Introduction To Adaptive Signal Processing

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Lecture No # 03
General Set of Random Variables

Ok. We start from where we ended last time that was, we discussed conditional density, conditional probability density. We consider two random variables X, Y. Now I just generalize it to three random variables and then you know you can yourself think of any number of random variables, jointly random variables. So, suppose we have got now three various cases will emerge. Suppose X, Y, Z, three jointly random variables.

So, there will be one axis for X, another for Y, another for Z. Suppose I consider a point x and then this width is dx. So, x, $x + dx$. Similarly, I consider up to this and then this much is dy ok. So, this is that box, we are familiar with these, because we consider two random variable case last time. But now Z also, we take up to some value, small z and then dz. So, we go up here ok. So, this much is dx, this much is dy, this much is dz. So, you know probability of this pair X Y Z falling within this cube, will be proportional to the volume of this.

$$Prob[x \leq X \leq x + dx, y \leq Y \leq y + dy, z \leq Z \leq z + dz] \; \alpha \; dx\, dy\, dz$$

If of course, and if the widths $dx\, dy\, dz$ are very small, then that proportionality can be taken to be linear that means, we can I am just generalizing you know same logic. So, probability of what capital X lying between $x + dx$ to x simultaneously with capital Y lying between $y + dy$ proportional I mean and then y and capital Z lying with between small $z + dz$ to small z. This will be proportional to the volume, larger the volume higher will be the chances, smaller the volume less will be the chances. And if the widths $dx\, dy\, dz$ each is very small, infinitely small, proportionality can be taken to be linear. So, it will be a proportionality constant P times $dx\,dy\,dz$, P will and I write here $x\ y\ z$ to indicate the random variables involved and this will depend on the location in which the box is located, that is this point ok.

$$Prob[x \leq X \leq x + dx, y \leq Y \leq y + dy, z \leq Z \leq z + dz] = P_{X,Y,Z}(x, y, z)\, dx\, dy\, dz$$

This point gives the full idea about the entire all the other vertices. If I move the box to some other place in this 3-dimensional space, dx remains same, dy may remain same, dz may remain same, the probability will change. Obviously, I mean there is no guarantee that you know, you have the same probability of this $x\ y\ z$ triplet occurring in the box, if it is here and also if it is there, then we differ it. So, it depends on the location of the box, that is why it will be a function of this location, which will take to be $x\ y\ z$, once $x\ y\ z$ is known other points are known. This is called again joint probability density.

As before, if we integrate it from minus infinity to infinity and there will be triple integral. That is certainty, because the pair will always lie within this 3-dimensional space, where capital X goes from minus infinity to infinity, capital Y goes from minus infinity to infinity, capital Z goes from minus infinity to infinity. So, that will be again 1 and all other things follow fine. So, this is the thing, but now I can bring in the concept of conditional probability density in various ways here ok. One way assume capital X is restricted to x to $x + dx$, that is we are considering only those cases where capital X is occurring from here to here, other cases we are not considering.

Other cases occur we are only considering this. So, this is kind of things within those cases, what are the chances of capital Y falling from small y to $y + dy$ simultaneously with capital Z falling from small z to $z + dz$. That is y in this much and dz in this much, $dy\ dz$. So, what is the probability that capital Y and capital Z lies in this plane this side, which could be anywhere here, you know depending on the particular value capital X takes. So, that here, here, here, here, the probability of the side, this side, this is dy this is dz.

So, it will be proportional to the area of this, right.

$$Prob\left(y \le Y \le y + dy, z \le Z \le z + dz \big/ x \le X \le x + dx\right) = P_{YZ/X}\left(yz/x\right) dy\, dz$$

So, subject to this, subject to the above, probability capital Y lying between small $y + dy$, capital Z lying between small $z + dz$, it subject to this way we write, subject to the condition that capital X lies between small $x + dx$ and x. So, that will be linearly proportional to the area of this and so, it will be a proportionally constant, which is P times $dy\, dz$ and this P again will depend on the location, because the entire thing will change if I move the box elsewhere, even if the size of the box remain same. So, this P will be a function of $x\, y\, z$, but that I write in this manner, we follow a convention, first we write those value variables $y\, z$ which are free, this by the constraint. capital X is constraint and then y and z are taking various values.

So, free variables, y is a constraint variable, this is the notation. But actually, it is a function of these three and that corresponding values are y, z small z by small x all right. capital X is small x capital Y takes small y, capital Z takes small z. So, this is the conditional probability, what kind of condition? That capital X is subjected to remain between x to $x + dx$, subject to that, probability of capital Y falling between y to $y + dy$ and z from small z to $z + dz$ is this times $dy\, dz$. So, this is one conditional probability density ok.

$$Prob\left(y \le Y \le y + dy, z \le Z \le z + dz \big/ x \le X \le x + dx\right)$$
$$= P_{YZ/X}\left(yz/x\right) dy\, dz\, .\, P_X(x)$$

So, then overall chance of capital $X\, Y\, Z$ lie within the box will be how much? First, we subject capital X to lie here and then find the probability of y z lying within this box, any box that times, now I have to I mean, I have to multiply that by the chance of capital X lying here, lying from here to here. So, the entire box will be covered. First, I fix it here, capital X I fix within this, subject to that, I find the probability of capital Y and z lying in any of these planes, any of this square plane that will be proportional to this. Now the chances of capital X really occurring from here to here, I have to take that, multiply that probability with this, then that will be the overall chance of capital $X\ Y\ Z$ lie within the box. Therefore, this thing, same probability of the same thing I am writing.

$$P_{XYZ}(x,y,z) = P_{YZ/X}\left({}^{yz}/_{x}\right) dy\, dz\, .P_X(x)$$

Instead of writing there I can also erase this, no need to rewrite the same sentence. And this is equal to then this probability or this quantity. This is equal to this in one way and the same thing will be, I have to multiply this by into the chance of capital X lying from here to here, which is we have seen this. So, if I now equate the two, we get this relation joint density, because dx dy dz common on both sides and, they are positive. So, you can take them common.

So, that is a product of either, here these are the two free variables then constraint X. So, value of capital Y is small y, value of capital Z is small z by value of capital X this times. So, if it is a constraint variable. So, now, there will be another probability density of this which is constraint variable. So, probability density of that alone will be here all right.

But I can view the thing in another way ok. I can further better think that suppose we are constraining not only capital X. we are constraining capital X and capital Y both, capital X will lie from here to here, capital Y lie from here to here. So, this is the zone in which capital X capital Y will figure. But capital Z can vary.

Ok capital Z can vary from here to here anywhere, like capital Z could be like this, from small z to z plus dz. So, these are lying. So, that chance will be that is subject to capital X lying here and capital Y lying here. That is the capital X comma Y pair lying on this square, capital Z lying from here to here. So, could be here to here, could be from here to here, could be from here to here, could be from here to here anywhere, but that will be that, but on this.

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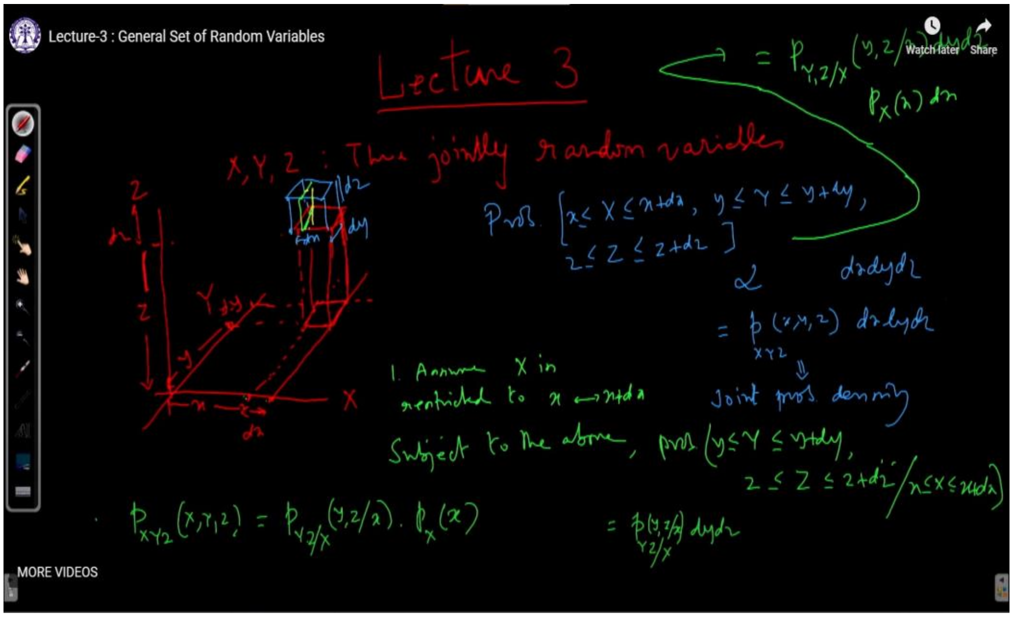

So, that will be proportional to the height of this ok by the same argument. So, you can also work like this let me go to the next page. So, with reference to that probability of just capital Z is the free parameter now, subject to capital X lying from simultaneously, that will be proportional to just dz and I take it to be linearly proportional because dz is so small.

$$Prob\left({}^{z \le Z \le z+dz}/_{x \le X \le x+dx\ , y \le Y \le y+dy}\right) \alpha\ dz$$

$$= P_{Z/_{XY}}\left({}^{Z}/_{x,y}\right)dz$$

So, this constant, proportionally the constant will be p and it will be again the three random variables are x, y, z, but now I write again in that form, first the free variable or variables, here only one is a free variable. So, z by the two constraining variables now which are constrained to remain here and here.

$$Prob(x \le X \le x+dx, y \le Y \le y+dy, z \le Z \le z+dz) = P_{XYZ}(x,y,z)dx\ dy\ dz$$

So, they come next. So, their values also capital Z is small z divided by small x small y times dz. So, then probability of X lying between small x to x+dx, capital Y from a small y to $y + dy$ and capital Z from small z to $z + dz$, which we already have seen in terms of joint density. This again, we can view this, this probability multiplied by the chance of this happening. First, I am assuming this is fixed, this has already happened and then what is the probability of this, this is this point. But what is the probability of this event happening, that times this.

$$Prob(x \leq X \leq x + dx, y \leq Y \leq y + dy, z \leq Z \leq z + dz)$$
$$= P_{Z/_{XY}}\left({}^{Z}/_{x, y}\right) dz\, P_{XY}(x, y)$$

So, that will take care of the overall thing. That is capital X lying here, capital Y lying here, capital Z lying here simultaneously ok. So, that will be, this probability, this chance factor, then this happening. This happening together means joint density, x lying here, y lying here. So, if you get the two and forget about dx dy dz they are common, see now you can write p x y z as p z by x y small z, previously you have seen that time only x was constrained.

$$P_{XYZ}(x, y, z) = P_{Z/_{XY}}({}^{Z}/_{x, y}) P_{XY}(x, y)$$

So, it was y z does not matter y z or z y, I am just writing y z by x and then, this is you see out of all the variables you can put some to be free, like y z were free, subject to the constraint. So, this was constraint. So, put like p probability of y z condition to capital X. So, y taking for the small y, z taking for the small z, this capital Z and capital X taking for the small x, times the probability density of the variable or variables which are constrained. Here I had y z free and x constraint.

So, it was multiplied by probability density of x only, there is only variable under constraint. Here I took x y together under constraint, subject to that capital Z was free to take values. So, z by x y then it is x y. If it is y z by x then it is x. Similarly, if it is x z by y then it is y, if I could have an x z by y, then it would have been y, capital Y accordingly small x small z by y and y small y all right.

$$P_{YZ/_X}({}^{y,z}/_x)P_X(x)$$

So, you can take these variables, take some of them to be free by some of some of them others which are constrained. So, that is a conditional probability, that is subject to this constraint variables taking certain ranges, what is the probability of the other variables falling within certain other ranges, that times now we have to take the probability density of this constraining constraint variables, joint probability density of the constraint variables ok this is a general result.

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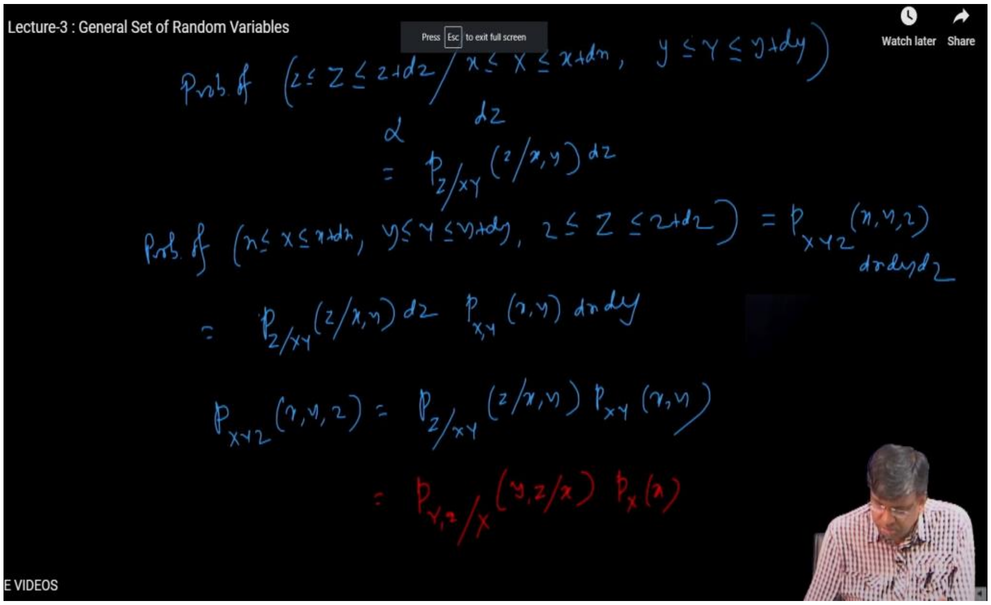

Now, certain things follow from here, suppose we are given a function $f(X)$. Ok it is only one variable, but I want to find out $E_{X,Y}[f(X)]$. In general, any function whether x or y or y of f, if I am taking E x y, it is nothing but, take the double integral of the function small x within integration, because it is a integration variable.

Capital Z of the general notation of the variable random variable, but small x is the actual integration variable the value on the x is x x is whatever the point. So, this is to multiplied by the joint density because expected value with respect to two variables. Even if inside right, this function is such that depends only on capital X, does not matter. What is the average value with respect to capital X Y? So, I have to multiply it by the joint density as before, then what I see this f x. So, I will now use this relation, I will take multiply by a conditional density, I will take y whatever is not figuring here that to be free variable y, whatever is figuring here that to be the constraint variable.

So, p y by x small y by x and if that be then I have to multiply it by p capital X small x, y by x and x, y by x that is first x is capital X is constraint, then probability density, conditional density and then this is multiplied by the probability density itself of capital X only ok. I will take it this way because you will see what happens after this if I take it choose this way.

$$E_{X,Y}[f(X)] = \iint f(X)\, P_{XY}(x, y)\; dx\; dy \;\; P_{Y/X}(y/x) P_X(x)$$

f x p x x, it is a function of x, it is a function of x ok. So, but they do not have any y. So, I take them out in the outer integral, inner integral is with respect to y, now what is the meaning of this integral? I know what is the meaning of this, meaning of this product is if capital X is constrained within the zone small x to x plus dx that is fixed, not changing, then what is the probability of capital Y lying between small y to y plus dy.

$$= \int_{-\infty}^{\infty} f(X) P_X(x) dx \;\; \int_{-\infty}^{\infty} P_{Y/X}(y/x) dy$$

So, that is proportional to dy, for this integral gives thus the same probability, but from minus infinity to infinity. That is I mean, this is capital XY, this much is small x this is dx, this is y and this much is dy. So, probability of this pair I mean subject to capital X lying between here to here, probability of capital Y falling from here to here is proportional to dy and this much was the this was the proportionality constant. So, this much is the probability. If I integrate from minus infinity to infinity means subject to this constraint what is the probability, subject to this constraint what is the probability net total probability of capital Y lying on this line from minus infinity to infinity.

Again, that is certainty because even if capital X we are constraining here and then observing capital Y, capital Y will always occur from minus infinity to infinity, it will it cannot have any other value. So, probability of capital Y taking a value from minus infinity to infinity, even with this constraint that is 1, because there is certainty even under capital X lying from only here to here, it does not matter. So, chances of capital Y occurring from minus infinity to infinity under that condition even that is 1, because there is certainty. Even if capital X is kept, we chosen to be from here to here only, capital Y will always occur from minus infinity to infinity, it cannot occur it cannot take any other value. So, total probability of that which is given by the single integral this is 1.

So, this is equal to 1 and therefore, I am left with only this much and this is nothing but original function multiplied by its probability density, the original function of random variable X multiplied by its probability density and integral from minus infinity to infinity. So, that is the expected value if you go back to the definition of the same function which is put to fx which is put to x. So, if there is no, I mean if there in the argument of the function if one random variable is not figuring and we are still taking E xy, then you can jolly well drop y and write like this, just y is gone here. It is like taking the expectation of f capital X with respect to capital X alone ok. Alternatively, another example, suppose you got 3 random variables X Y Z and there is one function still, suppose x y, so, this is again triple integral now, function as it is we put, but with small x small y and there is a joint density.

Here these are the 3 random variables x, these are the ok, these are figuring within the function, a capital X which takes a value small x here, capital Y which takes a value small y. So, x y the figure here. So, as I told you our policy will be to set z to be free and capital X y to be the constraints. So, this I will write as P capital Z free, sorry, P capital Z by x y, these are their capital X y, they are under constraint, they will take values from certain range, subject to that, probability density of capital Z, there is a probability of capital Z lying between small z to $z + dz$.

$$E_{XYZ}[f(X,Y)] = \iiint_{-\infty}^{\infty} f(x,y)P_{XYZ}(x,y,z)dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)P_{XY}(x,y)dx\,dy \int_{-\infty}^{\infty} P_{Z/XY}(z/xy)dz$$

$$= E_{XY}[f(x,y)]$$

There is this conditional probability. So, capital Z takes value small z by into this we have derived earlier. Now the probability density of these 2 constraint variables joints what happens then this integral I can write as outer integrals f x y and this P with respect to dx dy because they do not depend on z and the inner integral with respect to z that is small z dz. Again, this is equal to 1 because you are constraining capital X and Y to fall within certain area under that probability, the probability of capital Z occurring from small z to dz was this much and of the integral is the under the same constraint probability of capital Z lying from minus infinity to infinity. That will always happen capital X and Y may be constraint, but even under that, capital Z will always occur between minus infinity to infinity that is certain to happen. So, this is certainty its probability is 1 and if that be I am left with this much only, which is the function multiplied by the function of 2 random variables multiplied by the joint density of those 2 random variables and integral, which is nothing, but just E x y z gone.

This again shows even if we have a series of random variables here, but the function contains only few of them and then you are taking its expected value with respect to all then you can drop those random variables, which do not figure here, like z was not figuring here and you see z has gone.

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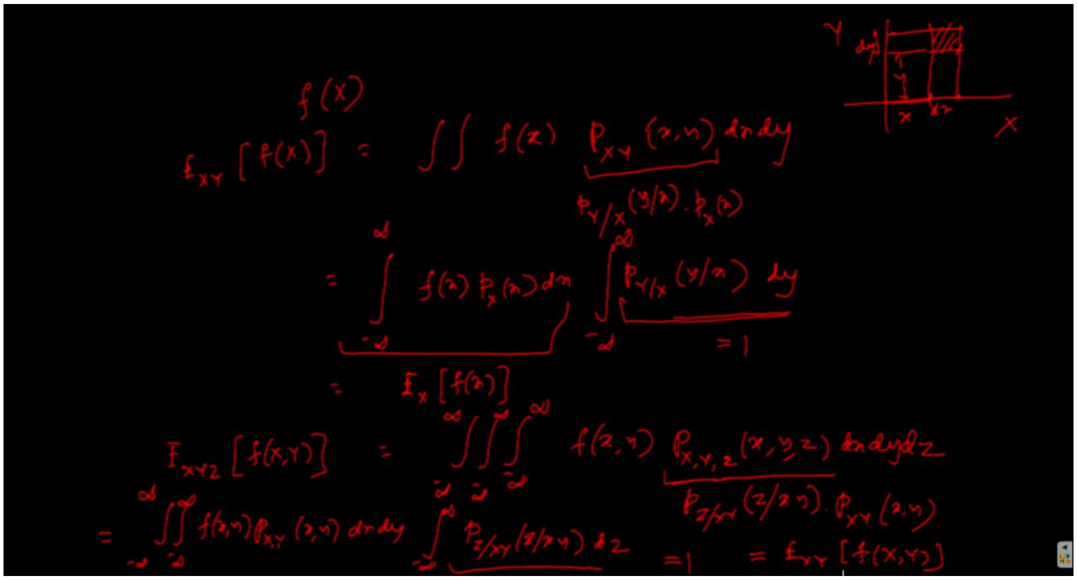

So, you can drop them. So, I stop here now and in the next class we begin from here. Thank you very much.