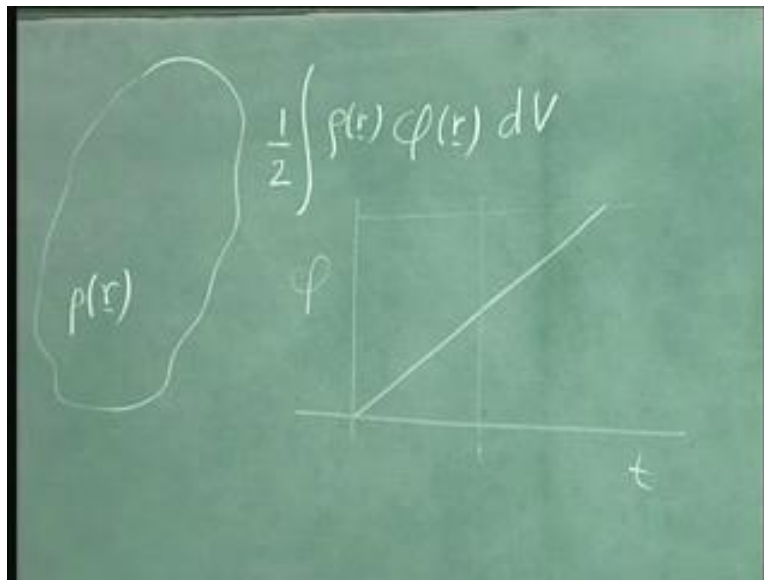
Now we had derived last lecture that the energy in any cloud of charge cloud described by rho of r is given by volume integral rho of r phi of r d V the factor of one half in front. We derived this by bringing charge slowly from infinity and building up this cloud and as a function of time, as we brought the charge the potential at every point built up linearly. And the amount of work we did was equal to the potential at any given time t.

So what happened was that, in a certain sense we were measuring the area under the triangle and that is where the factor of 2 came. Because, this rho times phi in a certain sense measures the area of the rectangle because this axis is time and not rho but the way I define my derivation charge was brought in in time.

So, in a certain sense this also represents rho. So, this was the derivation we did and we obtained that the total energy was one half rho times phi d V but in our current problem. We do not know rho. So, we would like to get a better expression for this. Let us see if we can do that.

Now we have one expression from Gauss's law which tells us that divergence E is equal to rho over epsilon naught. So, I can substitute for this rho by epsilon naught times divergence of E. Let me do that. So, this becomes equal to one half volume integral epsilon naught divergence of E times phi d V. Now this divergence can be combined with phi. The reason is, phi times divergence of E, can be written out. It is equal to phi times del E x del x plus phi times del E y del y plus phi times del E z del z.

Now I know from my calculation that when I have a product of terms and I take the derivative of the product, this is one of the terms it comes out. So, I can write each of these as del del x of phi E x minus E x del phi del x plus del del y phi E y minus E y del phi del y plus the third term del del z phi E z minus E z del phi del z. That is because if I take the derivative of this it gives me this term plus this term.

Now this term this term and the corresponding z term will give me. I will take the epsilon naught out. So, these 3 terms will give me divergence of phi times E because, you can see that is what it is phi times E the x component is phi E x the y component is phi E y the z component is phi E z and this term will become minus E dot grad phi. Well, we are almost done because I can now use a divergence theorem on this piece.

And here is what I get. Your book has the same derivation. So you can check with it what you get is epsilon naught over 2 surface integral phi E dot d S applying the divergence theorem here plus epsilon naught over 2 volume integral E dot E d V. That is because minus grad phi is nothing but E. If we go far away enough phi times E is going to go to 0. So this term you can drop. And so this integral which involves knowing both charge and potential can be reduced to a simpler form. It is equal to volume integral epsilon naught E square over 2 d V.

And the interesting thing is you can give an interpretation to this. Here the energy in the field was due to the fact that we brought charge in against a potential. But here, the charge is no longer visible. The potential is no longer visible. There is only a field here and we are saying that at every point in space due to the presence of field, you have a certain amount of energy present epsilon naught E square over 2. This particular form is very convenient and we are going to use it just now to solve our other problem.

But it will keep getting used so keep it in mind. I want to say supposing, I knew that there is some charge distribution rho in a metal, then I can apply Gauss's law again and what would I get, I would get that E of r is equal to Q over 4 pi epsilon naught for a less than

r… sorry b less than r and for r greater than less than a that is for regions inside and regions outside its nothing but coulomb's law.

In the region of the metal I am going to assume some charge density I do not know what it is. So, it is going to be Q plus integral a to r rho of r prime d V, okay? Divided by 4 pi epsilon naught r square that is what Gauss's law will give us. Because it is going to be charge enclosed and what is the charge enclosed its Q plus. Really I should write this as 4 pi r square, d r. I was writing it as dV to make it clear integral a to r rho of r prime 4 pi r prime square d r prime.

This is for a less than r less than b. This is completely general. If I knew rho, this is true. But let me write down the energy associated with this field. I will call it u it is going to be equal to integral 0 to a E r square epsilon naught over 2 d V plus integral a to b epsilon naught over 2 d V plus integral b to infinity E r square epsilon naught over 2 d V. So, there are 3 parts the part that corresponds to the inner air region. The part that corresponds to the middle region and the part that corresponds outside and regardless of what ever charge, I put down for the metal this region and this region are fixed.

That is the energy in the field is not a function of what I chose for the charge distribution inside the metal any charge distribution gives me the same coulomb's law in region one and region 3. The only thing that the charge distribution can change is region 2. So, I can write u is equal to u 1 plus u 3 these 2 regions plus u 2 and this is fixed and this is function of rho the second thing. That is very obvious from this integral is that I am talking about something that is always positive or it was 0 E square cannot be negative.

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So, when I take the integral of something, that is either positive or zero, the integral is always positive or worst case zero. So I have that the total stored energy is equal to some fixed part plus one more part that is either 0 or greater than 0. If you have studied anything about gravity gravitational systems and friction and all those kinds of mechanics problem, you know that systems like to reach a state of lowest energy.

If you want to ask what is the lowest state of energy for this system, the lowest state of energy would be if u 2 were 0 that would be the lowest energy state, it cannot be negative. So, best it could be 0, but what does it mean? If it is 0, it means that E r equals 0 for all a less than r less than b E r is 0 everywhere inside the metal which is nothing but what Gauss's law told us.

Gauss's law gave us this result. But, the result of Gauss's law is really saying nothing more or less than that if you allow an electrostatic field problem to stabilize, if you allow the charges to reach wherever they want to go subject to constraints, then the charges will arrange themselves in a minimum energy state. It is a very important idea and you will find that a lot of problems can be understood if you realize that ultimately electrostatics is about minimum energy states.

Okay, let me do a third problem now and this third problem is fairly interesting. I have a ring of radius a the ring has a charge lambda coulomb's per meter, this ring, the charged ring has a central centre of symmetry is the z axis and at t equals 0. I have put a charge q and I give this charge a velocity in the z direction velocity v naught the question is what happens, okay? What happens to the charge in time due to the electric field, let us work it out since the charge is on the z axis. It will remain on the z axis because the forces are all symmetric.

So, there is no force trying to send it away from the z axis. So, we need to know the force on the charge at any value z. I can work out the potential phi of z is going to be equal to. I am at a distance square root of r square a square plus z square. That is the length that is this it is given by Pythagoras theorem z square plus a square. I have one over 4 pi epsilon 0 and the charge the total charge which is 2 pi a times lambda.

Now this is one case where you can see how easy it is to work with electrostatic potentials, we took a lot longer to calculate the electric field. Whereas, the electrostatic potential I could write down by just inspection once I have got phi, I can calculate E the electric field is along the z direction E z of z is equal to minus del phi del z is equal to 1 over 4 pi epsilon naught and I will put this here 2 pi a lambda. Then I have to take the derivative. There is a minus sign coming from here and there is a minus sign coming from the derivative. So, those 2 signs get cancels out.

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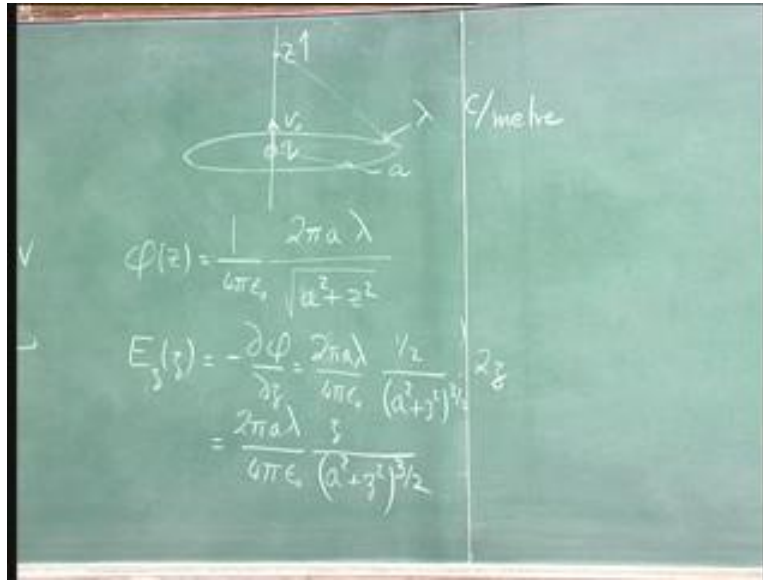
So, I get half over a square plus z square to the power of 3 halves times derivative of a square plus z square with respect to z which is 2 z the 2 cancels the half. And so I get 2 pi a lambda over divided by pi epsilon naught z divided by a square plus z square to the power of 3 halves. So, the electric field assuming that this lambda is positive is pushing away from the ring and its pushing away from the ring in both directions and the electric field along z is 0 at the origin. If you plot it E z of z, I will plot it on this plot itself in this figure itself it will be something like this.

So, it reaches a maximum the maximum value it reaches is approximately where z is of order a, that is where z over a square plus z square becomes the maximum and then it falls off. And you can see it falls off as one over z square because for large z. You can neglect the a square. So, it is z over z cubed z square to the power 3 halves is z cubed. So z over z cubed is one over z square. So it is coulomb's law very far away. But the electric field actually grows as you go close, to the come close to the ring.

Now let us assume lambda is greater than 0, okay? It is perfectly general there are 2 cases one is q greater than 0 if q is greater than 0 the charge starts moving and as it reaches any point it keeps seeing and accelerating electric field. The force on the charge is equal to q

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E z also greater than 0 is the force along z. So you have m d v d t along z is equal to F z is greater than 0 you started with v z that is greater than 0 and it keeps increasing. So, this is a kind of switch system where you are in z or in time I should say it started with a velocity v naught and it increased and saturated this particle with charge q escapes. By that I mean the charge starts leaving speeds up speeds up and goes away to infinity, it does not return.

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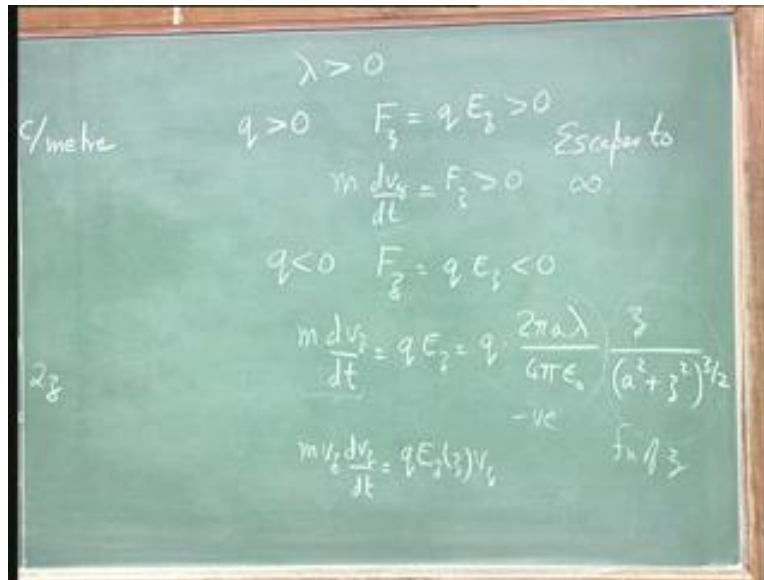
Now what happens if q is less than 0? Now you can see F z which is equal to q E z is less than 0. Therefore you start with some velocity. You start slowing down you reach some maximum amplitude maximum height z and then you start coming back then you go to a maximum negative height and you come back. And so you expect oscillations can we see that well let us write out the force equation force equation says m d v z d t is equal to q E z which is given by we already have the form for q.

So, m d v d z this is the force rate of change of momentum is equal to some constants which are negative lambda is assumed positive q is negative. So, this whole thing is negative times a function of z. This is the function of z you must be familiar with this

kind of equation when you have done the simple pendulum. This is the standard way of solving such problems which is multiplied both sides by v z.

So, you get m v z d v z d t is equal to q E z which is a function of z times v z but v z is nothing but d z d t m v z d v z d t can be written as d d t of one half m v z square that is rate of change of kinetic energy. The other side can be written as well. I wont write it out yet it is equal to the same expression q E d z d t. Now let me integrate both sides with respect to time, that is, I will just multiply through by d t so what do I get.

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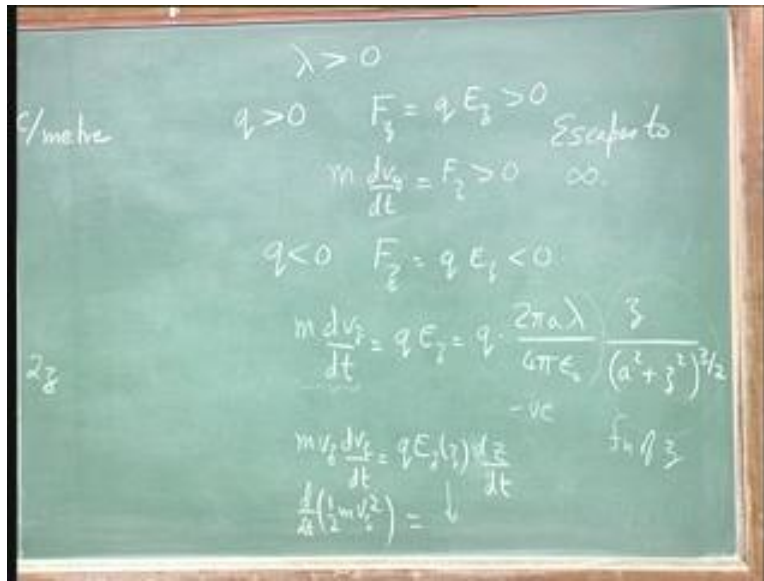
I get that d d t of one half m v z square is equal to minus q d phi d z d z d t which can be written as minus d d t of q phi. So, this is your kinetic energy this is your potential energy. So, you can combine the 2 together you get d d t of one half m v z square plus q phi equals 0 or something which is very obvious i can define a quantity E called m v z square over 2 plus q phi of z equals constant this would be my total energy this is kinetic energy this is potential energy.

So clearly what is happening is that the total energy of the particle is remaining constant as the particle moves in z it can exchange kinetic energy for potential energy or potential

energy for kinetic energy. But the sum of the 2 must be a constant this is extremely similar to Kaplarian orbits - very similar to gravitational orbits, is exactly the same problem. So, also the problem of a pendulum, now given all this what can we say about the orbits initial kinetic energy was v naught it was velocity. So, initial kinetic energy was m v naught square over 2 initial potential energy was q phi at z equals 0 which we can work out here.

We already have the form for q of z the final velocity is one half m v z square and the final potential is q phi of z. What this really means is supposing I want to look at orbits that manage to reach infinity at infinity this becomes 0. The potential energy due to the electric field keeps reducing as you go further and further away. So it is equal to 0 when you reach infinity this piece becomes what is called escape velocity. The same idea, as escape velocity for a satellite leaving the earth, there you would be talking about gravitational potential energy and you would be talking about kinetic energy of the satellite.

So, this is the velocity I would not escape velocity is probably the wrong term or say it is a terminal velocity or asymptotic velocity. This is my starting velocity and this is my starting potential. So, what it means ultimately is, I am inside a potential. Well, because my particle is being attracted back by the charge, that the charged ring, so I can draw a potential energy curve in z the potential energy curve in z says that very far away the particle has 0 potential energy no charges have any effect very close.

I have particles of opposite charge they actually have negative potential energy this is my potential energy. I now take a particle with some kinetic energy. Let us say I took so much kinetic energy. This would be my one half m v naught square if I take it, then this is my total energy. My system conserves total energy E equals one half m v z square plus q phi of z is constant. So, as the particle moves in z this is not perfectly parallel, but it should be the total energy of the particle does not change.

So, the particle initially has a strong velocity. But as at most further and further in z this velocity keeps reducing or rather it is kinetic energy keeps reducing. And finally the kinetic energy reaches 0 when it reaches 0 the particle has reached its highest point at that point it has no velocity left it just has potential energy. So, it turns round and falls back so the particle goes up goes up turns round comes back down goes up turns round and does an oscillation.

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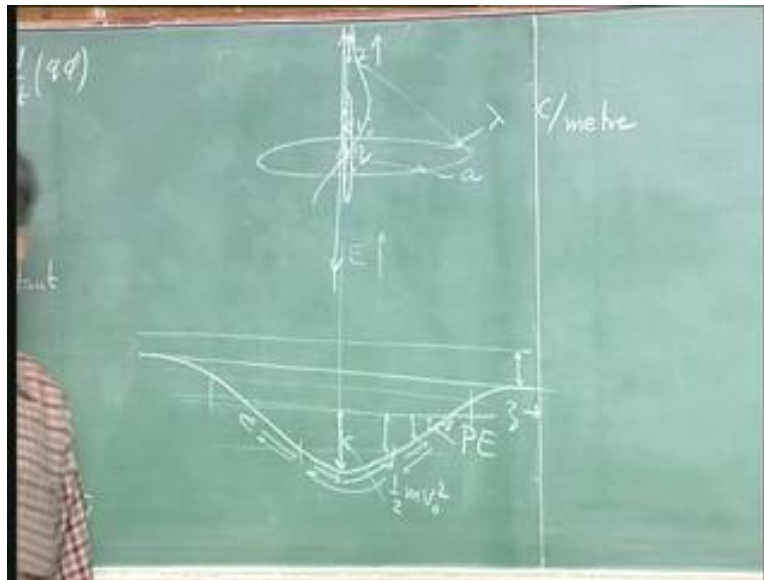
If, on the other hand, I had more energy the points were at turns or further apart, if it has less energy the points were the particles turns, turns back or closer apart. But supposing the particle had too much kinetic energy, if it had kinetic energy greater than this amount, then even no matter how far away it goes there is always some amount of velocity left. Some amount of kinetic energy left and this is the particle that has greater than escape velocity with respect to the hook. So, you will get trajectories where particles leave come back and oscillate and you will have other trajectories where the particle just goes away. All of this is very analogous to mechanics is very similar to what you see in a pendulum. It is very similar to what you see in satellites and it is very similar because the equations are very similar and it is always important to know the different parts of physics have the same equations because it makes it easier to study.