Introduction To Adaptive Signal Processing

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Lecture No # 09

Spectral Decomposition of Hermitian Matrices

Okay. So, again I start from where we stopped last time. we have discussed the Hermitian matrices, their eigenvalues, eigenvectors, properties and all those. Okay. We also told this thing that all the eigenvalues are real for a Hermitian matrix and if there are two distinct eigenvalues lambda 1 lambda 2 corresponding eigenvectors, they are mutually orthogonal and you can normalize each. So, that the norm square is 1.

In that case, there will be orthogonal orthonormal right. This is what I said. Now, we understand how you find out the eigenvalues and all if you are giving an equation like Ax for a general matrix not necessarily Hermitian equal to lambda x. This is all known stuff, but still it is A minus lambda I A matrix minus lambda identity matrix that into x equal to 0.

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

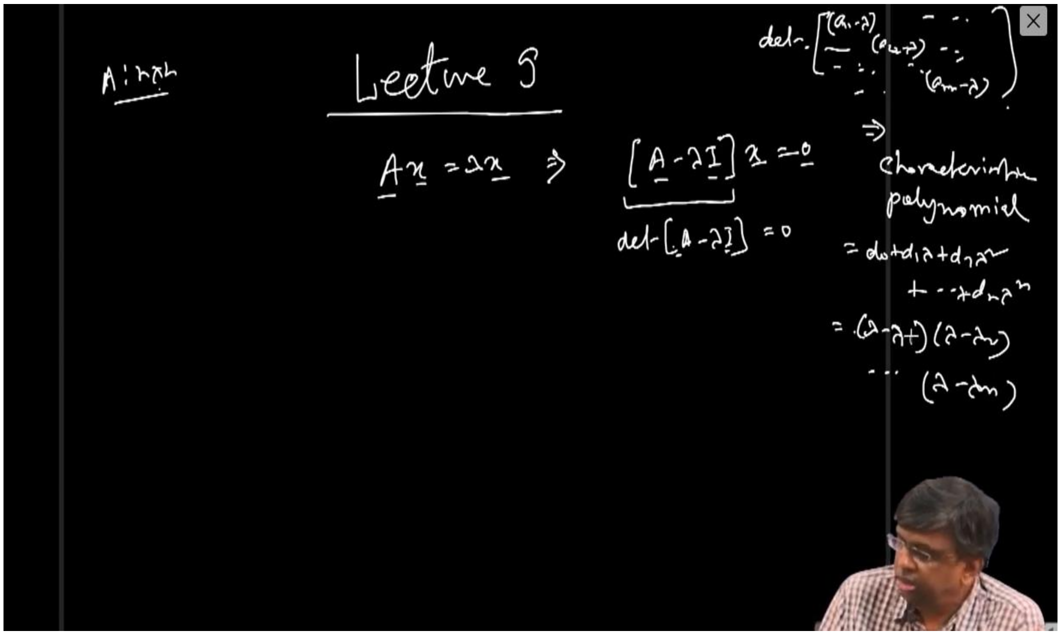
$$\left[\underline{A} - \lambda \underline{I}\right]\underline{x} = 0$$

So, therefore, for this matrix we have if this is A minus lambda I, if this matrix is invertible then obviously, x is 0 because if you call it B, B x is 0, B is invertible, So, B inverse 0 which is 0. But we are looking for non-zero solutions because eigenvector is non-zero. That means, this must not be invertible ok, which means this determinant must not be 0 ok. So, we should find out where it is determinant, I mean determinant at least for some points it should be 0. So, what we do we find out the determinant of this A minus lambda I all right and equate to 0 and find out for which values of lambda after all if you take the determinant what it will be? It will be a matrix A minus lambda I means diagonal entries what is lambda I matrix? It is a diagonal matrix with all diagonal entries lambda.

So, A minus lambda I means diagonal in from each diagonal entry of A lambda will be subtracted and then you find out the determinant. So, the determinant will be actually function will be a polynomial in lambda ok and polynomial order will be same as the size of the matrix if it is n cross n is because it will be like a 11 minus lambda a 22 minus lambda dot dot dot a n n minus lambda and this side other elements and if you take the determinant finally, you know these times I mean determinant of this will come as one top again that determinant will have these times determinant of this and like that. So, that product will be the will involve all of them and other products will have less number of such factors, but nevertheless I mean everything will be you can understand I mean you will have a polynomial in terms of powers of lambda and highest will be if it is n cross n lambda to the power n. So, if I equate that to 0 I can solve a polynomial I can factorize into n number of first order factors lambda minus something lambda minus something and all that. So, I will get n values of lambda maximum n values of lambda for which this is 0.

So, those values of lambda for which this is 0 then the matrix will not be invertible in that case only I can have non 0 solutions of x right. So, those will give you the eigenvalues. Now it can so happen that when you factorize the polynomial which is called characteristic polynomial. Determinant of this gives rise to characteristic these are known stuff polynomial. It could be something like you know some constant may be d 0 plus d 1 lambda plus d 2 lambda square dot d n lambda to the power n.

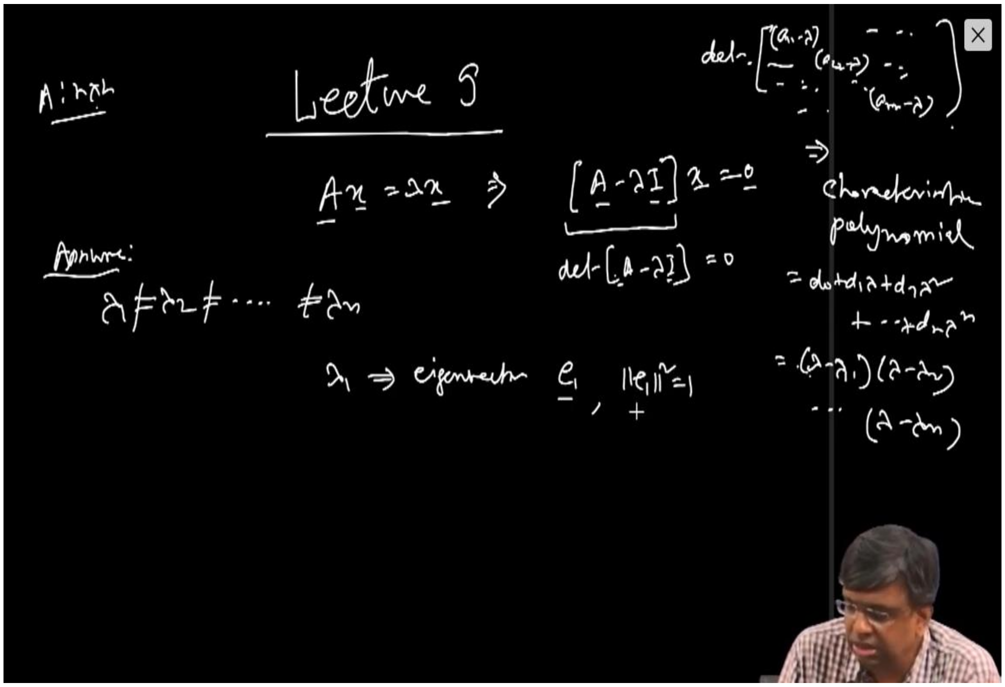
a is n cross n. So, that it will be you know having n terms. So, you factorize in first order factors. First order factors could be like you know I mean if you take it could be lambda minus may be some lambda 0 or lambda 1 rather.



In general lambda 1 lambda 2 lambda n will be distinct, but they can repeat also lambda 1 and lambda 2 could be same.

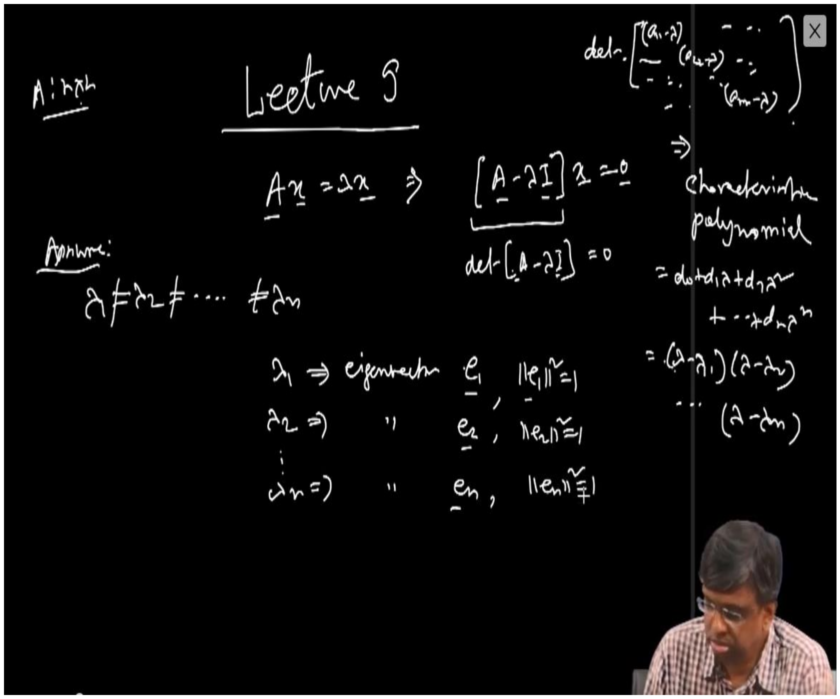
So, you can be in that case this factor will have power 2 lambda minus lambda 1 again lambda minus lambda 1. So, when you factorize maybe you have you have distinct factors all distinct. So, you get distinct eigenvalues lambda 1 lambda 2 lambda n because if you equate to 0, either lambda equal to lambda 1 will be eigenvalue or lambda equal to lambda 2 will be another one in which this is for which this is 0. So, eigenvalue lambda 1 eigenvalue lambda 2 etcetera. So, in a general case lambda 1 not equal to lambda 2 not equal to dot dot not equal to lambda n, but there could be cases where I mean some of the eigenvalues can be same, to the factor will repeat.

Now I will prove something for which I will assume the eigenvalues to be distinct. That is and I will prove something, but that result holds even when eigenvalues are not all distinct some are repeating, but that is beyond you at this stage. So, I will not prove that all right. And say suppose lambda 1 has eigenvalue, eigenvector say E 1 and E 1 norm square is 1 normalized. You said you have seen give me any vector if it is a non-zero vector in norm square is 1.



So, you can divide it by just the norm that is take every element of the vector divide them by the norm resulting vector has unit norm. So, that is doable. So, suppose the eigenvectors are already and if E 1 is an eigenvector is some eigenvalue lambda 1 any scalar times this eigenvector also is another eigenvector with the same eigenvalue that was we have seen. So, normalization does not change the eigenvalue. Eigenvector just changes to 1 whose I mean whose length is 1.

So, suppose lambda 1 has this lambda 2 has eigenvector E 2 again dot dot dot lambda n has eigenvector E n all right.



So, now what I do and none of them is a 0 vector and remember norm square of any any any vector is x h x. Yesterday we have seen h is the Hermitian transformation. And in a dot product between any x y is x h y these things were seen yesterday. Suppose I form a matrix T by putting the eigenvector side by side E 1 eigenvector as a first column, E 2 second column of the matrix, dot dot dot, E i ith column, dot dot dot, E n nth column all right.

$$||\lambda||^2 = x^H x$$

$$\underline{x}.\underline{y} = \underline{x}^H \underline{y}$$

So, n vectors and every 1 is length n cross 1 because matrix is n cross n. So, eigenvectors are length n. So, length is n total number is n. So, it is n cross n square matrix right. This matrix has a beautiful property if I do T H T, T H means remember this first I mean you have to take transpose and then conjugate.

So, transposition you can easily see E 1 is a column vector, now it will become after transposition it will be the first row. Then E 2 is the second column after transposition it will become the second row and likewise. So, after transposition it will be E 1 column vector. So, now, row version is E 1 h because you have to transpose and conjugate. Because T H, T H means you have to take apply h on this ok.

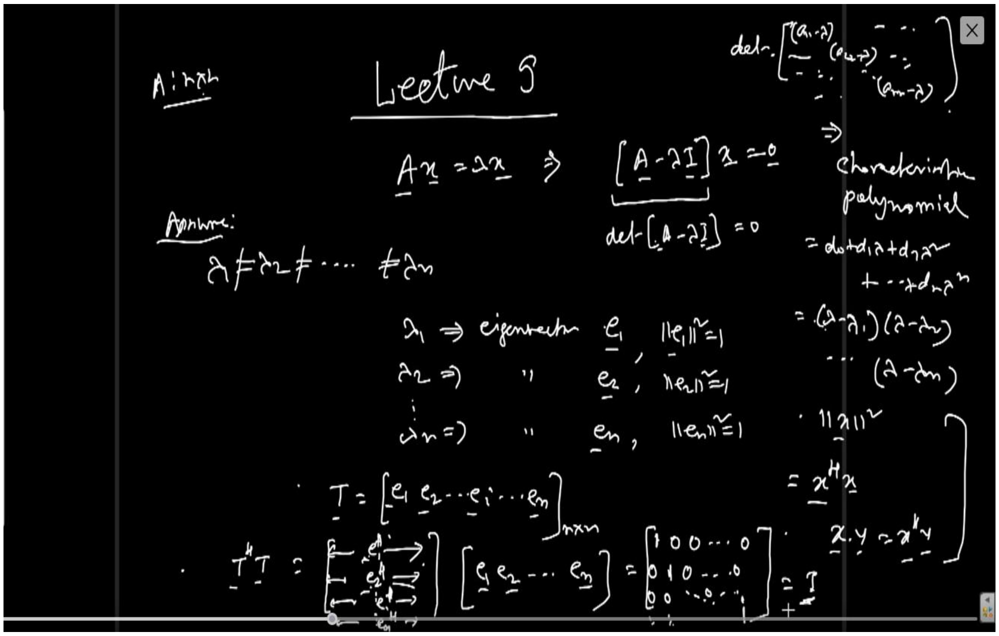
That is you have to, I mean, you have to take transpose of this matrix and conjugate every element. Ordinary transposition means what, you can easily check that all the elements here, they will go here. So, first column will become first row and then conjugate because I am looking for T H. So, not just conjugation not just transposition, but conjugation. So, it will be E 1 made into a row from column and then conjugated every element conjugated.

So, that is nothing, but E 1 h it will be a row vector. Then second column all these elements there will be the second row after transposition and then conjugating each element. So, it will become E 2 h dot dot dot E i h i th row and dot dot dot E N h N th row. This is your T H and then T is this. So, what will be the product let us start with this.

E 1 h E 1 row vector column vector that will be the 1 comma this element, this row times this column, but E 1 comma, E 1 h E 1 is norm E 1 square ok x h x is norm x square. So, E 1 h E 1 is norm E 1 square which is 1 normalized. E 1 h E 2 these are dot product between E 1, there is inner product between E 1 and E 2. x dot y means x h y. So, E 1 h E 2 means E 1 dot E 2 we have already seen.

So, E 1 corresponding to lambda 1, E 2 corresponding to lambda 2, and they are not same therefore, they are orthogonal. So, E 1 h E 2 will be 0, next will be E 1 h E 3 that will be 0, dot dot E 1 h E N will be 0, then E 2 h E 1 that will be this guy, again it is 2 it is 1 and corresponding Eigen values are distinct. So, for E 2 h E 1 which is the dot product between E 2 and E 1 that will be 0. Then E 2 h E 2, E 2 h next column E 2, E 2 h E 2 is the norm square of E 2 which is again 1. Then E 2 h next column E 3 that is 0 dot dot dot dot E 2 h E N 0.

Similarly, it will be like this it will become 0 dot dot dot 0 dot dot dot 0 dot dot dot all. So, it will be basically identity matrix.



So, this is a property that is T h T is identity.

$$\underline{T}^H\underline{T} = \underline{I}$$

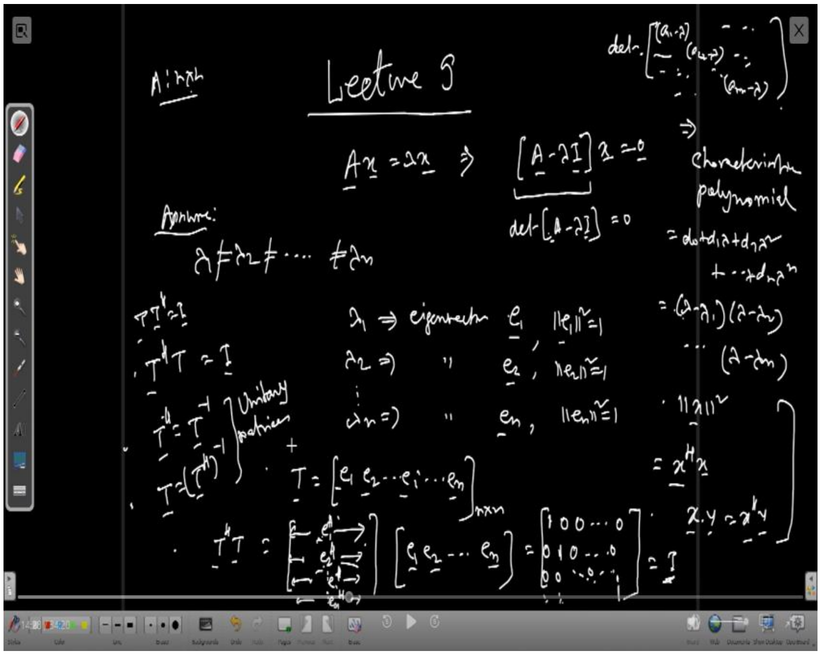
So, T is a square matrix. So, T h also square because you take transposition.

So, it was originally square N cross N. So, after transposition also N cross N and then just conjugated. So, square matrix square matrix. I told you the other day if there is a square matrix such that if A and B are suppose two square matrices and their product is I. then one is the inverse of the other.

That is B is A inverse, So that, A inverse is I and A is B inverse, So that, B inverse B is I. They must be square and if their product should be I, then one is the inverse of the other ok. So, that means, that T h is T inverse and T is T h inverse. Such matrices are called unitary matrices, U N I T A R Y. The beauty is if you want to calculate the inverse of this matrix you do not have to carry out all those you know in detail steps, just take the I mean given T take the Hermitian transpose that is conjugate every element and transpose or transpose and then conjugate.

Resulting matrix will be the inverse unitary matrices. That is for T is a unitary matrix means its inverse is nothing, but its own Hermitian transpose. That is T h T is I and therefore, T T h that is also I. Because one is the inverse of the other.

T h T I T h is the inverse of T. So, T T inverse here which is I ok. This is called unitary matrices and very very useful in signal processing all right. So, if the eigenvalues are all distinct I get N number of normalized eigenvectors and they are mutually orthogonal. So, if I construct a matrix T by putting them together side by side, T h I mean T product is identity that means, this T becomes unitary matrix right.

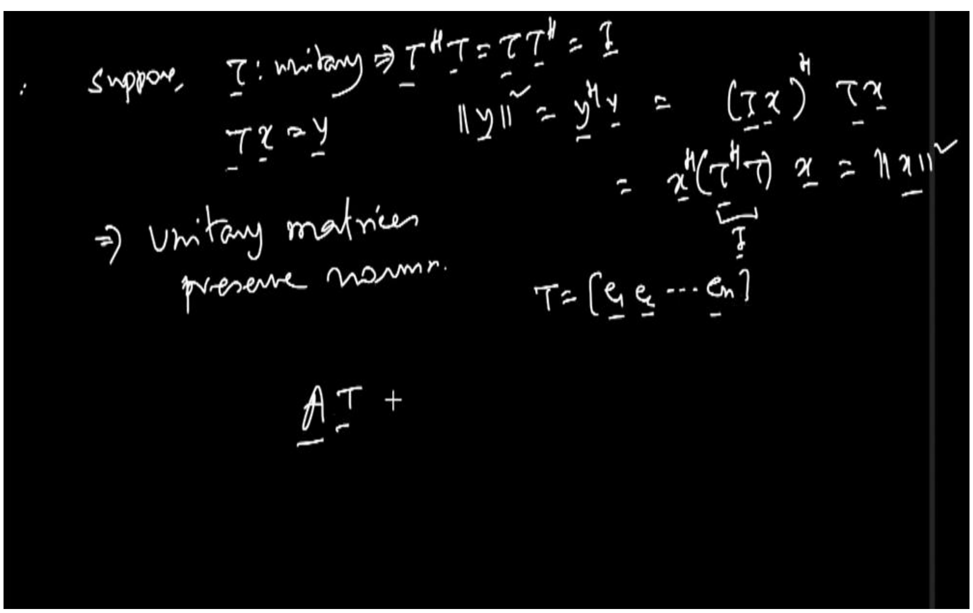


Next, another property of unitary matrix, suppose T x, x is a vector T is unitary, suppose unitary T or y unitary then T x, if it is if you are given x you apply T on this T on it call it y, Norm square of y will be what we all know y h y.

There is a length square of the norm or square of the length of y will be nothing, but y h y and y is obtained from x by this transformation, T working on some given x giving you y. So, if I replace y by T x h and again T x. So, it is x h T h then T x. But T is unitary means T h T and T T h are identity. So, this T h T it is identity matrix, identity matrix T x is x x h x is back to the norm square.

So, if y is obtained from x by applying on x a unitary matrix then norm square of y and norm square original norm square of x then even same. So, unitary matrix is preserved norm. It can change a vector x to y, but length does not change. This is another property all right. So, now suppose a matrix was given to me which is Hermitian and A T I am taking that same a matrix as earlier it has n number of distinct eigenvalues it is a n cross n matrix and I am assuming all the eigenvalues to be distinct.

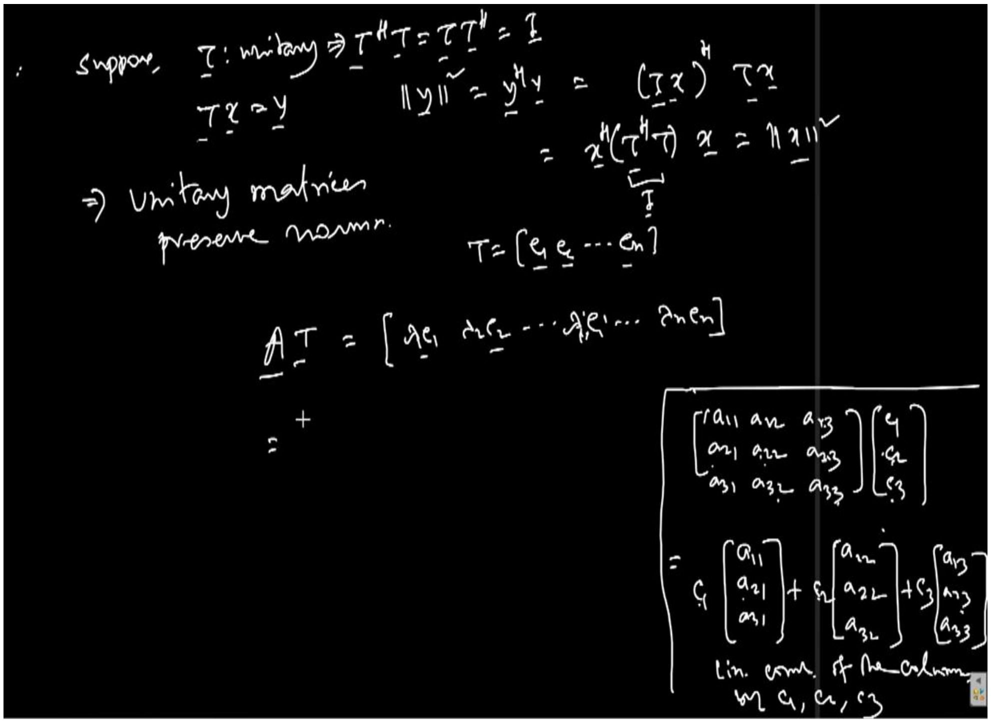
So, eigenvalues were lambda 1 lambda 2 dot dot dot dot lambda n corresponding eigenvectors were E 1 E 2 dot dot dot dot E n all these vectors are normalized this norm is 1 norms are 1 and they are mutually orthogonal, So, they are orthogonal, Then using them I construct the T matrix by putting those eigenvectors side by side T was E 1 E 2 just I am repeating for your sake E n this T is unitary.



Then what happens to a T, a T means a on E 1, a 1 E 1 is lambda 1 E 1. Then next column will be a 1 E 2 which is lambda 2 E 2 dot dot dot lambda i general column E i dot dot dot lambda n E n ok. Now I do something the side story. Suppose I give you a matrix a 1 1 a 1 2 just 3 by 3 a 1 3 a 2 1, a 2 2, a 2 3, a 3 1, a 3 2, a 3 3, and I multiply by some elements c 1 c 2 c 3.

we all know how to do it a 1 1 times c 1 plus a 1 2 times c 2 plus a 1 3 times c 3 will be the first element of the relative vector column vector, then a 2 1 into c 1, a 2 2 into c 2, a 2 3 into c 3, summation will be the second element and a 3 1 c 1 plus a 3 2 c 2 plus a 3 3 c 3 summation will be the third element. But we actually do not I mean carry out matrix into vector multiplication in those manners anymore. After I mean when we reach this stage we do not and from today I hope you will also not. You will see that this is nothing, but if you take the first column of this matrix a 2 1 a 3 1 if I multiply this by c 1 all the elements multiplied by c 1 then take the second column a 1 2 a 2 2 a 3 2 multiply all the elements by c 2 and take the last column a 1 3 a 2 3 a 3 3 multiply by c 3. Then if you add you will see a 1 1 c 1, a 1 1 c 1, a 1 2 c 2, a 1 2 c 2, a 1 3 c 3, a 1 3 c 3.

So, you get the first element which you are getting here then second a 2 1 c 1 here a 2 1 c 1 ok, a 2 2 c 2, a 2 2 c 2, a 2 3 c 3, a 2 3 c 3 they are summation. So, you get the second element like that. This is what the actual meaning of matrix into vector multiplication, that is you take the columns and linearly combine them by those coefficients. Let us take the first column multiplied by the first element take the second column multiplied by the second element take the third column multiplied by the third element like that and add it is called linear combination, linear combination of the columns by c 1 c 2 c 3 all right.

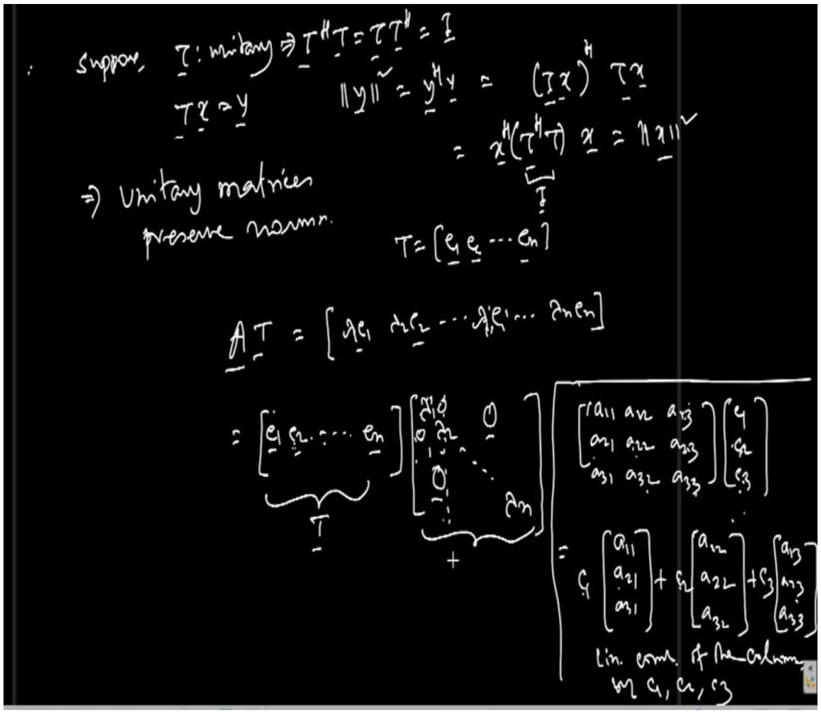


So, using this right hand side I can write like you just see e 1 e 2 dot dot dot dot e n and I form a diagonal matrix at this side 0 that is all the elements here are 0 this is a big 0 all the elements here are 0 that is the diagonal elements.

You can verify this equal to this because we will be, so just consider this matrix times the first column like this matrix times the column vector this matrix times is first column what it will be it will be lambda 1 times e 1 then below is 0, 0 times e 2, then below is 0, 0 times e 3 like that and their addition 0 0 0 lambda 1. So, it will be lambda 1 times e 1 other side you know getting multiplied by scalar 0. So, they go so lambda 1 e 1 that is what you have here. Then again take the second column the second column first guy is 0 here. So, this matrix times is second column vector will be what is 0 times first column e 1 lambda 2 times second column e 2 then again 0 times next dot dot dot 0 times last.

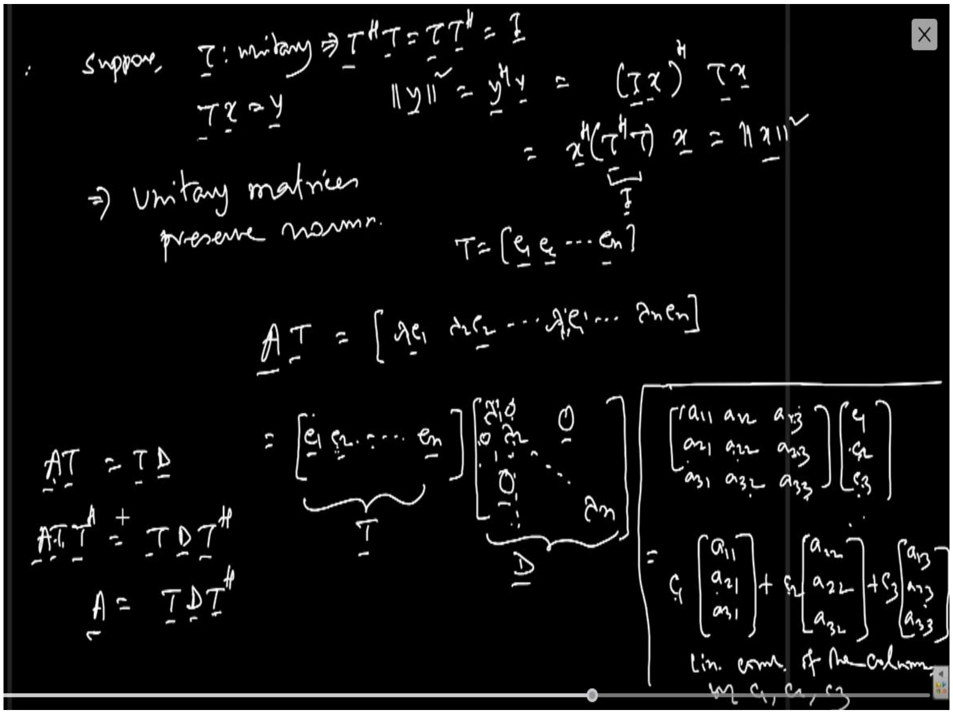
So, lambda 2 into e 2 only that will remain and likewise ok just use this, but what is this matrix this is good old T by definition the T I constructed e 1 e 2 e n and this is the diagonal matrix with the eigenvalues distinct eigenvalues of A and they are all real because A is

Hermitian. So, it is a diagonal matrix for all the distinct eigenvalues ok and eigenvalues are real.



So, I call it D D for diagonal matrix T so that means, I have an equation A T is T D. So, suppose I do A T I multiply this by T H.

After all A T is a matrix T H is a matrix. So, I can multiply them so here also T D T H, but T T H is identity T T H is identity because they are unitary. So, it becomes i A into i is A. So, in this case A can be written in this product form This is a very important result product form or sometimes called factorization of A. So, in the product of three matrices as though I factorize into a product of three factors T factor T is a factor D is a factor T H is a factor factorization, where D is a diagonal matrix T is a unitary matrix all right.



This is a beautiful result I will be using it later. If A is Hermitian this is always doable. This implies two things given A equal to T D T H number 1 Determinant of A will be you know determinant of this product which is determinant of T into determinant of D into determinant of T H. But T H is same as T inverse and determinant of T inverse is 1 by determinant of T. And that cancels with this, So, you are left with determinant of D and what was D? D was the diagonal matrix of eigenvalues this side 0 this side 0 being 0. So, determinant of this matrix is nothing, but product of the eigenvalues product I denote like this I equal to 1 to N lambda I.

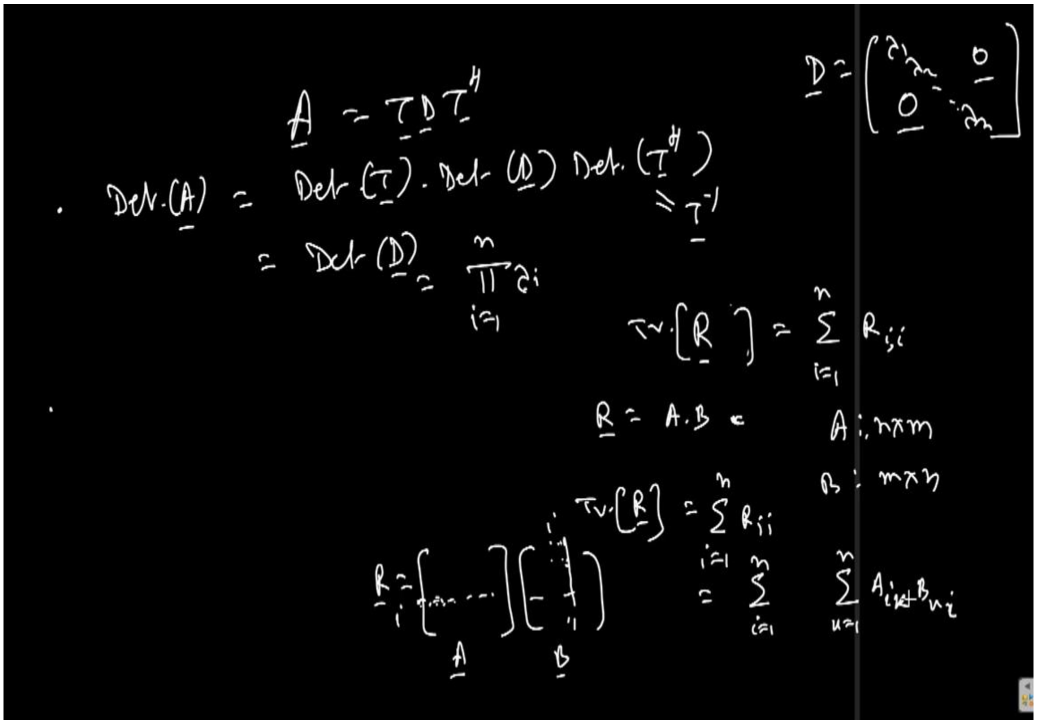
This is lambda 1 lambda 2 dot dot dot lambda N which means if every eigenvalue is nonzero, the determinant is nonzero which means the matrix is invertible. On the other hand if A is Hermitian and it has at least one eigenvalue 0 then determinant is 0 which means the matrix is not invertible understood ok matrix is not invertible. So, A is Hermitian and in that case if the A is given to be Hermitian in that case if all the eigenvalues are nonzero, positive or determinant does not matter, but if they are nonzero, determinant is nonzero. So, they are invertible, but if any eigenvalue or even a single eigenvalue becomes

0 means I mean determinant is 0, So, matrix is not invertible. Another property called trace, trace of a square matrix if I give you a matrix say D not D D is used here.

So, if I give you a matrix say R say N cross N ok It is a square matrix its trace means trace t r a c e trace t r a c e trace I am writing t r dot means you just sum up all the diagonal elements that is R its ith ith diagonal element you can forget this M cross N here you just indicate the size. So, I equal to 1 to N there is r 11 r 22 r 33 44. So, those are the diagonal elements sum of the diagonal elements is called trace. Now, suppose R is a b where a can be, a is maybe N cross some M and b is M cross N. So, N cross M, M cross N, product is N cross N, that is what R then trace of a b will be what trace of R is this summation r ii, but what is r ii? R is obtained R is like this a and b.

So, r i comma i will be what i th row of a times i th column of b ok. i th row means a i 1 a i 2 a i 3 like that and i th column means a 1 comma i 2 comma i 3 comma i and they are multiplying and getting added. So, r ii any r ii will be what a i 1 b 1 i a i 2 b 2 i like that. So, they are summation. So, a i, bigger index k, k 1 to N, a i 1, So, then b 1 k is 1 then b 1 b k i.

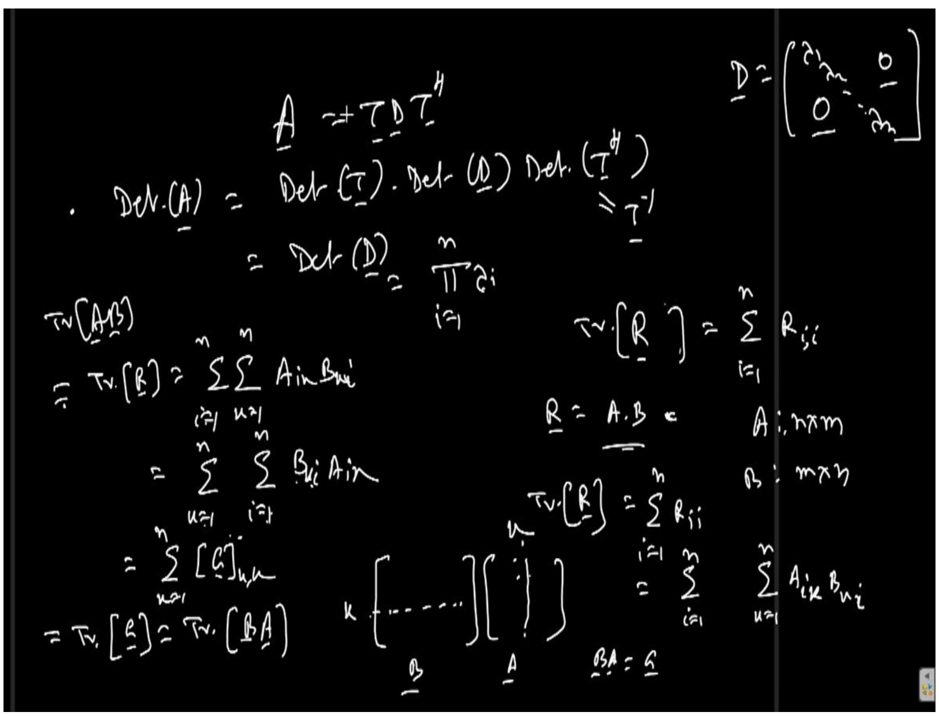
If k is i k is 1 a i 1 here b 1 j here 1 i then if k is 2 a i 2 b 2 i here and like that. So, this times this next times next next times like that and their summation all right This is a trace.



Now that means, trace of R is double summation i k b k i. Now see this double summation has a property, what is happening? First you are fixing i at 1 then for that i you are putting 1 here then you are running k from 1 to N there is 1 to N 1 to N then you are taking i equal to 2. So, first you took i equal to 1 i equal to 1 and you run k from 1 to N k from 1 to N freeze this summation then move i to 2 again run k from here k from here ok.

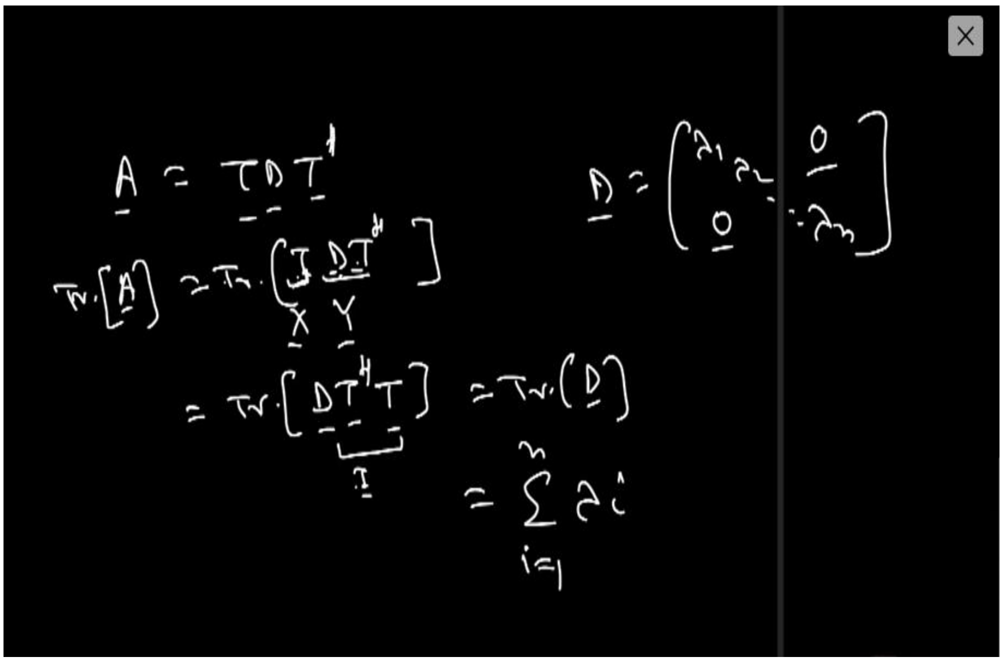
freeze and like that and then you are adding. You will get the same thing if you first fix k at 1 and move i over this range then again k here move i over this range and same here ok. So, you can interchange that too instead of fixing i first at a value and then moving k over the range you can fix k over a value then move i over the entire range then take the next value of k again move i over the same range and so and so forth. You will cover the same grid i equal to 1 to N k equal to N to all the points integer points you know for i and k they will be covered. So, which is nothing, but called interchanging the summation order that is first you take summation over k which means every time you give a value to k hold that fix for that move i i equal to 1 to N again another k hold that and then move i to 1 to N like that and this does not change, but i for convenience write B first B k i A i k what does it mean? It means this suppose I have instead of A B I had B A kth row and kth column. So, product matrix I am finding out say k comma k kth row kth column here multiply will give me the product matrix is something called G.

So, G k comma k kth diagonal entry I am finding out suppose. what will be the expression B k comma 1 see k k 1 and then 1 k A 1 k then B k 2 A 2 k again B k 2 because k is fixed A 2 k and likewise isn't it? So, this will be if you call it this matrix B A let us say G This will be G matrix its k comma kth element where G is B A ok. So, if you protect the product call it G its kth diagonal element and now you are summing all the diagonal elements 1 1 2 2 3 3 4 4. So, it is nothing, but trace of G and G is B A. So, trace of B A this proves the result and R was A B. So, this trace of A B was equal to trace of R, R is A B and up to here.



So, trace of A B is same as trace of B A ok trace of A B is same as trace of B A and A is this. So, next thing that we do today is you are giving the matrix A is T D T H ok and if you have to find out the trace of this square matrix which is Hermitian trace of A it is trace of T D T H ok. You take this matrix to be may be x and y this is dT H suppose is y and this is x, but we have seen trace of any x y is same as trace of y x you can bring dT H first and T second, but then now T H T if you combine this identity this is usually matrix D into identity is D. So, trace of D what was D? D was a diagonal matrix with this eigenvalues of A other elements are 0. So, trace is just summation of the eigenvalues and remember eigenvalues are real.

So, if A is Hermitian is another property its trace is summation of the eigenvalues and therefore, trace is real ok trace of R, trace of A is this summation of the eigenvalues all right.



So, I stop here today and we have covered some good properties of this Hermitian matrices There is a special class of Hermitian matrices called positive definite matrices. So, I will just cover the minute it will take 10 minutes or so and then we will go to random process random waveforms and all that with that background later we will move to adapt optimal and then adaptive filters and then some examples and then again back to adaptive filter higher versions of adaptive filter and all that ok that is all with that. Thank you very much and good way from here. Bye bye.