Introduction To Adaptive Signal Processing

Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture No # 07

Introduction to Hermitian Matrices

So, in the last class we discussed Hermitian Transposition of matrices. that is conjugate transposition as we first conjugate take the complex conjugate of each element and then transpose the matrix or first transpose the matrix and then take complex conjugate. And it was denoted by a superscript capital H Hermitian transposition when the matrix is real it becomes usual transposition convectional you know ordinary transposition which we which we denote by a superscript t on the matrix right. And today we will consider a very important class of matrices called Hermitian matrices which also can be called conjugate symmetric. We know ordinary symmetric matrix X in the case of real elements that is conjugate symmetric ok. Given say m cross n matrix A it is Hermitian if A matrix and its Hermitian transpose they are same ok.

$$\underline{A} = \underline{A}^H$$

That is if you take A conjugate every element then transpose whatever you get that is on the right hand side on the left hand side A and if they are same then A is called Hermitian ok. Now, you see LHS left hand side is a m cross n matrix, right hand side because there was a transposition in the involved in the Hermitian transposition it will be n cross m and since they are equal and m cross n matrix and n cross m matrix they are in general not equal they can be equal if n equal to m ok. Here I have got m number of rows and here I have got n number of rows and they are equal, the two matrices, they are same. That means, m must be equal to n which means Hermitian matrices are square matrices.

Next suppose I take LHS left hand side of this, I find out its i comma jth element and right hand side also i comma jth element and then the two matrices are same because of the equality. So, i comma jth element on the left hand side i comma jth element on the right hand side they are same, but i comma jth element of the left hand side is nothing but A ij ,ith row jth column that element the usual matrix notation A matrix ith row jth column that element is i comma jth element A ij and RHS i comma j will be, first there was a transpose ok and then conjugate. So, transposition means current i comma j, current i comma j after transposition is j comma i before transposition forget about this h, if I just give you an ordinary matrix, real matrix A you take its transpose transpose. So, i comma jth if the element goes to j comma i th position and j comma i th guy comes to i comma jth position. So, right hand side after the transposition now i comma jth will be original matrix its j comma i and then a conjugate because Hermitian transposition.

$$(LHS)_{ij} = A_{i,j}$$

$$(RHS)_{ij} = A^*_{i,j}$$

So, it will be A ji star and these two are same that means, in this case A ij because of the equality the two matrices are equal. So, i comma jth element on the left hand side for any i any j will be same as the i comma jth element on the right hand side that means, A ij and A ji they are conjugate of each other. And that is why it is called conjugate symmetric, i comma jth position and j comma i th position. They are conjugate of each other like you know you can take a matrix say, and one more thing, if it is i comma i th i comma i th element j and i are same that means, A i i and again A i i star they are same and that happens when this is real because a real number and is conjugate they are same. And i comma i th element means i th diagonal element which means every diagonal element of a Hermitian matrix is real.

$$A_{ij} = A^*_{ji}$$

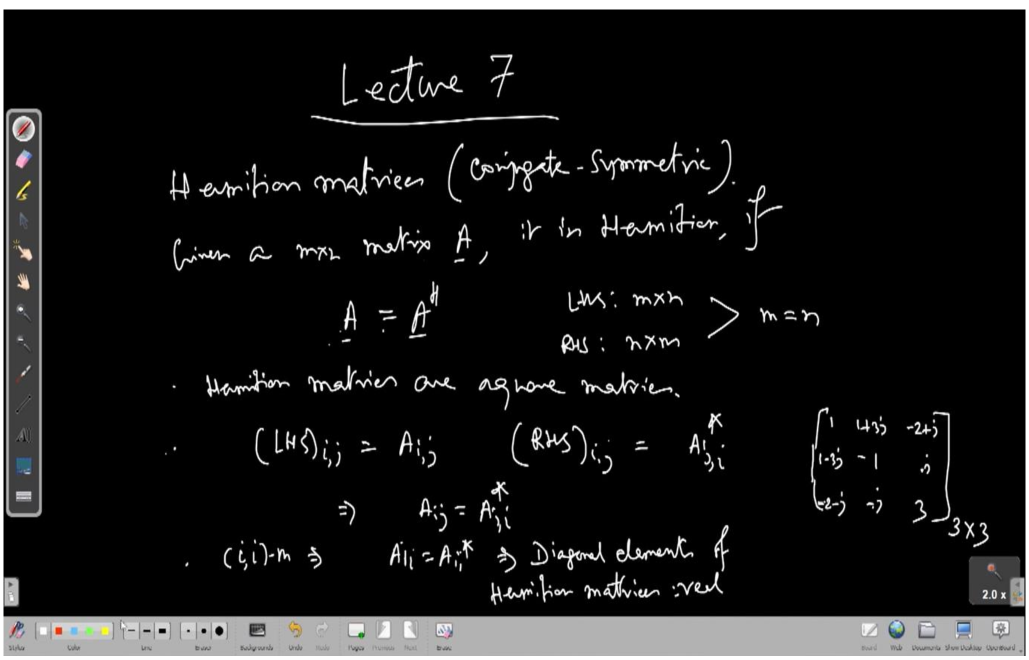
$$A_{ii} = A^*_{ii}$$

Diagonal element sorry diagonal element Hermitian are real. So, typically a matrix say a 3 by 3 it has to be square matrix. So, a 3 by 3 element 3 by 3 a 3 by 3 matrix Hermitian matrix and just as an example diagonal elements are real they could be anything 1 minus 1 3 like that then 1 comma 2 and 2 comma element 1 they are conjugate of each other. So, if it is say maybe 1 plus 3 j it will be 1 minus 3 j if it is then 1 comma 2, 1 comma 1, 1 comma 2, 1 comma 3, row 1 column 3, if it is minus 2 plus j. So, 3 comma 1, 1 comma 3 first row third column and third, first row third column and now third row first column, this, they are conjugate of each other.

So, it will be minus 2 minus j and again second row third column and third row second column these two are conjugate of each other. So, if it is suppose j it will be minus j like that. This is a typical example of a Hermitian matrix all right.

$$\begin{bmatrix} 1 & 1+3j & -2+j \\ 1-3j & -1 & j \\ -2-j & -j & 3 \end{bmatrix}$$

(Refer Slide Time: 7:41)



Then now before I go into this Hermitian matrix why you are doing the Hermitian matrix, let us come to that. Last time I was doing then I took a branch I went into this side Hermitian and transposition and all.

Now, consider p complex, in general complex, because that care of that care of real also, complex valued continuous random variables x1 dot xp with min mu1 dot mu p. Last time I wrote mu x1 mu xp, just I am making it simpler mu1 to mu p, because there is no other y or z random variables here this is fine ok. Then you from this vector incremental vector x1 minus mu1. So, mu1 is the mean, x1 minus mu1 is the increment or decrement around the mean, x2 minus mu2 dot dot dot dot xp minus mu p, you can give a name small x maybe tilde, all right, that is equal to this. So, they are also random because mu1 is constant.

$$X_1, \ldots X_P \rightarrow \ \mu_1, \ldots \mu_P$$

So, this is increment can go in the up direction or below direction, but this is random because x1 is random. So, these are random increments or decrements ok. Then now I define this covariance matrix C maybe xx was expected value I am not writing anything on the subscript you know whichever random variable comes under the parenthe under this operation only those variables will be part of the subscript other drop out. So, that is up to you and now I think we have done it enough. So, yourself can work it out which random variables will come when.

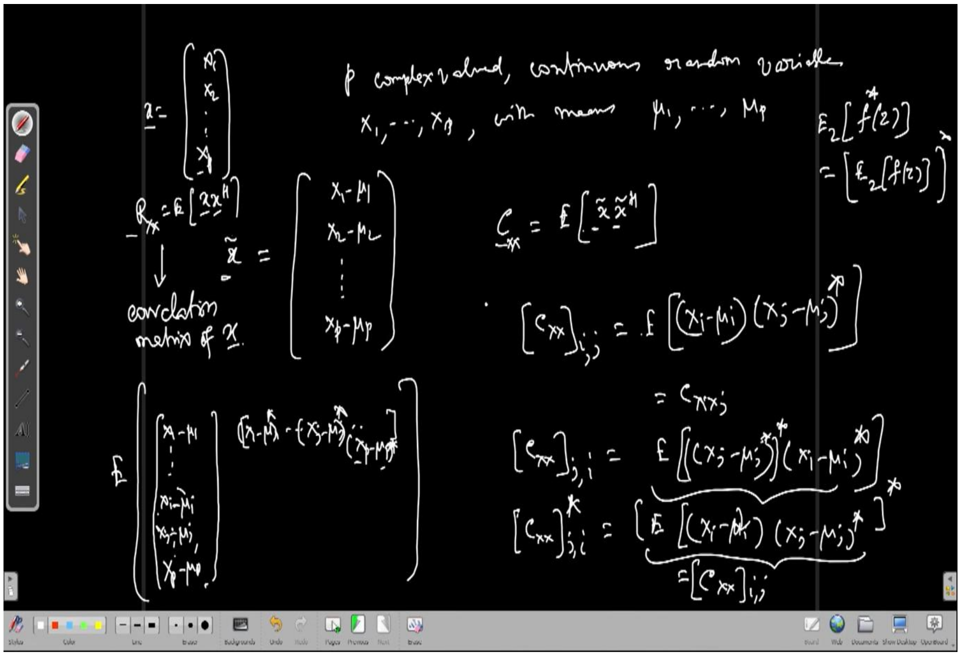
$$\tilde{X} = \begin{bmatrix} X_1 - \ \mu_1 \\ X_2 - \ \mu_2 \\ . \\ . \\ X_P - \mu_P \end{bmatrix}$$

So, I am not writing anything here it is suppose I take x tilde all right. So, typical i comma jth element x tilde is this, column vector x tilde h, h means first is a Hermitian transformation. So, you take the transposition means column vector becomes row vector ok and then it is like this , x tilde, as it is you put here and then the column vector, a row vector, x1 minus mu 1 dot dot dot dot xj minus mu j x p minus mu p. If you want you can even make it this way you write the full of this x tilde here, this x tilde Hermitian is this transposed and then conjugate I have to put a conjugate on them. xi minus mu i all right and then expected value this is the meaning of this.

So, this column vector into row vector it is a matrix and then you have to apply E on everybody. If xi and xj and they are present in the ith row jth element jth column of this matrix will be what, this element times this element. Because that is how they are form, now this into this will be the first guy ith row first column, this into next will be ith row second column, this into this will be ith row jth column and so on and so forth. So, this two multiplied and E and E will have only xi and xj in the subscript other variables will drop off ok. That means, cxx i comma j will be what E xi minus mu i this fellow and this xj minus mu j star.

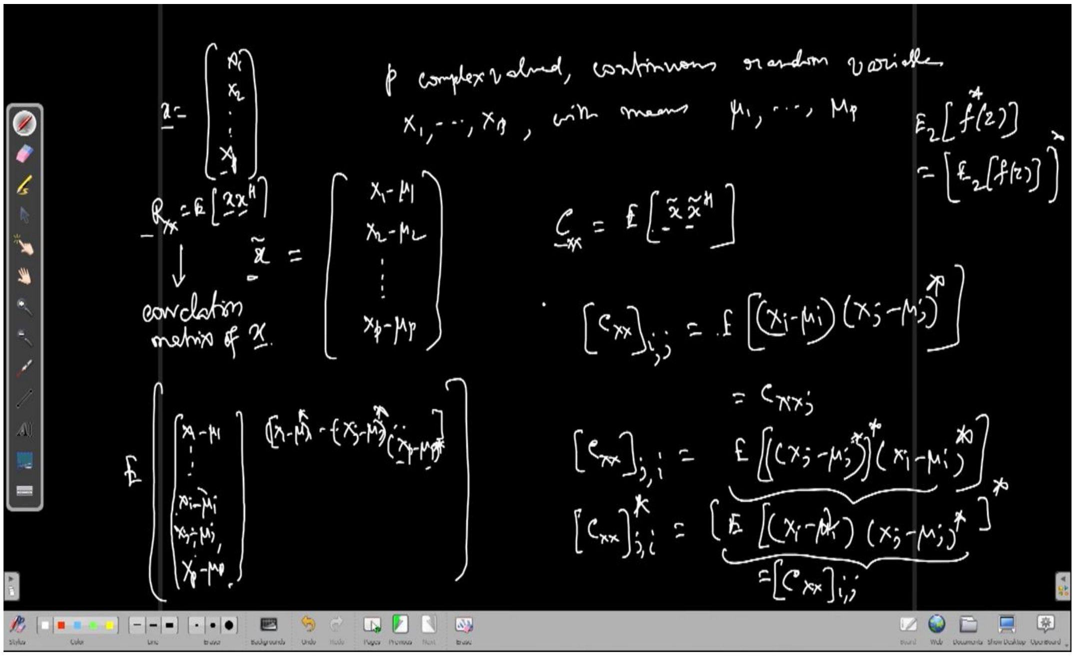
This is nothing, but covariance between xi and xj two random variables. It is the increment around the mean this into its conjugate expected value. By definition there is a covariance between xi xj. And by the same way cxx if you take j comma i, it will be xi sorry it will be maybe xj minus mu j, this times xi minus mu i star j comma i. So, i and j their roles are just interchange xj minus mu j come here no conjugate and xi minus mu i goes here which conjugate alright.

So, therefore, if I take conjugate of this, suppose I take conjugate, complex conjugate of this, what happened the entire thing here have to apply conjugate. But we have already seen earlier in the case of complex random variables E of some function of the complex variables then star outside is same as E of the function its conjugate Like E remember E z f z may be vector of complex random variable ok. And then if you put star f z conjugated then E z, it will same as original E z f z and then star. So, it is like this f z star, f z was ok, f z is what? you can write it as there is no star here right. So, you can assume as though there is a star and outside there is still a star.



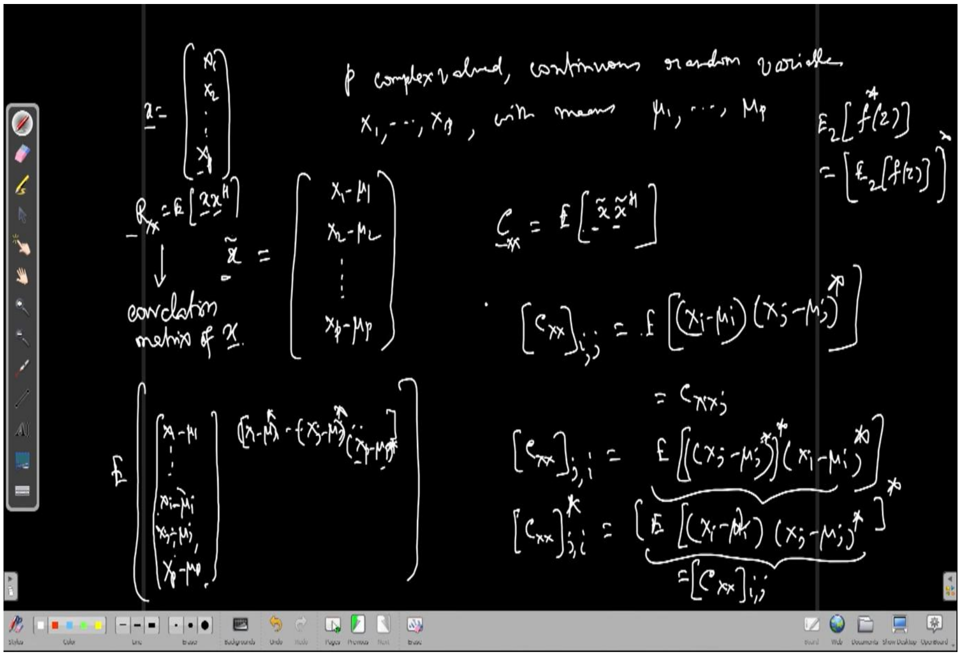
star star cancels and if you call this random variable z1 if you call it z2 z1 star z2 star means z1 z2 star ok. So, expected value z1 z2 star it will be same as expected value first z1 z2 outside star. So, it is like expected value z1 z2, z2 is this much z1 with the star first inner star z2 this much then star is same as expected value of these two guys then outside star ok. So, xi minus mu i star goes out of it and here xj minus mu j there is a star ok, but this quantity is same as what we have here xi minus mu i xj minus mu j star expected value that is what we have. So, that is nothing, but cxx i comma j.

So, that means, this matrix is such its j comma ith guy and i comma jth they are conjugate of each other. So, that means, in summation i comma jth conjugate symmetric i comma jth element and j comma ith element they are conjugate of each other ok. So, it is hermitian and when i and j are same they are real you can easily see because if i and j are same i comma i it is xi minus mu i again xi minus mu i star which means mod of xi minus mu i whole square which is real not negative and expected value of that which is the variance. So, diagonal elements are real and in this case in this case not only just real they are in fact, non negative they cannot be negative because mod of, if you take j equal to y xi minus mu i, xi minus mu i star. So, it is mod of xi minus mu i whole square.



So, mod is real and then square. So, it cannot be negative either 0 or positive an expected value of that which is variance ok. xi minus mu i mod square is the power of the increment xi minus mu i in the particular trial when you got xi minus mu i and now you mod square and then you are averaging expected value. So, there is a average power which is variance and variance is real non negative because there is a mod square inside the expectation. Ok if this guy is mod square expectation cannot be negative because expectation means what You have to multiply by the joint density which is again non negative.

Probability density cannot be negative dx dy d that dxi dx j say are integral ok. xi means actually you have to take the real part imaginary part real part imaginary part. So, 4 times integral, but there is again that cannot be negative because probability density is non negative. Mod of this xi minus mu i square ok it will be actually double integral because you have not xj you have got xi only. So, mod xi minus mu i square xi has 2 component real and imaginary.

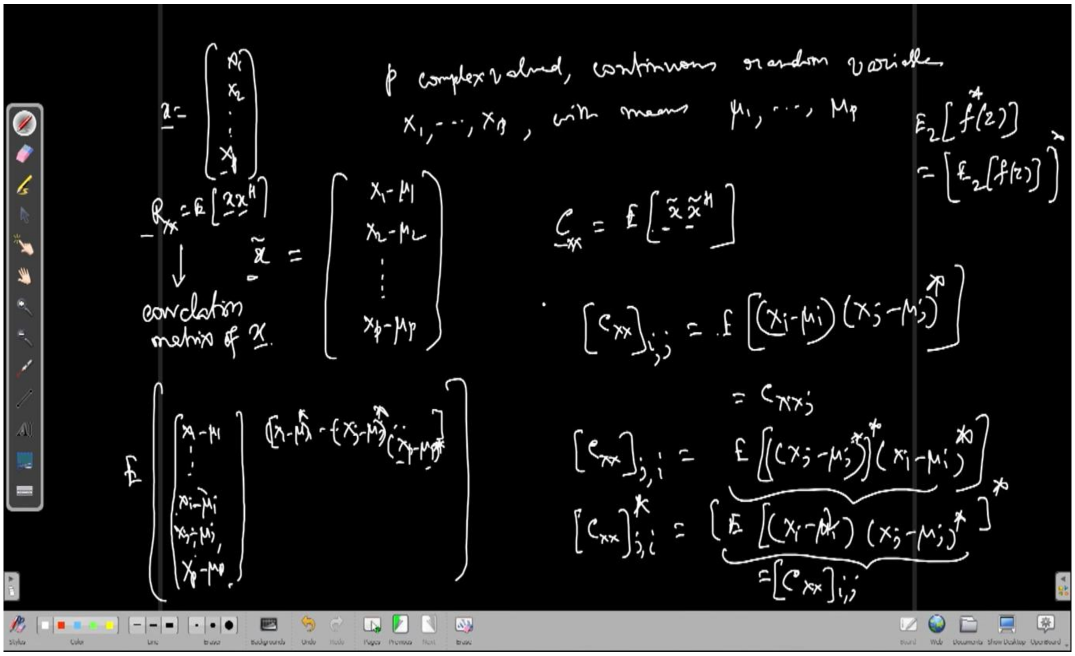


So, multiply these by joint density of the real part imaginary part and double integral. So, this cannot be negative because mod of xi minus mu i square under the double integral non negative can never be negative. Joint density cannot be negative d differential you know dxi dz d of the real part d of the imaginary part they cannot be negative. So, this is not only real. Hermitian matrices required this to be real, but this is not only real in the case of covariance they are also non negative all right.

So, covariance matrices are Hermitian matrices and in fact, mean is 0 it will be just a vector x1 to xp and its Hermitian transpose expected value of that and that is called correlation matrix that is if you just take x1 x2 xp, call it x. Then this correlation matrix is expected value of the just x no subtraction of mean when the mean is 0 both the terms are 0. By the same logic this is again Hermitian ok. You can take i comma jth and j comma i they are conjugate of each other. This is called correlation matrix of x.

This gives you the motivation for why you are going for Hermitian matrices all right. Now, Hermitian matrices have some beautiful properties, we will next study them.

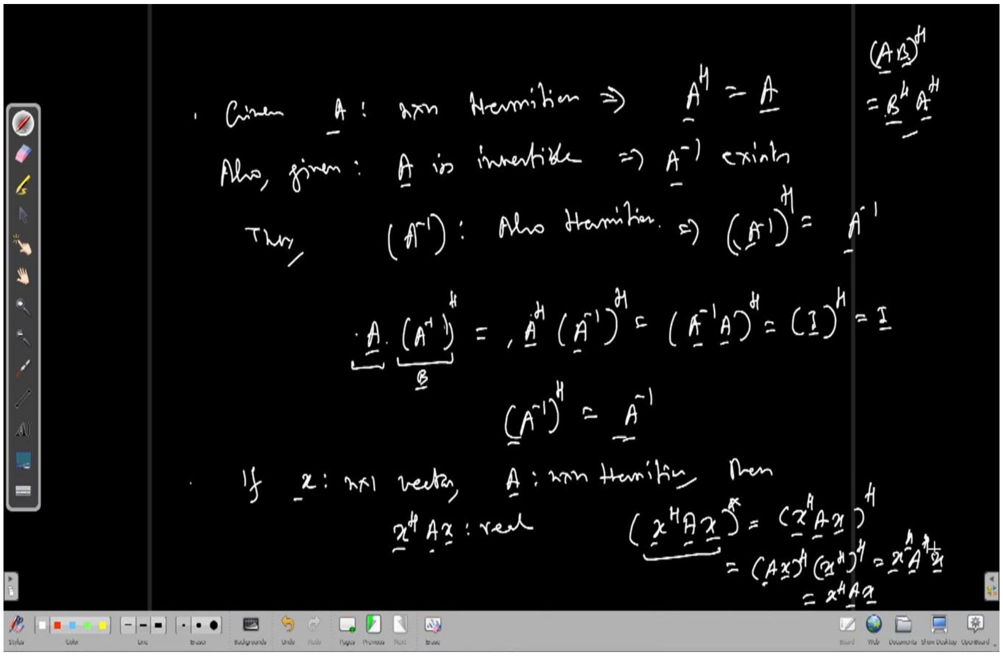
(Refer Slide Time: 20:00)



Given A is a n cross n square matrix that is a n cross n or n cross n or whatever, Hermitian meaning if you take Hermitian transpose of it conjugate and transpose, you will get the same matrix A. Also given A is an invertible matrix, see all matrices cannot be may not be invertible, but in this example suppose A is given to be invertible that is its determinant is nonzero you can find it inverse. That is A inverse exists Then A inverse also Hermitian, it is a claim.

I will show it if it is Hermitian that is if you take A inverse and its Hermitian transpose you should get back the same A inverse. So, this matrix is Hermitian transpose is A inverse this is a claim we are making and easy to prove. Suppose I take A, A is going to be Hermitian. A and this A inverse H, there is no underscore here. This I can write as A is nothing, but AH, because AH is A and then A inverse H as it is.

We have seen ABH is BH AH. You can go in the opposite direction if it is BH AH it will be ABH. So, like BH AH, so, it will be ABH. It will be A inverse which is taking the role of A, is A inverse is taking the role of A here AH, AH is taking the role of B here first guy is B, second guy is A. So, BH AH means ABH. So, this comes first this goes here AH.

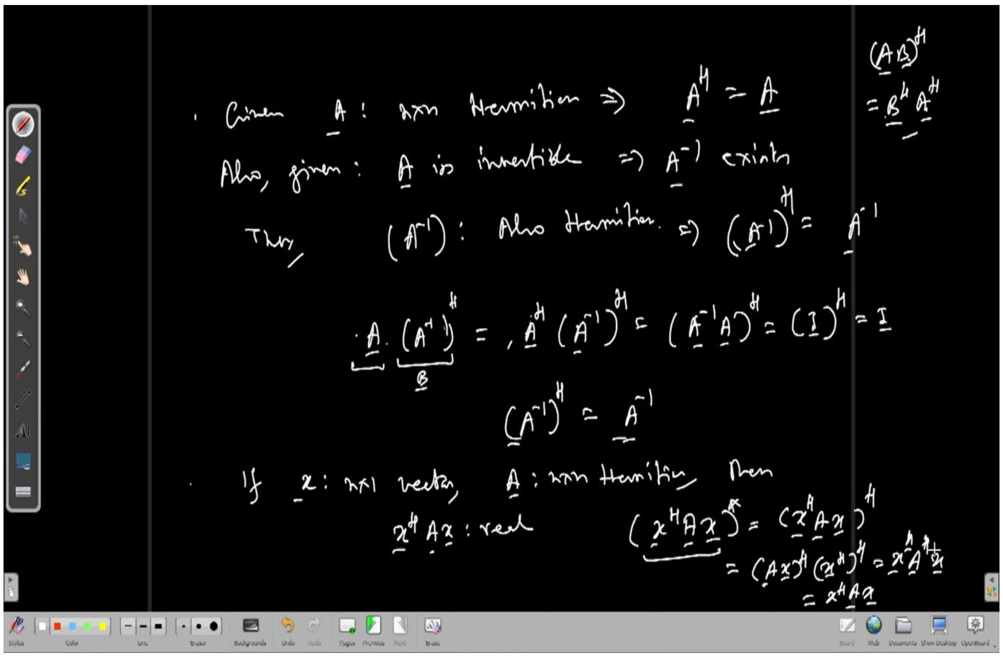


And A inverse A is identity. Identity is already Hermitian all the diagonal entries are 1, they remain there after Hermitian because i comma i after transpositions remain at i comma i. So, diagonal entries remain 1 if any i comma jth element is 0 because it is identity matrix. After transposition it will go to j comma i and conjugate has no meaning 0 is real. It will become 0. So, identity matrix if you take is Hermitian transpose it remains identity zeros from i comma j go to 0 j comma i and zeros from j comma i go to i comma j A diagonal matrix diagonal elements remain same 1 1 1.

So, it remains identity. Now in matrix theory we already know that if I give you 2 square matrices say A and if you call it B after the Hermitian, if AB both are square matrices product is identity then 1 is the inverse of the other. So, B is A inverse A is B inverse. So, that means, if I write B that is same as A inverse because AB identity means 1 is the inverse of the other B is inverse of A A is inverse of B A B i. So, A A inverse is i or A inverse A is i or B inverse B is i ok and this proves what you wanted A inverse A is same as A inverse.

So, inverse of a Hermitian matrix is Hermitian also alright ok.

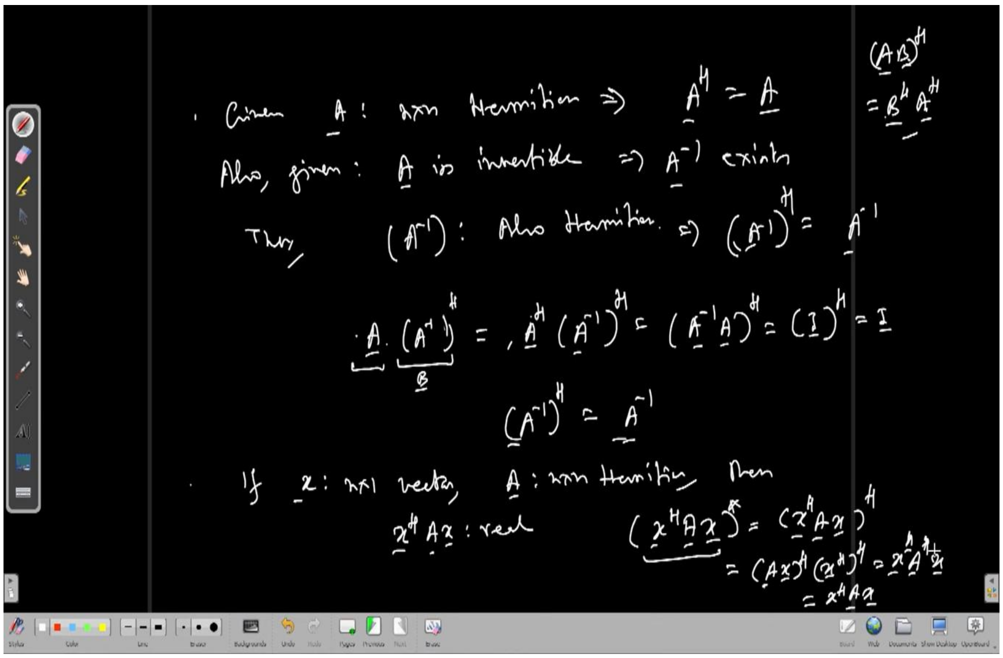
Another thing if x is a n cross 1 vector may be complex A n cross n Hermitian then if you take this quantity x h A x, A times x, x is a column vector, x is a column vector, A is a matrix, A into x is a column vector x h is a row vector because x was column. So, after transposition Hermitian transposition has a real transposition plus conjugation. Transposition it became row A x is a column. So, row into column is a scalar.



It is not a matrix, it is a scalar. So, this our claim is this is a real scalar even if x is complex A has complex. How to show any real number is what if I take complex conjugate of the real number I should get back the same thing then only it is real ok. If x is a real number then x star is same as x. Now here left hand side if I take star this is a scalar star. Now here is a trick any scalar also you can view as a 1 by 1 matrix.

If I do a scalar number say alpha is a scalar it can take 2 or 3 or j or whatever value, but scalar any scalar number is also a 1 cross 1 matrix like a n cross 1 n matrix n columns n rows it is a 1 cross 1 matrix ok. So, 1 cross 1 matrix so this I can also write as let me first write H because if it is a 1 cross 1 matrix H will be in first transpose, but a 1 cross 1 matrix after transposition remains as it is because only one element and then conjugate. So, there is a conjugate I wanted. So, I can make it instead of writing star I can put H here, because it is a 1 cross 1 matrix 1 cross 1 matrix under the one element. So, very few take transpose of that matrix you get back the same matrix then conjugate all right and conjugate is what I have already put.

But now I know a b H is b H H that is what I will write here. Suppose this is your a this is your b. So, it will be a x H, a x H and x H H, x H H will be bring back x because x was a column vector now x H will be row vector with conjugates. Again H will make it further column vector transposition again and conjugate on conjugate will remove the conjugate. So, you will get back x and here again a b H is b H a H.



So, it will be your x H a H and here it is x, but a is given to be Hermitian. So, a H is same as a this a H is same as a which is x H a x. So, that is what this is. So, this scalar is conjugate is same as the scalar itself. This scalars conjugate is same as the scalar itself.

$$(A^{-1})^H = A^{-1}$$

$$\left(\underline{x}^H \underline{Ax}\right)^* = \left(\underline{x}^H \underline{A}\ \underline{x}\right)^H$$

$$= \left(\underline{A}\ \underline{x}\right)^H \left(\underline{x}^H\right)^H$$

So, this scalar is real. So, in subsequent lectures I might use this kind of expressions and we will take it to be real. So, you understand why it is real, provided a is Hermitian, a H is a otherwise not all right. Now, next I will go to what is called eigenvalue and eigenvector which is a new topic. So, that I will do in the next lecture. Thank you very much.