# Introduction To Adaptive Signal Processing

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## Lecture No # 06

## Complex Random Variables

Ok. So, in the last class we started with complex random variables right. So, now today I take two complex random variables $X$ which has a real part $X\ R$ which is a real continuous random variable plus $j\ x\ i$ for imaginary. $x\ R$ and $x\ i$ two real continuous random variables. So, they are combined like this. So, capital X is a complex number, it's a complex random variable and there is one more $y\ R$ plus $j\ y\ i$.

$$X = X_R + jX_I, Y = Y_R + jY_I$$

So, two complex numbers mean two complex random variables x y mean actually four real random variables right. So, also given $E$ of $x\ R$ whether we put $x\ R$ comma $x\ i$ here or just $x\ R$ does not matter because $x\ i$ does not figure here that will drop off. So, it will be $E\ x\ R$ of $x\ R$ which is mu R mu not mu R let me change the notation to something else mu of this random variable. So, mu x R plus j, sorry not plus, I am very sorry, I am wondering whether this is a good notation or there can be something better.

$$E[X_R] = \mu_{X_R}$$

$$E[X_I] = \mu_{X_I}$$

ok. Let it be as it is. Similarly, x i has this mean mu x i, E of y R is mu y R E of y i is mu y i. Then consider a function of x and y or equivalently a function of x R, x i, y R, y i. This function, it is a function of x y and x has two components, y has two components, it is basically function of four components, and it is this E either you write x R x i y R y i and then x minus.

$$E[Y_R] = \mu_{Y_R}$$

$$E[Y_I] = \mu_{Y_I}$$

So, mu x is expected value of this and x you write as x R plus j x i. So, E x equivalently now I have got two variables x R x i when this works on x R x i drops off. So, it is E x R of x R which is mu x R plus j when E x R x i works on x i x R drops off. So, E x i of x i which is mu x i, So, j mu x i.

$$\mu_X = E_X[X]$$

$$= E_{X_R X_I}[X_R + jY_I]$$

$$= \mu_{X_R} + j\mu_{X_I}$$

Similarly mu y will be mu y R plus j mu y i all right.

$$\mu_R = \mu_{Y_R} + j\mu_{Y_I}$$

So, here I write like this x y, x minus mu x complex and sorry it is not, y minus mu y, but I put a star here conjugate ok.

$$C_{XY} = E_{XY}[(X - \mu_X)(Y - \mu_Y)^*]$$

I am just constructing a function x minus mu x y minus mu y star ok. What does it mean that x is a complex random variable its mean is mu x. So, x minus mu x is the increment what the mean y minus mu y is the increment, complex increment, over its mean and star and you are multiplying them and then taking average. So, suppose x and y they are highly correlated.

So, when x minus mu x and y minus mu y you multiply ok. When so, you will get a in good product ok. Product if it is a magnitude of actually actually magnitude wise and if you average them it will be high. On the other hand, if you, I mean, because what happened is it has got, maybe you understand it better if I expand it right like. So, x you replace by x R plus j x i are like that.

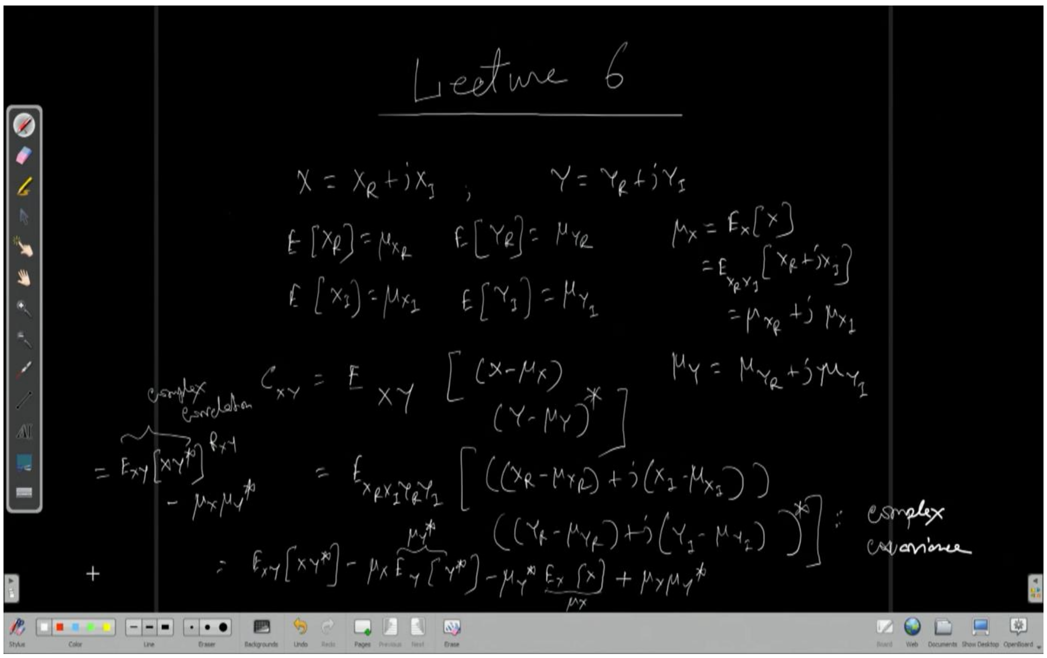
So, it is x R minus mu x R from here plus j x i this is x minus mu x and y R if you multiply them and there is a star here sorry. So, if x R and x i they are highly correlated y R and y i then why this is positive and this is positive this also be positive, it is going up over the mean, it is also going up over the mean same here same here suppose. So, products will give high positive value and you will have high a quantity of high magnitude or if they are going down the mean when this also goes down this also goes down. So, negative when you multiply you still get a positive number. So, you get and then you will be average your overall you know magnitude of the average will be high.

But if they are not correlated then sometimes this is positive half of the cases this could be negative this could be negative and another half cases this could be positive this could be positive same here. So, you multiply and average out you will be getting close to 0. This is a more general definition of covariance is a complex covariance between x and y. It is a complex covariance all right. When x and y are same why we need the star because when x and y are same then it is x minus mu x if I put star then only you will have mod square because if it is vary covariance and when x you are taking covariance of x with itself then it is variance and variance cannot be complex is the power average power.

$$E_{X_R X_I Y_R Y_I}[\left(\left(X_R - \mu_{X_R}\right) + j\left(X_I - \mu_{X_I}\right)\right)\left(\left(Y_R - \mu_{Y_R}\right) + j\left(Y_I - \mu_{Y_I}\right)\right)^*]$$

So, unless you put star there you would get real thing here. if you put star it will be x minus mu x and this is now x, x minus mu x again star. So, complex number into its star. So, it is mod of the complex number square which is real and so, expected value is real, that is where the star has come. All right, this complex covariance and as before if x and y are uncorrelated this covariance will be 0 ok.

This covariance will be 0 if this is uncorrelated in that case if you expand it without breaking like this it will be E x y x y star minus mu x is a constant it will go out of expectation. So, E x y y star so, since x does not figure inside x will drop off. And then minus mu y star, E x y of x and again y does not figure here. So, y will drop off plus mu x mu y star expected value, So, that will come out is a double joint density you have to multiply this by the joint density and double integrate. So, this will come out of the integral and joint density double integral from minus infinity to infinity equal to 1.




So, it will be minus minus plus mu x mu y star, but this is equal to, this is expected, what is this, other day I told you if E y y is mu y. So, E y like I told E I mean if I doing E z of a function f of z fine, but if you do E z of a function f star z there is nothing, but star of E z of f z. So, f z you can take to be z. So, E z of z star it star of E z ok. So, it will be star of E y of y that mean E y of y is mu y.

So, it will be mu y star and this is simple mu x. So, mu x mu y star mu x mu y star mu x mu y star. So, 2 cancels. So, you are left with this quantity minus just mu x ok.

$$= E_{XY}[XY^*] - \mu_X \mu_Y^*$$

This is called the complex correlation $R_{XY}$ minus this.

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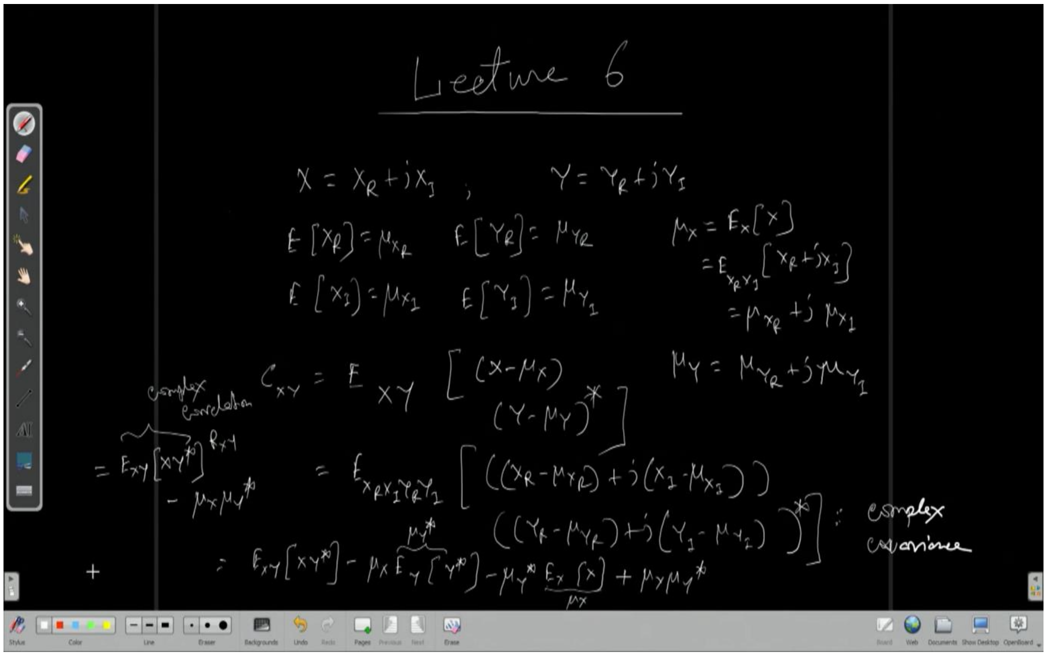


So, it is a more general result there is a extension to the complex case. Extension of the co very earlier previously discussed covariance and correlation case and all that uncorrelatedness all those things to the complex case all right. One more thing now suppose we are given P complex random variable x 1 which has x 1 real part plus j x 1 imaginary part dot dot dot x P which has real part, imaginary part, expected value E. Expected value will be mu x 1 here mu x P expected value on this fine. So, as before I form this increment vector x 1 minus mu x 1 dot dot dot x P minus mu x P ok.

$$X_P = X_{1R} + jX_{1I} \rightarrow \mu_{X_1}$$

$$X_P = X_{PR} + jX_{PI} \rightarrow \mu_{X_P}$$

Let me give it a name x, any column vector will be an underscore lower case letter, any matrix will be an underscore upper case letter and if it is not underscore it will be scalar whether does not matter if it is lower case or upper case, but underscore means either vector when it is lower case or matrix when this is upper case. So, suppose this is this and I call it x tilde. Then x tilde x tilde ok before this.

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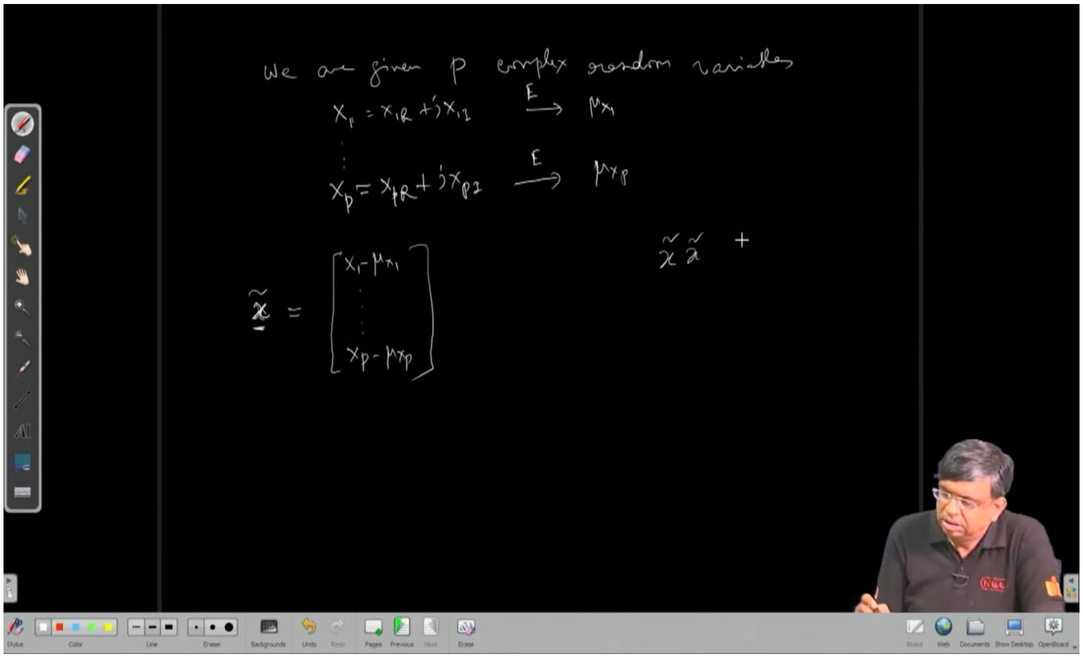



Before this I need to do something called Hermitian operation fine. I will come back to this and if I will come back today next class.

Given the separate thing I am considering Hermitian transpose of a matrix or a vector in fact, a vector is also matrix, n cross 1 column vector is actually n cross 1 matrix. So, no need to write a vector ok because there is any vector also is a matrix whether row vector or column vector ok. This first we discuss Hermitian transpose then I will go to the previous page there I discussed a matrix x tilde. I will take its Hermitian transpose then I can proceed that is I am doing this here. So, given a general complex valued matrix A m cross n its Hermitian transpose Hermitian transpose, You know ordinary transpose it is indicated by a superscript t, here it will be A matrix that is your upper case underscore, A not t here, but capital H it is called Hermitian operation.

$$A^H_{n\times m} = \left(\underline{A}^*\right)^t = (A^t)^*$$

$$\left(\underline{AB}\right)^H = [C^*]^t$$

A each will be either you conjugate take complex conjugate of each entry then on that matrix apply ordinary transposition or alternatively you can first transpose and then conjugate you will get the same thing. First you conjugate everybody then transpose or you first transpose and then conjugate everybody this obviously, they will be same. So, this is equivalent. So, if it is m cross n after this it will if it is m cross n this will be n cross m because transposition all right. Now, one thing we know now, suppose we are given 2 matrices, A which is say maybe m cross n and B which is maybe n cross r.

So, resulting is m cross r m cross r. There is a matrix if on that matrix I apply H. So, m cross r matrix after this H will be r cross m fine r cross m but what would be what it will be we have to say this. Now, if you call it this matrix to be C which is m cross r m cross n n cross r m cross r C H will be you first take C star one of the two, I take them this for C star and then transpose. C star means everybody in the C matrix, what is the C matrix? typically i th row of A and j th column of B when you multiply them you get i comma j th element of C ok.

$$C_{ij} = \sum_{N=1}^{n} A_{iN}B_{Nj}$$

$$\left[C_{ij}\right]^* = \left[\sum_{N=1}^{n} A_{iN}B_{Nj}\right]^*$$

Then you have to start this is A this is B and then you have to 1 minute and then outer is transpose A B C C star. Now, every element of C is to be conjugated. So, i th row of A and j th column of B when you multiply them term by term and add you get C i j right. So, C i j is what a i 1 b 1 j plus a i 2 b 2 j and like that. So, it is a i you can bring an index k and b k j k from 1 to k will go like this how many n n columns k is 1 to n, a i 1 b 1 j a i 1 b 1 j i 1 b 1 j then a i 2 b 2 j then a i 3 b 3 you like that you add up to n times n times.

So, if I want to star you have to put conjugate every element of C. So, i comma j th element of C also is to be conjugated. So, if I want to conjugate it you have to conjugate this, but you know conjugate of a summation if I give you two complex number z 1 plus z 2 you will sum and then conjugate it is same as this you can verify first you conjugate and then sum and you can extend it to any number of random I mean complex variables. So, star over a summation is summation over star of this 1 minute and one more thing maybe here only I can say if you have given two complex number z 1 z 2 product and you want to star you can verify that it will be z 1 star z 2 star. Very simply z 1 you write in the polar form some r 1 times e to the power j theta 1.

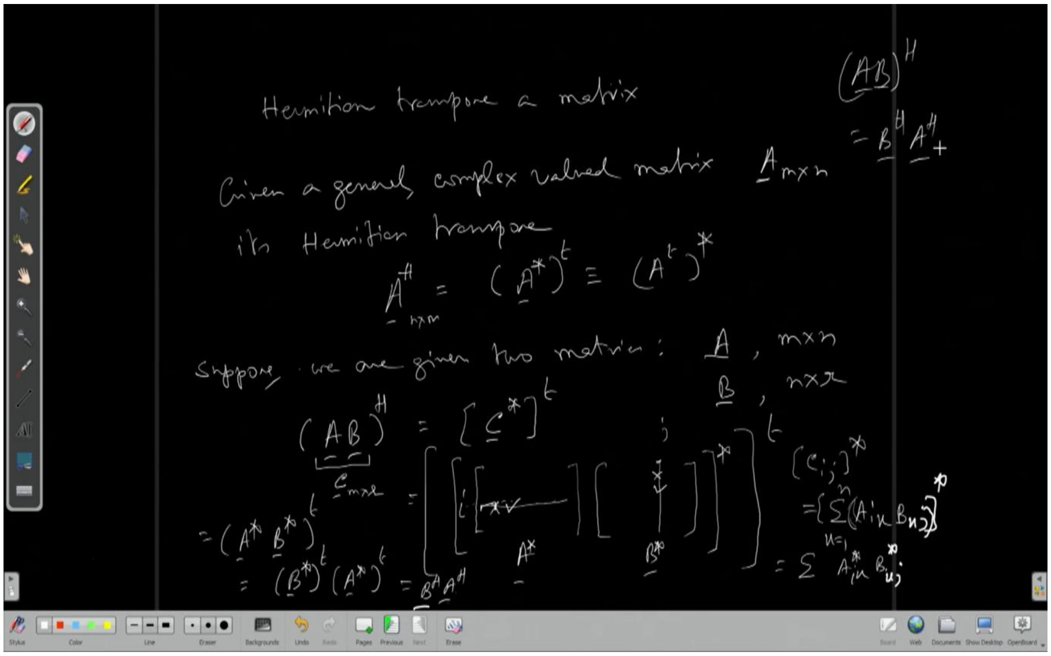
$$(Z_1 Z_2)^* = Z_1^* Z_2^*$$

This you write as r 2 times e to the power j theta 2. So, product is r 1 r 2 into e to the power j theta 1 plus theta 2 conjugate means it will be minus j theta 1 plus theta 2. So, r 1 e to the power minus j theta 1 r 2 e to the power minus j theta 2 z is z 1 star z 2 star. So, when you apply star star of a summation is summation of conjugated when you apply star over this for every k is a product of two complex numbers star over that is product I mean you have to star each in each one like z 1 star z 2 star. So, you have to put a star here put a star here.

So, first you conjugate then multiply all right. That means, as though a i star k as though this element. So, k equal to 1 say a i star 1. So, this one is like if you do a star b star then what will happen it will not be a i 1 it will be a i 1 star and it will not be b 1 j it will be b 1 j star and that is what I have i 1 star b 1 star then i 2 if you have a star i 2 also every element is conjugated. So, i 2 star and b 2 j star and that is what I have here i 2 star k is 2 b 2 j star and like that.

$$= \sum A_{iN}{}^* B_{jN}{}^*$$

$$= A^* B^*$$




So, it means every element of a is to be conjugated and every element of b is to be conjugated and multiplied. It is a matrix it is a matrix. So, a star b star then transpose. So, it is basically a star b star then transpose. But this is a matrix this is a matrix you know from basic matrix results if you multiply two matrices and they transpose it is first you have to transpose this second fellow then you have to transpose this first fellow.

These are very basics you have studied two matrices a b multiplied then transpose a b transpose is b transpose a transpose here instead of a b it is a star b star. So, b star transpose a star transpose, but b star transpose is by this definition b star transpose is b h and a star transpose is a h. So, this gives us the result a b you multiply then take h that is conjugate everybody and they transpose that matrix will be same as if you first apply h transpose on b that is conjugate everybody of b and then transpose and again conjugate everybody of a and transpose or vice versa then multiply you will get the same thing all right. That is this main result a b h is b h a h.

$$(A^* B^*)^t = (B^*)^t (A^*)^t$$

$$= (B^H A^H)$$

$$(AB)^H = B^H A^H$$

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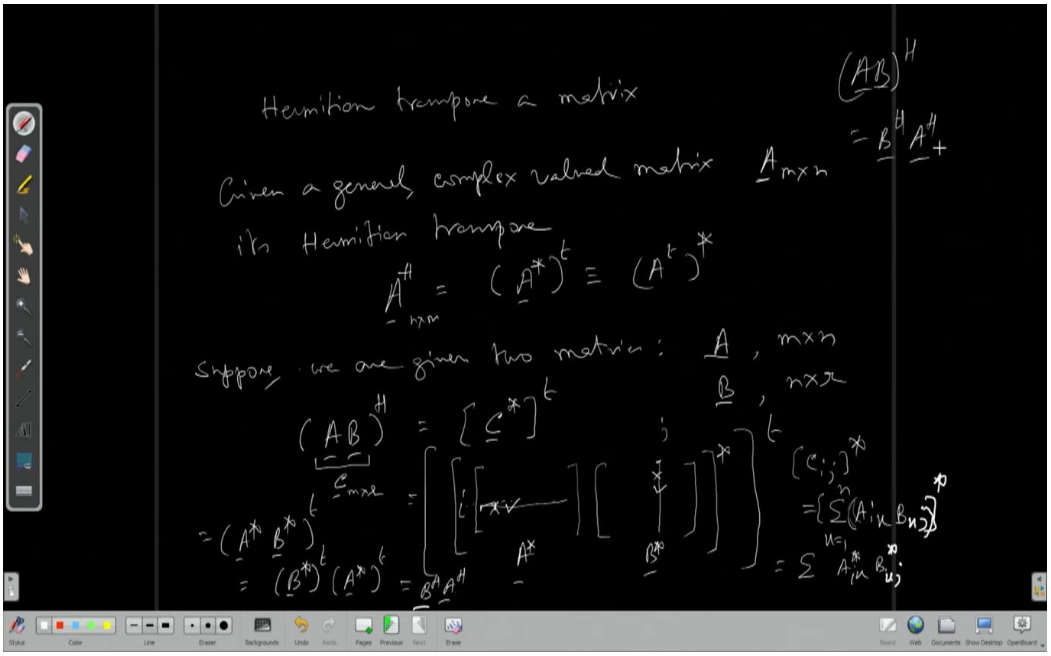


So, now, I go to the previous page I take this increment vector with a column vector n cross 1 and I take its Hermitian.

So, if it is n cross 1 this will be 1 cross n. So, column vector row vector and everybody after transposition everybody will have to be conjugated that is and then I take a expected value I am not writing in the super subscript anymore, it is with respect to all the real and imaginary real and imaginary. So, 2 p variables here this has to this has to everybody has to real imaginary real. So, total 2 p number of random variables real random variables they come here that is invisible I am not writing anymore ok.

Now, you should be able to understand. So, this is e of these in general x i minus mu x i and then Hermitian means transpose and conjugate if you transpose it you get a row vector and then conjugate. So, this will be every element will be conjugated take a jth guy take a j jth fellow x j minus mu x j star dot dot dot plus guy x p minus mu x p star and then expectation right. So, column vector into row vector will be matrix in that matrix i comma jth element will be what i th guy from here and jth guy from here they will multiply expectation. So, this matrix if I call it c actually it will be a complex covariance matrix c c x x complex covariance matrix. So, c x x it is typical i comma jth element will be what this fellow times this fellow i th guy how the matrix is from this into this expected value first 1 comma 1 then this into next 1 comma 2 this into next like that.

So, i th row i th fellow here and j th fellow here when you multiply you get the i comma jth element of c x x and what is that. That is e I am not writing anything in the subscript, it is up to you. You understand now it will be only the two random variables, like real part of x i imaginary part of x i real part of x d imaginary part of x d just four of them present or thus drop off and I have got x i minus mu x i x j minus mu x j transpose which is covariance by your definition between these two random variables. That is c may be x i this one matrix this scalar c x i x j. So, that is why it is called complex covariance matrix ok.

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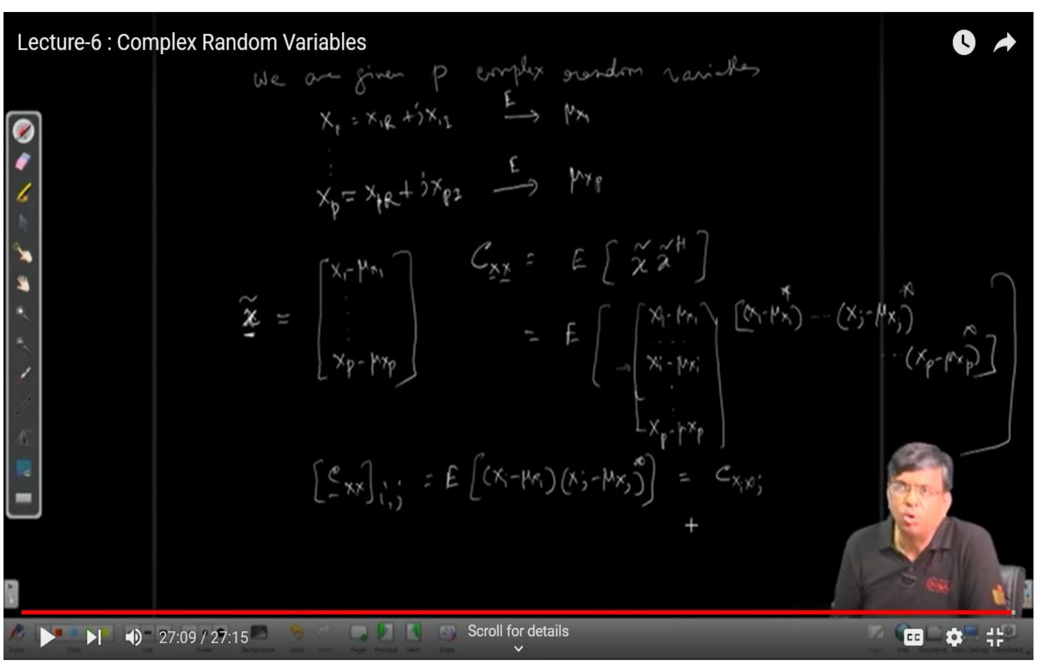

We will study properties of this matrix later in the next class. Till then good bye from here.
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