Introduction To Adaptive Signal Processing

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Lecture No # 08

Eigenvalues and eigenvectors of Hermitian Matrices

So, we continue from where we stopped last time, but before that, So, let us assume that you have some idea about eigenvalues eigenvectors of a matrix, but nevertheless for any square matrix A say n cross n x and x is not 0 Must this is very important it is not a 0 vector is an eigenvector with eigenvalue lambda if you do this product A x. See in general x is a vector A times x will be another vector y. There is no guarantee it will be in the same direction as x, but if x is an eigenvector the resulting thing will be just x. All the elements are either amplified at or contracted by same scaling guy lambda. So, x remains x just its magnitude changes all the guys, all the coordinates' elements they get a scaled up or down by common factor lambda.

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

Then it is called eigen lambda is an eigenvalue corresponding eigenvector is x, but x must be non 0.

For 0 vector it is trivially satisfied A 0 is 0 for any lambda, lambda into 0 is 0. So, trivially satisfied. x equal to 0 ruled out. This actually means that of course, you are a 3-dimensional vector this you have studied any vector here any vector here will have said this is a x is x 1 x 2 x 3.

So, any vector here we have coordinate sub x 1, sub x 2, sub x 3, x. If you take some 3 by 3, A 3 by 3 matrix times x, some given matrix A 3 by 3 times x you may get another vector may be in this direction this is your A x, but if x is an eigenvector, then resulting vector also be some lambda times x. That means, x 1 what is the value that will get scaled up or down by lambda This also this also, So, direction will not change. May be this much will be your this. So, direction does not change.

So, in general it does not happen in general a vector when you multiply it by a free multiply by matrix you change direction and magnitude both you go elsewhere like this, but in the case of eigenvector direction remains same. There is a magnitude scales up or down by a confactor lambda alright. Then claim for Hermitian A there is if A is a Hermitian matrix all eigenvalues are real. Now, here suppose take Hermitian transposition of LHS and RHS of 1. So, that means, ABH is BHH.

So, here x x H AH is now lambda is a scalar, but as I told you can view it as a 1 cross 1 matrix. So, that into x it will be x H lambda H, but lambda H is actually it is a 1 cross 1 matrix. So, transposition will keep it as it is and only conjugate. So, it will be lambda star only and lambda star is a scalar that is you can write as lambda H, but lambda H is lambda star. In the next slide we write that and now here x H AH, but A is given to be Hermitian.

$$\underline{x}^H \underline{A}^H = \underline{x}^H \underline{\lambda}^H$$

$$x^H \underline{A} = \lambda^* \underline{x}^H$$

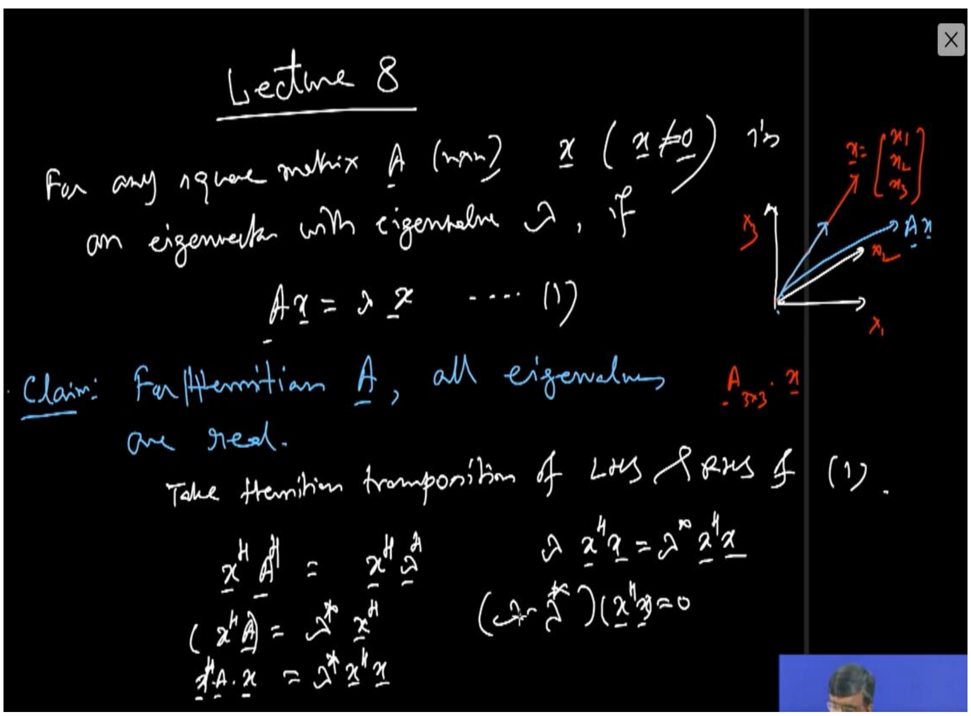
So, AH is same as A. So, x H A and here lambda H is lambda is a 1 cross 1 matrix. So, a transposition is as it is only star. So, it will become lambda star and lambda star is a scalar. So, scalar I can write in the front also this is a row vector times a scalar or you can write this also lambda star x H.

So, now this is a row vector times a matrix. So, it is a row vector, this also, this is also a scalar times a row vector. So, now what I do I take this row vector x H A as it is and multiply by a column vector x. So, row vector x H A is a row vector that times column vector x. So, row into column will be a scalar here also lambda star x H x, but now x H A x I can take this A x together first and we are given A x is lambda x.

$$\underline{x}^H \underline{A}.\underline{x} = \lambda^* \underline{x}^H \underline{x}$$

So, A x I can write as lambda x lambda can be brought out. So, it will be lambda x H x, A x earlier I had x H A x, but if I carry, I can carry out this product first A x first and then x H with that, but A x is lambda x. So, I write lambda x lambda is a scalar number. So, I can write it in the front, lambda can be 2 3 4 anything, that I can write in the front and x H x row vector column vector. So, it is a scalar that is same as as before lambda star same x H x.

So, I take this right-hand side this quantity to the left-hand side x H x is common. So, I can make it like this equal to 0. So, row vector into column vector this is a scalar and this is a scalar one scalar into another scalar equal to 0. So, either this is 0 or this is 0.



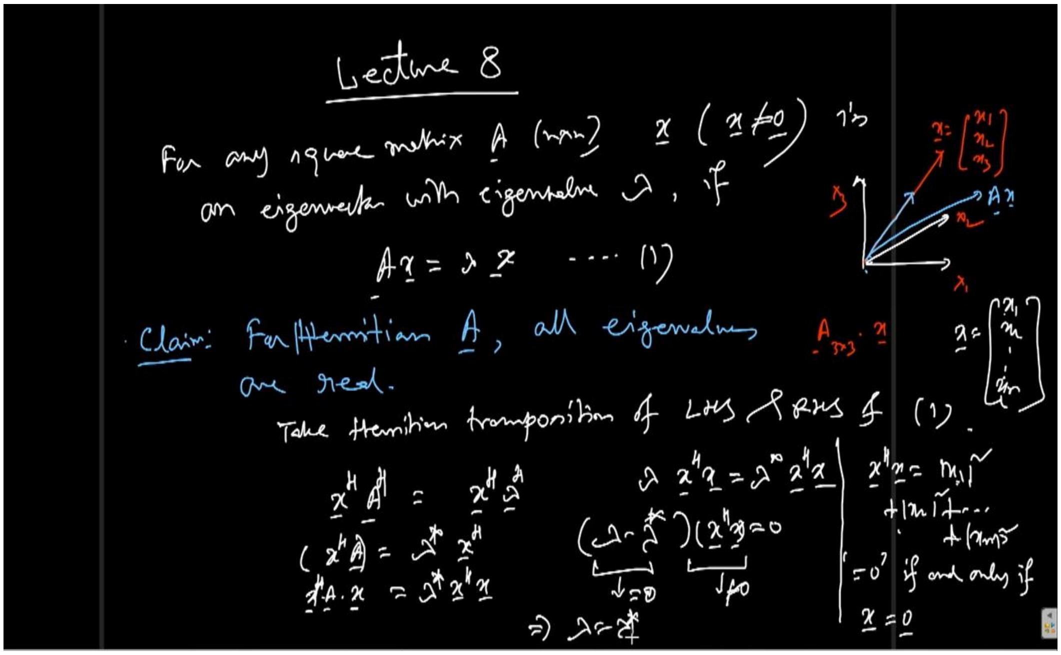
Now, let us see x H x, x is this vector x has these components like x 1 x 2 x 3 in general x 1 x 2 dot x N.

So, x H means it will be a row vector, this will be instead of column a row and then x 1 star x 2-star x N star. So, row vector is those and again this x, we carry out the product, it will be x 1 star with x 1 then x 2 star with x 2 dot dot dot ... So, this is very simple you can try yourself just make it a row.

So, it will be a row vector x 1 with a star x 2 with a star dot dot x N star and then a column vector this column vector if your product multiply x 1 star with x 1, x 2 star with x 2 that is what you have and their summation all right. Now, you see look at this thing mod makes it real and square. So, this cannot be negative, this can be always positive or 0 same here, either positive or 0 or positive or 0 and there is always plus sign. So, can this summation be negative can be 0, if it is 0 what happens? Because they are all contributing positively plus plus plus it is not that somebody is plus another person also plus, but there is a minus sign, so, they cancel each other no. all are plus, this plus, this plus, this plus, this ok.

So, everybody either is positive or 0 and their contribution is getting only added not subtracted from one to other. Therefore, if whole thing is to be 0, everybody has to be 0 then 0 plus 0 plus 0 is 0. Because there is no question of cancelling out contribution of one term from another term because all terms are added only. So, if they are the whole thumbs of thumb has to be 0 everybody has to contribute in a 0 way, 0 plus 0 plus dot dot 0 then only 0. That means, this is equal to 0 only if and only if everybody is 0, x 1 0, x 2 0 that means, x is a 0 vector, but x is an eigenvector.

So, it cannot be 0 ruled out because x is an eigenvector. Here if x is an eigenvector by definition, it cannot be 0 and if it is not 0 this whole thing has to be positive it cannot be 0. So, therefore, this quantity is non 0 which means this quantity is equal to 0, which means lambda equal to lambda star which means eigenvalues are real all right very important.



So, eigen I mean if A is a Hermitian matrix it is all eigenvalues are real. Next we have seen in the case of the 3 dimensional world, suppose you have got a vector x it has coordinates x 1 x 2 x 3.

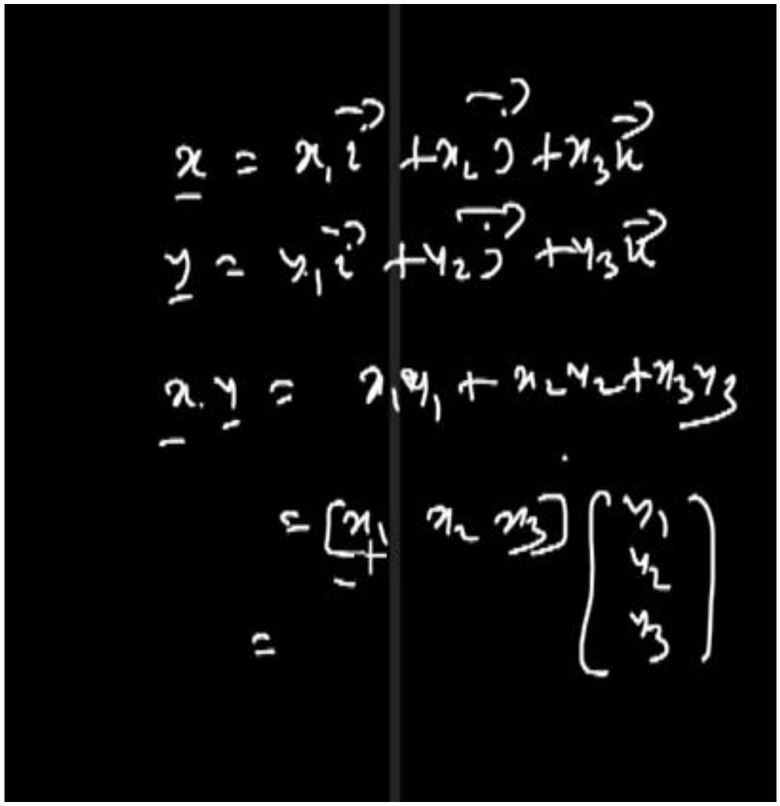
So, your vector algebra you write x 1 i, you need vector in the x direction x 2 j and x 3 k. Another vector supposes y, y 1 i, y 2, j y 3 k. Then you have got the dot product that you do will be x 1 y 1, i dot i 1, i dot j is 0, i dot k is 0 likewise. So, it will be x 1 y 1 plus x 2 y 2 plus x 3 y 3. This term is nothing, but you can write like this x 1 x 2 x 3 y 1 y 2 y 3.

$$\underline{x} = x_1\,\vec{i} + x_2\vec{j} + x_3\vec{k}$$

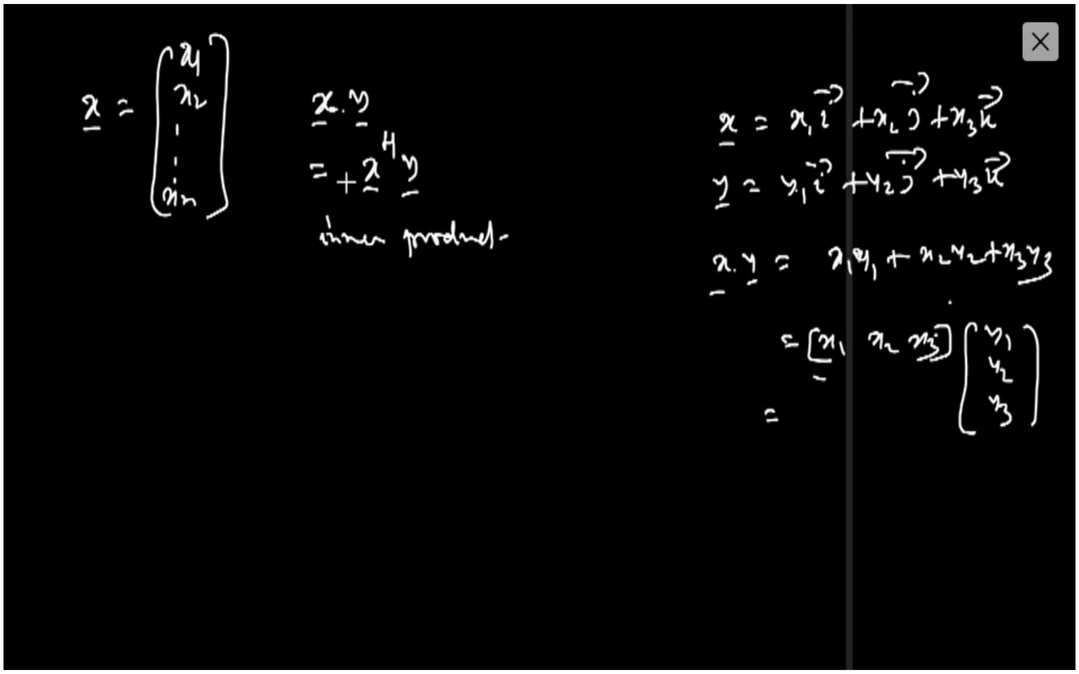
$$\underline{y} = y_1\vec{i} + y_2\vec{j} + y_3\vec{k}$$

$$\underline{x}.\underline{y} = x_1y_1 + x_2y_2 + x_3y_3$$

There is a coordinate vector, this you can, if it is a coordinate vector of x, it is a coordinate vector of y, say x transpose y. You can also say x Hermitian y because they are real here. So, conjugate has no meaning you can be as well instead of ordinary transpose you can write as x h y and instead of, now in our case instead of just 3 coordinates we will take general x 1 x 2 x 3 dot x n, they can be conjugate also. So, this dot product will be not just ordinary transposition, the more general case, coordinate vector x 1, x 2, x n, that is we will define x as not just x 1, x 2, x 3, it will go up to x n and then we will take x dot y. Actually, this is in our case denoted by another notation, but its okay, I will not bring that.



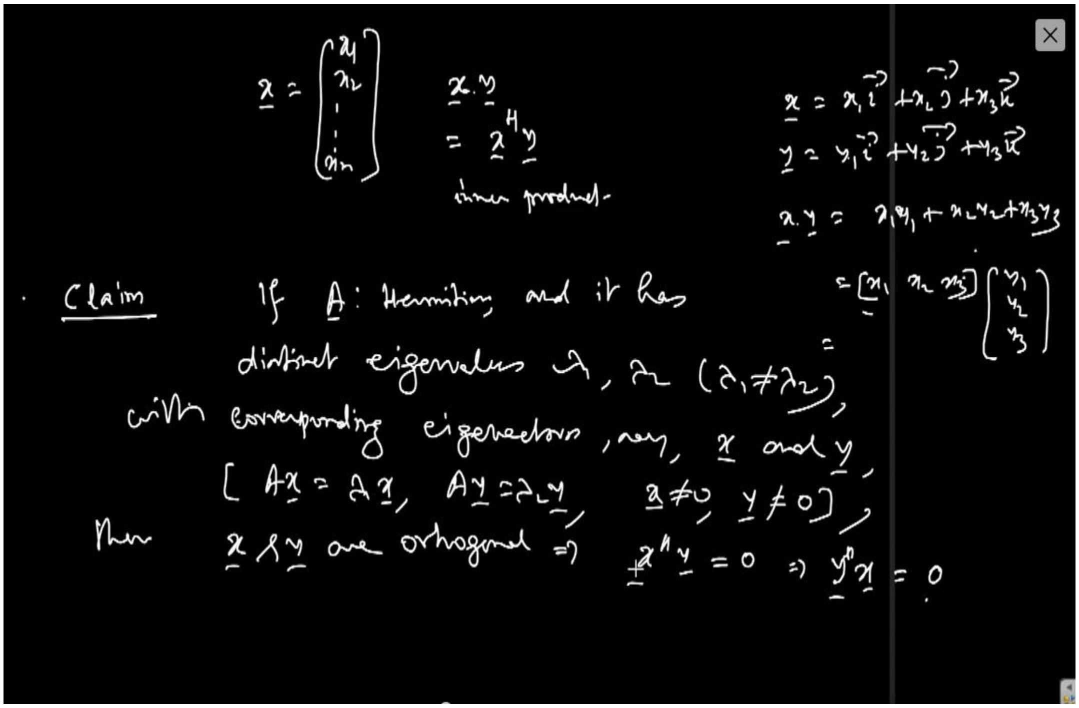
x like dot y it will be x Hermitian y. So, it will be like x was a column vector, it will be a row vector, everybody will be conjugated. Here there is no conjugate they are real, but I am I have to bring this conjugate and y and this is the definition of our generalized dot product called inner product all right inner product. So, if here we know that if we x and y they are orthogonal to each other at 90 degree, orthogonal to each other then the dot product is 0. Then this and this will be 0 all right.



So, in general x h y if that is 0 we see the 2 vectors are orthogonal all right. Then my claim is if a Hermitian and it has distinct eigen values lambda 1 lambda 2, that is lambda 1 not equal to lambda 2, but they are real. We have already proved that they are real. Then we it has distinct eigen values and corresponding eigen, with corresponding eigen vectors say x and y, that is a x, lambda 1 x, a y, lambda 2 y and none of them is a 0 x non 0 we always required.

Then x and y are orthogonal. That is if they are eigen vectors corresponding to different distinct eigen values then they are orthogonal x and y. These are claim, we show it, they are orthogonal. That is x h y equal to 0 and if I take the Hermitian of this side and this side, this side will be y h a b h is b h a if I take the Hermitian transformation of this side y h that x h h it will be x. So, y h x by right hand side this also will be 0. This is a scalar; sorry this is not a this is not a matrix this is scalar.

So, no underline there. So, 0 is a scalar, 0 Hermitian is first transpose, it remains 0, it is a 1 by 1 matrix. then conjugate 0 is real so conjugate has no meaning to 0.



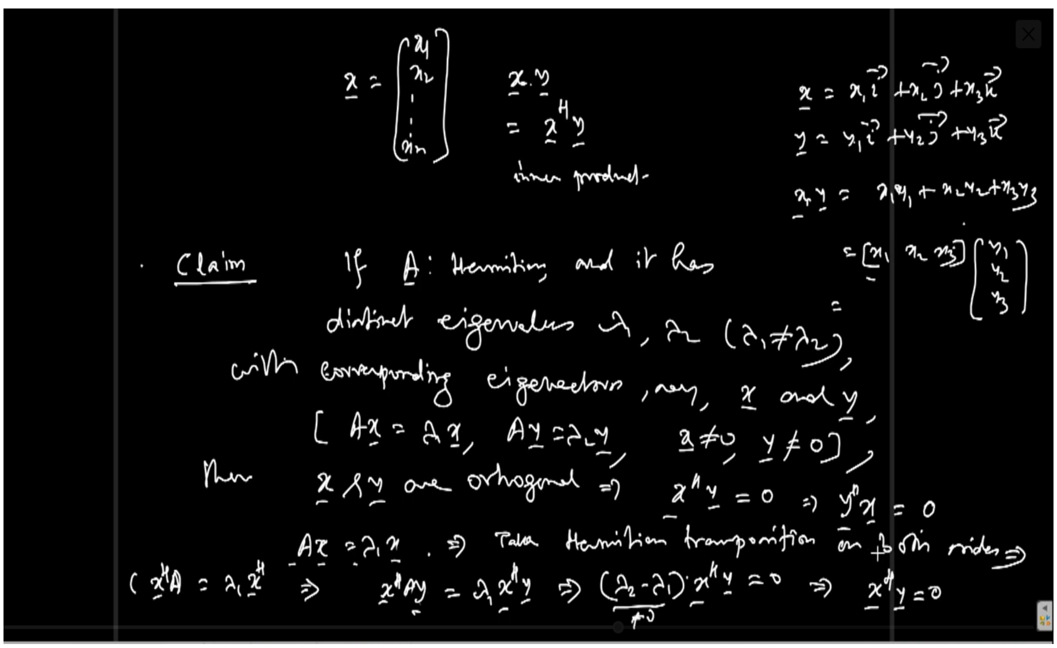
So, you either write x h y 0 or y h x 0 it is like x dot y 0 means y dot x 0. So, this claims we have to prove all right. So, now, we know a x 1 is lambda 1 x 1 is given here sorry a x dot x 1 I am very sorry a x is lambda 1 x.

I take the take Hermitian transposition on both side, that will give me x h a h, but a is Hermitian. So, I will write is a I now understand that because same exercise I did earlier where I first wrote x h a h they have said a h is a because a is Hermitian. So, that I am not writing I am jumping over those steps because we already have done, I think you can understand. a h is a that is why x h a h instead of a h I write a directly. And this will be again lambda 1 h x h lambda 1 h lambda 1 is a 1 cross n matrix.

So, its transposition will be lambda 1 itself and only conjugate, but lambda 1 is real. So, lambda 1 conjugate is lambda 1 it remains as it is and I since it is a scalar, I bring it to the front. So, it becomes this. If I treat lambda 1 as a 1 cross 1 matrix this into x. So, overall h means x h lambda 1 h, but lambda 1 1 cross 1 matrix its transposition is itself that is real.

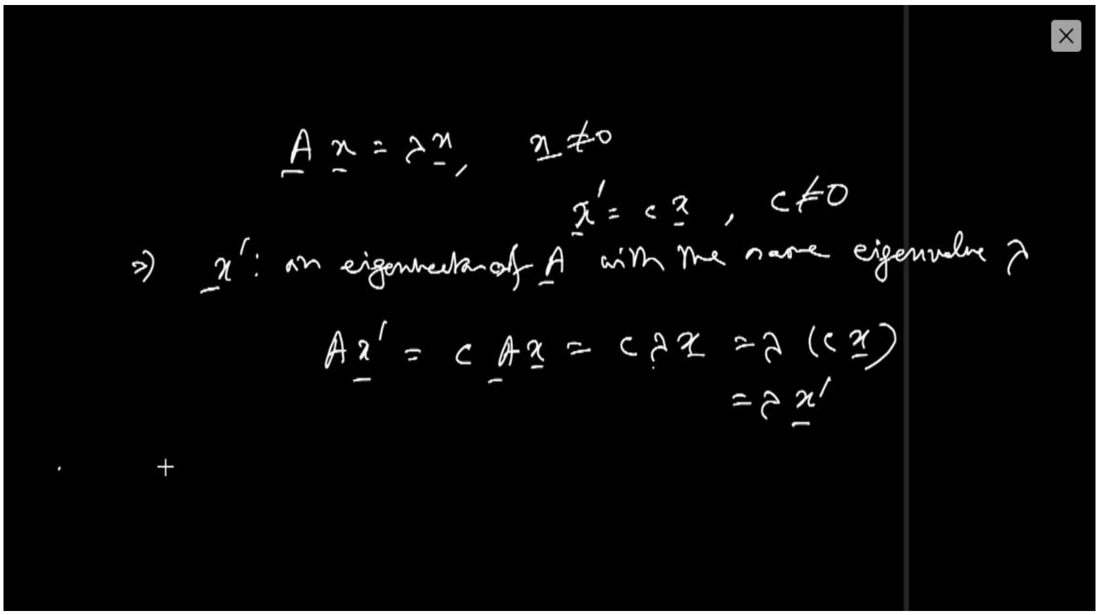
So, conjugate has no meaning. So, it remains scalar lambda 1 at the end, but since it is a scalar, I can just pull it in the front also. Then this is a row vector this is a row vector. So, here I bring y this row vector into y. So, again this side also y. So, x h a x a sorry a y is lambda 1 x h y, but now I take a y first a y as you have seen y is also an eigenvector lambda 2 is the eigenvalue.

So, a y is lambda 2 y lambda 2 is a scalar. So, lambda 2 I pull in the front. So, it is lambda 2 times x h y it is lambda 1 times x h y. So, it will become I can take both on the left-hand side, x h y x h y that is common equal to 0. So, x h y is a scalar row vector column vector is a scalar either that is 0 or this is 0, but this is non 0 because lambda 1 and lambda 2 they are distinct not same which means x h y is 0.



That is as I told x and y they are orthogonal to each other alright. Next, we have seen a x for any matrix not necessarily orthogonal a x is lambda x, x non 0 and lambda is the eigenvalue, x is the eigenvector. Then suppose I say new vector x prime is just some scalar times x. Then x prime also an eigenvector of A with the same eigenvalue lambda. Because if I write a x prime do I get lambda into x prime? Like a x is lambda x we will see yes, x prime you replace by c x put c x here c is a scalar, So, c I can pull in the front.

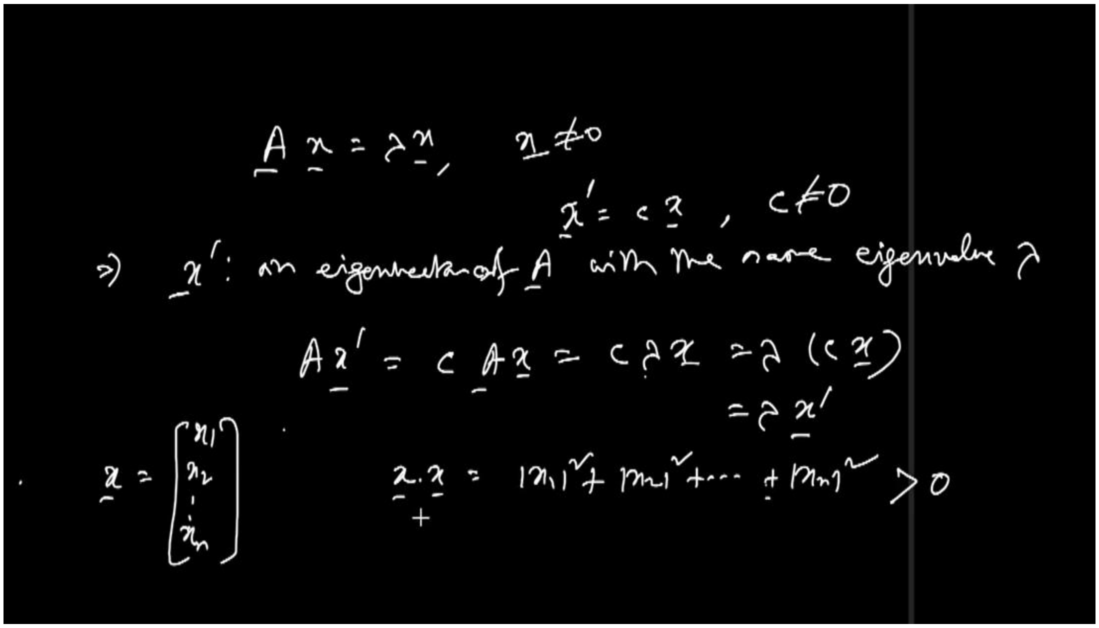
It is a x a x is lambda x. So, c lambda x the c is a scalar lambda is a scalar. So, c I can club with x just like lambda c x and c x is as before x prime, it will lambda times x prime. So, a x prime is lambda x prime and take c to be non 0 of course, because x is non 0 c you should not take 0 because x prime if x prime is to be an eigenvector this should not be 0 vector if you choose c to be 0 it turns out to be 0. So, c equal to 0 ruled out. So, then it is under that it is an eigenvector with the same eigenvalue lambda.



Now, we have seen vector x is dot product with itself. That is inner product with itself will be what, earlier it was x 1 star y 1 x 2 star y 2 inner product between x and y in the previous page x dot y. So, x 1 star times y 1 x 2 star times y 2 dot dot dot x n star times y n they are all added. Here x and y I mean y also x, So, x 1 star x 1. So, it will be mod x 1 square we have already seen x 2 star x 2 star dot x 2 mod x 2 square and dot dot dot since x is a non 0 vector all elements cannot be 0.

Then it is a 0 vector if even if only one element is non 0 mod square of that is positive. Others can be 0 or positive, but all are added there is no minus sign. So, as long as x is a non 0 vector even if only one fellow is non 0 here this whole sum is positive it cannot be

0. Because of all of them at least one mod square is positive even if others are 0 summation is positive. There is no negative there is no subtraction because all that plus.



If that be and this, I called this is like you know in a three dimensional vector in the real world, in the three dimensional world, it is x 1 square plus x 2 square x 1 is real there. So, mod x has no meaning x 1 square plus x 2 square plus x 3 square is what? It is the square of the length of the vector x 1 square x 2 square x 3 square. Summation is the length square, square of the length of the vector. So, if I take square root positive square root of that summation is the length not length square, but length instead of length we use a better term called norm square. Norm is like the length and we denote it by which is nothing, but in a dot product of x with itself in this summation which is square of the length that is square of the norm.

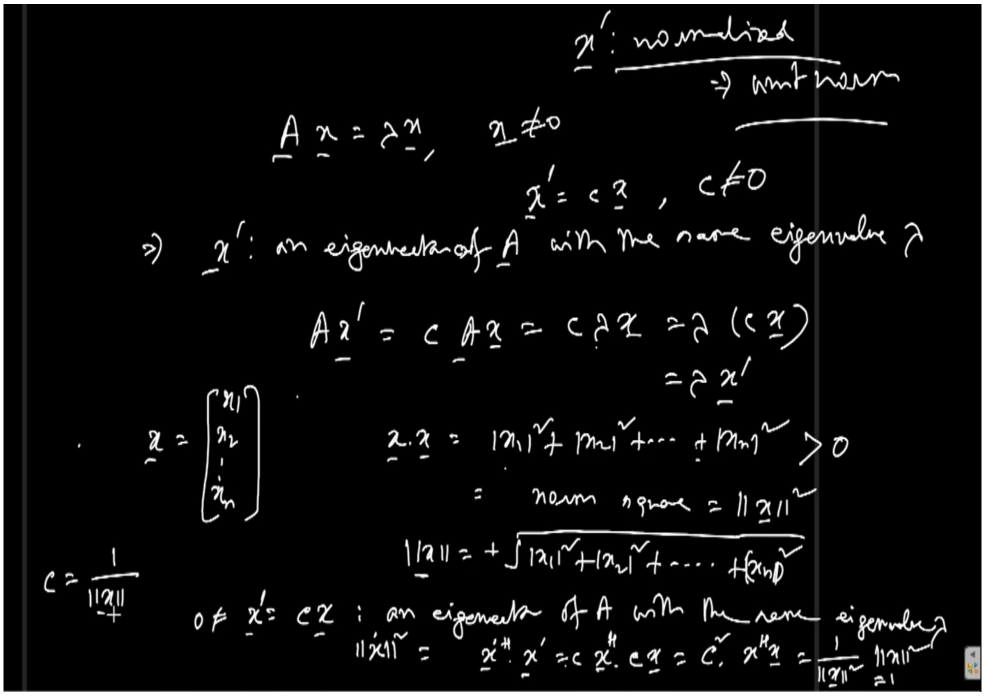
So, norm is square root of this thing where square of the length to take the square root and of course, the positive square root alright. Now we are given x to be an eigenvector ok. So, for x to be an eigenvector this is positive. because this is positive. Suppose i take c to be a scalar is just 1 by norm, not norm square, just norm.

This norm is, norm square is positive, norm square is positive, its positive square root is again positive. So, 1 by that no division by 0. So, I can generally will construct c. So, therefore, a into instead of x if I take c x we have just seen little while ago that c x for any c which is non 0 and this is non 0 and you see an eigenvector again with the same eigenvalue this c x if you call it x prime. So, a x prime is lambda x prime and x prime non 0 this is non 0 because c is non 0 no element here is infinity, So, it is ok c is non 0.

So, but what is the norm because I have scaled up or down every element by c. So, norm square that is a length square and therefore length has changed from the length of x because every element has been scaled up or down by constant c. So, overall length either has gone up or down by some factor. So, what is the new length, new length means take first square, square means x prime Hermitian dot product is for the complex case x prime Hermitian dot x prime we have seen already x prime Hermitian x prime is c x, c is real, that is very important because norm x is real all mod square mod square mod square, they make it real. And if x is eigenvector, they make it also positive because all elements cannot be 0 simultaneously because the vector is non 0, but it is real.

So, norm x is real, so c is real. Therefore, if we take x prime h x prime h that is c x h it will be x h c h, c is a 1 by 1 matrix scalar. So, c transpose which is c and conjugate is c because c is real Again it takes c in the front, So, it will be c times x h all right. c times x h there is this and again x prime c x, c is scalar. So, write c in the front it will be c square and x h x. c is this much now c square will be 1 by square up and x h x is norm square and they cancel ok.

So, they cancel and I get 1. So, this is a vector whose norm square is 1 and therefore, norm that is length is 1. We say x prime is normalized that is unit norm. So, generating this is no problem you find out any eigenvector that is non 0 find this norm take 1 by norm multiply the original vector eigenvector by this you get that unit norm eigenvector.



We will now onwards consider eigenvectors which are unit norm having unit norm and now if you have got 2 eigenvectors if suppose x y are unit norm eigenvector of a Hermitian matrix A at their unit norm, we know they are corresponding to of course, sorry corresponding to distinct eigenvalues lambda 1 and lambda 2 respectively. Then we have seen their x and y are mutually orthogonal meaning x h y and y h x it is 0 and other thing is x h x which is norm square of x equal to 1 y sorry y h y is norm square 1.

So, now the 2 vectors they are having unit norm and they are mutually orthogonal then we use the term orthonormal x and y mutually orthonormal unit norm and if I get 2 eigenvectors corresponding to 2 different eigenvalues x and y they are orthogonal normalizing is no problem everybody has non 0 I mean if a positive norm square. So, take the norm divide every element of x by its norm every element of y by its norm sees you you get normalized versions u x u y and then they are orthonormal. So, in the next class we proceed from here we will show how to diagonalize a Hermitian matrix and some properties that follow from that alright. So, till then goodbye from here.