Introduction To Adaptive Signal Processing

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Lecture No # 27

Second Order Analysis of LMS Algorithm (Contd.)

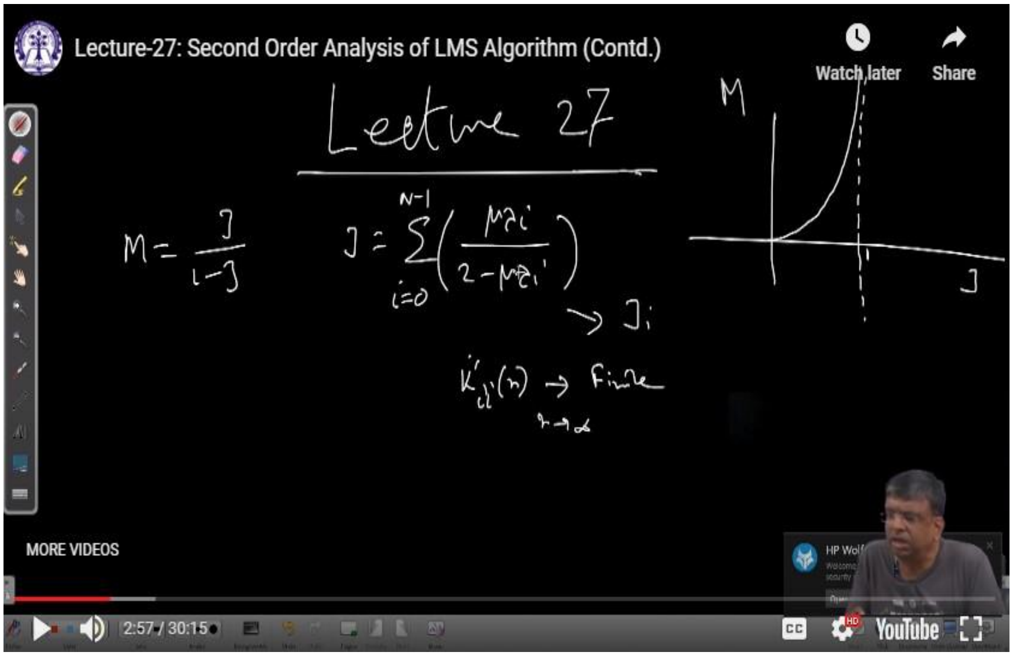
So, in the last class we had these results. Missed adjustment M was J by 1 minus J, J was summation of i equal to 0 to N minus 1 mu lambda i by 2 minus mu lambda i. This whole thing is called function Ji. So, J is a summation of N number of such separate functions ok. We have seen M, M is miss adjustment. So, it is strictly non-negative ok and when it is stable it should be finite.

$$M = \frac{J}{1-J};$$

$$J = \sum_{i=0}^{N-1} \left(\frac{\mu\lambda_i}{2-\mu\lambda_i}\right)$$

If we plot M versus G this is what we had obtained. Similar zones we are not interested because to the right of this point J equal to 1, M becomes negative and negative M is of no importance to us ok. Similarly, if you go to the left M becomes negative. So, this is the zone and at J equal to 1 M becomes infinity.

So, at least J equal to 1 and if you put J equal to 1 you solve it, numerator will have a polynomial in mu, denominator will have a polynomial in mu. You solve it with if it is equal to 1, numerator equal to denominator solve it you get a polynomial in mu Nth order. So, you get N roots at those values of mu J is 1 and therefore, M does not exist. So, those values of mu are outside the region of convergence ok for the algorithm because if the algorithm if it converges that is it goes k prime i i i N if it goes to finite as N tends to infinity then only it is stable algorithm is stable and if it converges if it leads to finite terms then that summation was I mean M will become this a finite. No other it cannot take any other value.

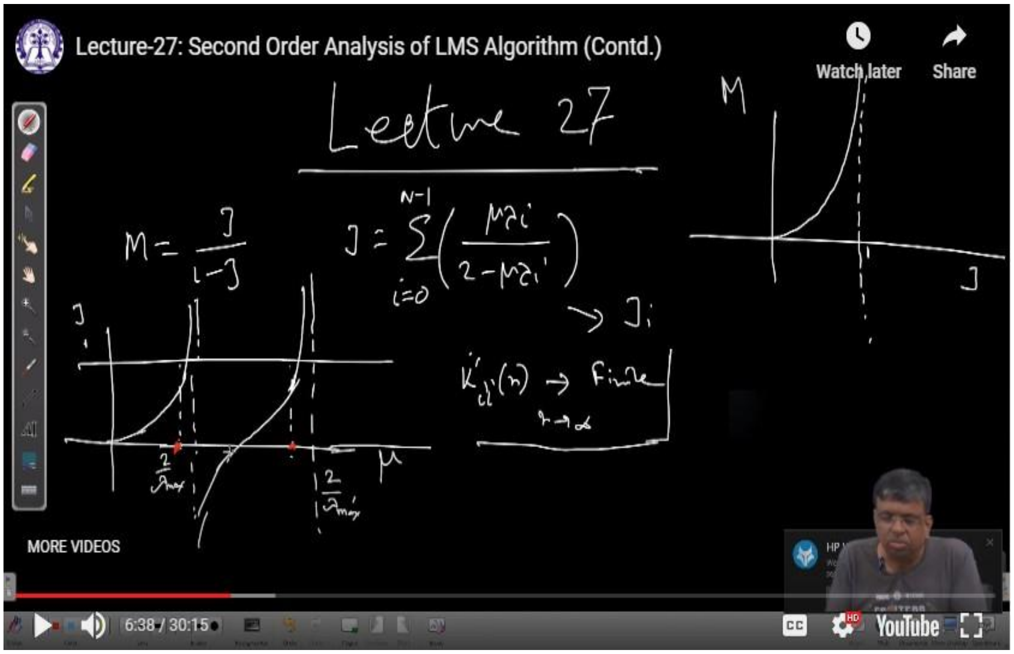


So, if it table then the misadjustment takes this value only there is no other value that is possible all right. So, what we did we are plotting J. So, J0, J1, J2 we consider the case where lambda is a lambda max all lambdas are positive because they are corresponding to the eigenvalues of the input autocorrelation matrix which is positive definite. So, if you consider the lambda max case see this Ji for any Ji it suits up to infinity only when u denominate is 0 that is when mu equal to 2 by lambda i at that value. So, if it is lambda max the 2 by lambda max will come first if we start from origin and I am not going to explain again this J versus mu was like this.

So, 2 by lambda max will occur first then again next one maybe 2 another 2 by 2 by lambda max prime that is next to lambda max the next value and like this and J equal to 1. So, these are the values of mu by J equal to 1 that is if you solve that polynomial thing ok you get this J equal to 1 and then solve factorize the polynomial and get first order factors. You understand what I am saying if we equate this to 1 then if you have J0, J1, J2 all these kinds of things you know if you do algebraic addition numerator denominator will be a product of these factors. So, it will be a polynomial in mu, numerator also will be a polynomial mu because mu lambda i times 2 minus mu lambda j 2 minus mu lambda k like that like that and so on and so forth in other cases. So, numerator will be also a polynomial in mu.

So, numerator by denominator equal to 1, numerator minus denominator 0. So, you have a polynomial equal to 0. So, you factorize the polynomial into first order factors of mu those are the values of mu for which equal it will be left hand side will be equal to 0 means J will be equal to 1 these are those values. Clearly these are the values at which M will suit up to infinity. So, these are outside the region of this thing convergence all right.

After that I did something maybe you know I brought some numbers a, b, c which are between 0 to 1 and I showed one inequality I am coming back to that later. But before that one thing is sure if M is infinity that is j equal to 1 those values of mu cannot be part of the region of convergence that is this figures k prime i i n cannot converge to finite value for those mu because if it at all converges to finite value it will converge to this expression where j is not equal to 1 because j equal to 1 means it if it is it will not converge, converge means converging to finite value or we have seen if it converges it will converge only to those ok this figure with j not equal to 1. So, that is it is a finite value therefore, those mu for which j equal to 1 they will be outside the region of convergence. So, these are the mu's let me use some other ink. So, this is one mu this is one mu these are the points which lie outside.



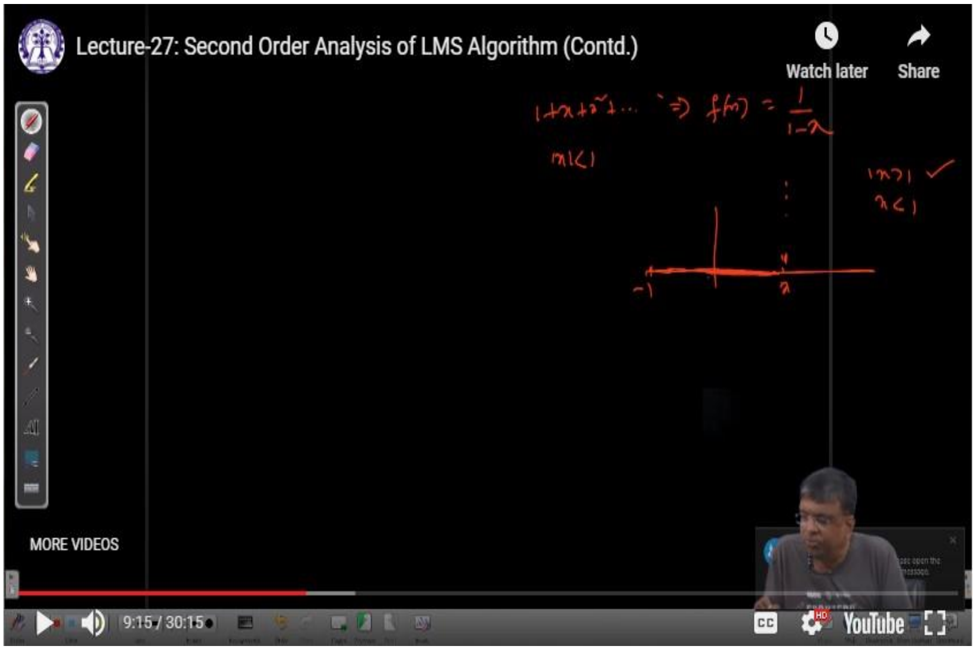
Now what happens to the intermediate regions between two red dots? If I take that mu does this converge to finite value or not that we do not know, but if it converges to finite value, it will be finite ok it will be finite you will take this expression all right this much. So, I know that at these points this function suits up to infinity. So, k prime i i n does not converge ok it has to not only converge to M it has to converge to finite M. So, those are the values where this will not converge, but the intermediate range and all I do not know that is because as I told you the other day let me go to a new page. If you take an expression like say suppose 1 plus ax plus x square dot dot dot it gives you a function fx equal to 1 by 1 minus x for this mod x should be less than 1 ok.

$$1 + x + x^2 + \cdots \Rightarrow f(x) = \frac{1}{1 - x}$$

$$|x| < 1$$

So, x equal to 1 so this zone 0 to 1 ok suppose x is positive a real and positive to be less simple. So, 0 to 1 for that region this summation converges to a finite value and if it converges will take this value, but when you look at this function forget about the series this function does not exist at x equal to 1. So, x equal to 1 cannot be a point of convergence here, because if it is a point of convergence it will converge to this value with a finite value for this functional is a finite value at x equal to 1 that does not happen. So, x equal to 1 cannot be a point of convergence, the point of convergence means this summation will converge to this function with a finite value that is what happening here. So, x equal to 1 is ruled out, but x less than 1 greater than 0 I am assuming x to be say positive otherwise you have to go from minus 1 to 1 if you want to be precise mod x to be less than 1.

So, any of these values here take any of the value this will converge, but when you look at the function, function does not exist at x equal to 1, but it exist at x greater than 1 also at x less than 1 also. Only thing is at x equal to 1 function does not exist means that is not part of the region of convergence of this.



So, this much from the looking at the function this much we can say that the that point where it is should it is does not exist where x equal to 1 that cannot be region of convergence, but the function I know giving the function it exist for x greater than 1 also less than 1 also ok. In fact, it is better if I put it just x greater than 1 alright only it exist only x equal to 1 does not exist. So, from the function by looking at the function I can make out that it will not converge here ok or here, but function exist on this side also function exist on this side function exist on in this in the region also ok.

Of all these places some zone may be a region of convergence for this series. Now I see one thing if I put x equal to 0 in this series it is 1. So, it is a finite value. So, x equal to 0 is ok it is a region of what are the region of convergence and 0 to 1 is 0 to 1 then what will happen if it converges at a value of x x equal to 0 and then I increase x by very small amount epsilon very small amount ok. It will still be it will not suddenly should up to infinity because epsilon is very small.

So, there is a neighborhood and it will go on like that till we find the point where the summation will finally, should up to infinity its value will be increasing when you give x equal to 0 value is 1 if you give x equal to epsilon may be some positive number value will increase still increase, but finally, at certain value this function will this will this will be shooting up to infinity. So, it will not take this function is a finite value that happens at x equal to 1 before that. So, it will continue to be like this ok going up. So, any this from 0 till we reach here we do not touch this, but in this point, this is convergence ok. So, we find out one point of convergence that is equal to 0 and from that we go to the point of discontinuity point of you know divergence not convergence till that point not touching the point this will converge ok this will converge to this form.

So, this is the thing we bring here. We know that the LMS algorithm wn plus 1 wn plus mu xn en. So, w1 if you take n equal to 0 n equal to 0 w1 is w0 and suppose I take mu equal to 0 then this is 0 then w2 will be w1 which is again w0. So and so because mu is 0 means no updates. So, initial value w0 continues at w1 w2 w3.

$$\underline{w}(n+1) = \underline{w}(n) + \mu \underline{x}(n) e(n)$$

$$w(1) = \underline{w}(0); \quad \mu = 0$$

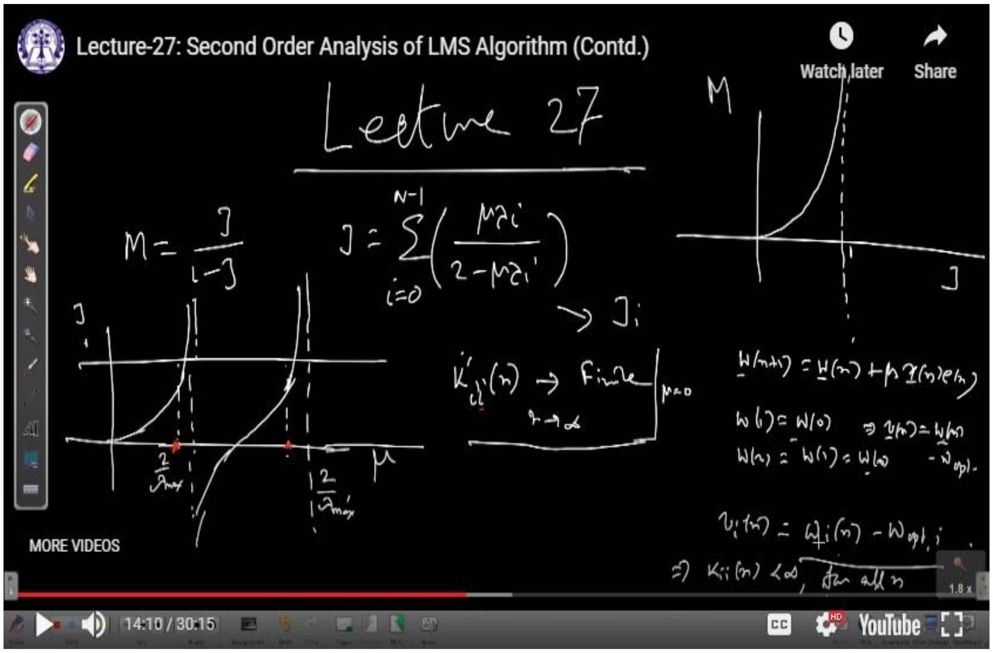
$$w(2) = w(1) = w(0)$$

So, error error was that component vn ok, vn is we know wn minus w opt, but wn whether wn is 1 n is 2 n is 3 it is constant which is w0. So, vn also constant initial value minus w opt.

$$\underline{v}(n) = \underline{w}(n) - \underline{w}_{opt}$$

So, any component of n vin will be win, win minus w opt i is constant because win is constant i th w1 is same as w0 if mu is 0, w2 is same as w1 if mu is 0, w1 is w0 so on and so forth. So, all the filter weights remain constant at w0 ok in this case mu equal to 0 case mu equal to 0 then what happens to that error vector weight error vector with respect to the optimal weight. So, w at nth point of time we know it is wn minus w opt, but wn does not change it is same as w0, w1, w2, w3 are all same.

So, that is why vn also constant if I take any ith component of this vector say vin it will be ith component of the filter weight vector minus ith component of this and there is a constant because it is not changing with time. So, if that be the variance of this does not change it is finite ok. That means, ki in less than infinity for all n because variance does not change it is just a constant difference win minus w opt i and win is nothing, but wi0 because at 0th point of time that initial value whatever I gave wi0 that remains with that remains the case for all n. So, wi0 minus w opt i it is a just finite number it does not change as n changes. So, as a result variance also does not change is a constant variance is just the square of it ok that also remains finite for all n and if it is for ki in same for same for this is just a transform version of that alright which means mu equal to 0 this always remains finite.

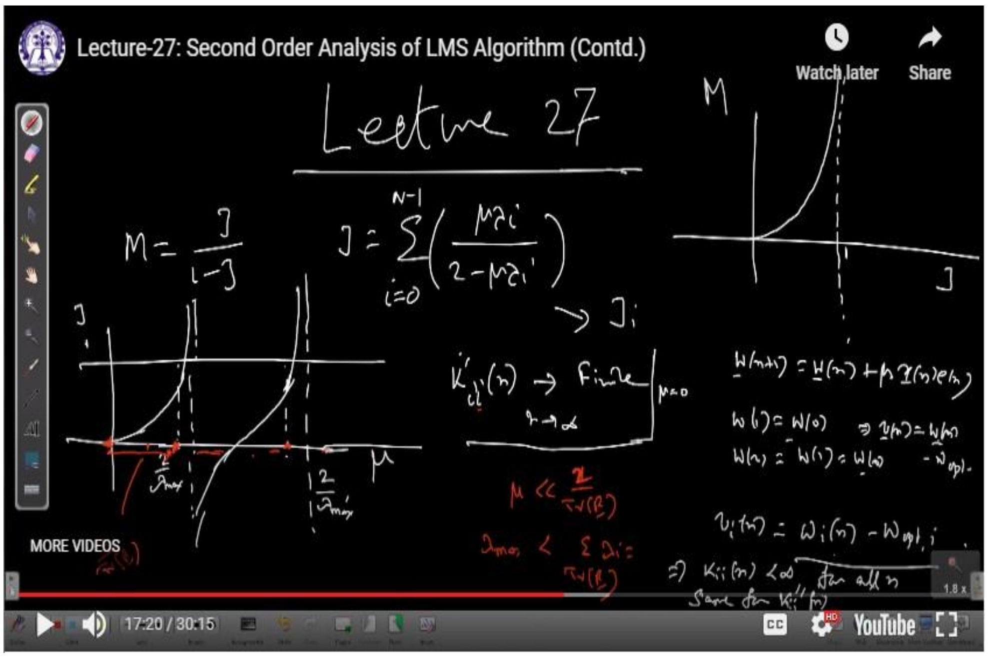


That means, mu equal to 0 mu equal to 0 this is a point of convergence and not only that we have already seen if mu is much less than 1 by tracer then it is algorithm is stable m remains finite and all that we have seen earlier alright. In fact, you can see mu is much less than 2 by tracer also that will fine,

$$\mu \ll \frac{1}{Tr(\underline{R})}$$

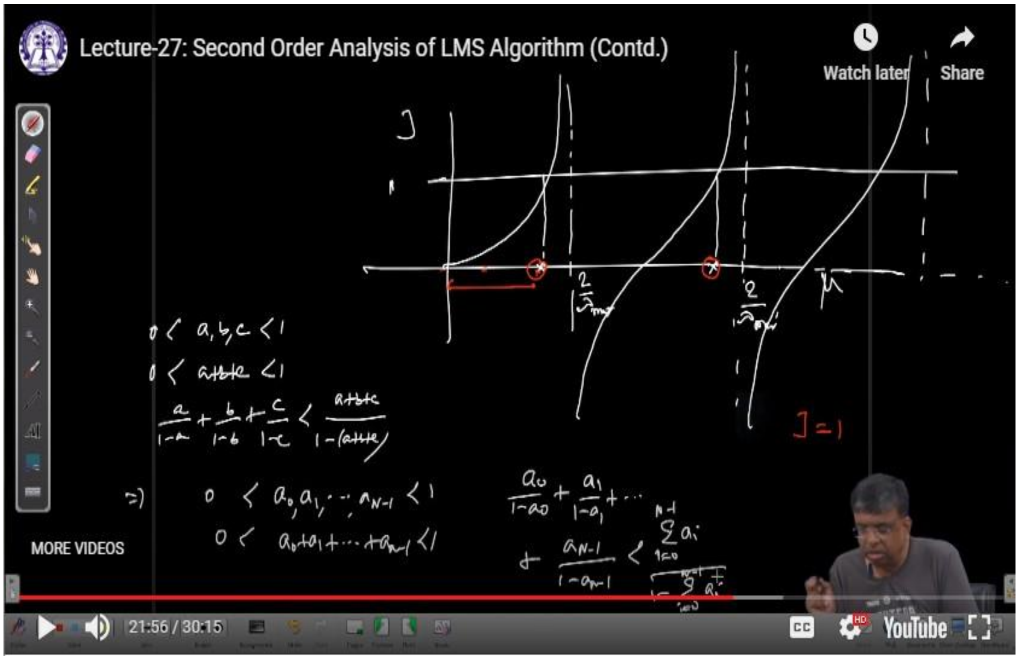
But 2 by tracer because 2 by lambda max, we know lambda max is less than sorry it is less than summation of all the eigenvalues right all eigenvalues are positive one of them is lambda max other eigenvalues are also positive. So, summation and that is tracer. So, lambda max less than tracer by r less than tracer. So, 2 by lambda max is greater than 2 by tracer.

So, 2 by lambda max 2 by tracer will be somewhere here. So, if mu is less than this, we have seen already m is finite. So, that means, from mu equal to 0 if you go further to the right we are still in the region of convergence and this will remain so till we hit this point first point where j turns out to be 1 and therefore, m shoots up to infinity. So, at least this much we can take to be our region of convergence there may be other this place is also possible maybe mu if I take from this region or the some other region maybe it is possible that m also will remain finite for them, but I am not sure, but about this I am sure because mu equal to 0 is a point of stability up to in fact, as long as mu remains much less than 2 by tracer they are all points of stability and therefore, I can go up to I can increase, increase, increase till the point I time I hit this point where m shoots up to infinity because j touches 1 all right. So, I will be just finding out this range that means, I will find out at what point this will become 1 let me draw again.



So, these are the points these are the points ok let me draw again this by j this by mu and j will shoot up to infinity at 2 by lambda max then next time 2 by lambda max prime maybe like that. So, this will be like this next one again next one like this j equal to 1 is the case at point. So, this is this fellow this fellow these are the points of discontinuity, but I have seen at least one thing if I remain to the left from 0 to this till I hit this point because this mu equal to 0 is a point of stability and then if I go further to the right at least as long as mu remains much less than 2 by tracer m this is stable because m remains finite and if I go further and further this increases till we hit this point. So, this will be at least a guaranteed region of convergence in mu there may be other places also ok there may be other places also like from here if I go up to the next point maybe here also it is fine if I take mu from here m will be finite, but I am not sure I cannot work that out that is very difficult math very tedious, but I can work for this part there at this mu from here to here if I take mu from here to here I will be within the region of stability where mu equal to 0 is a point of stability and as you go along till 2 by tracer stability and you can extend further till it becomes mu equal to this point for which j is 1 alright. That means you have to equate that j equal to 1 and find it out what are the values of mu for which j equal to 1 take the 1 which is smallest, but that is again very tedious ok that is why we will not be using this I will do some further I would say approximation I will apply further tricks we have seen already that if a, b, c they are between 1 to 0.

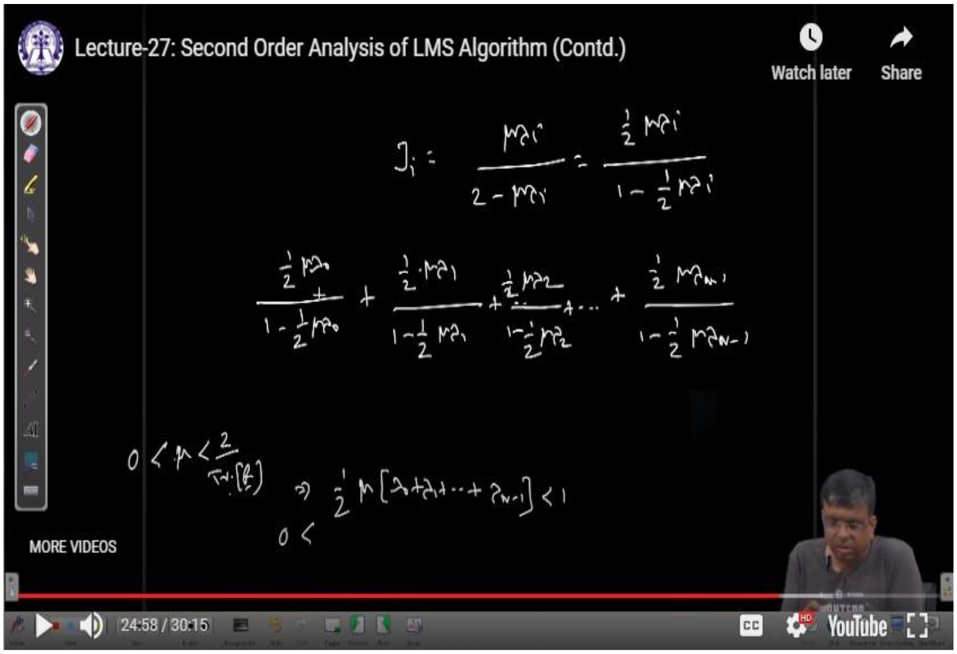
So, that a plus b plus c also between 1 to 0 which means a plus they are positive. So, that means a plus b is between 1 to 0 a plus c between 1 to 0 a plus c between 1 to 0 then a by 1 minus a plus b by 1 minus b plus c by 1 minus c is also less than a plus b plus c by 1 minus a plus b plus c. In fact, you can generalize suppose a1 a2 dot dot dot an maybe a0 a1 sorry there are some constants between 0 to 1. So, that a0 plus a1 plus dot dot dot an minus 1 is between these then we can extend this a0 by 1 minus a0 a1 by 1 minus a1 plus dot dot dot an minus 1 by 1 minus this is less than summation of these terms a plus b plus c now it is summation ai by 1 minus the same summation right. Now, we are doing this what is the expression of J? J i is mu lambda i by 2 minus mu lambda i.



So, you take two common it is half mu lambda i by 1 minus that is we have got this was J0 then right. We have seen mu should be less than 2 by tracer we transform the convergence in mean condition right that means, what is tracer? Summation of the eigenvalues. So, mu into tracer should be less than 2 or half that means, half if you take 2 here 1 by 2 half mu into tracer tracer means lambda 0 plus lambda 1 should be less than 1 and of course, greater than 0. So, mu is greater than 0 tracer is greater than 0, but what does it mean? Half mu lambda 0 plus half mu lambda 1 half mu lambda 0 half mu lambda 1 they are summation less than 1 ok and they are in the visual also they are between 1 to 0 ok. Because when you take all the eigenvalues then half mu and summation of the eigenvalue it is less than 1 if you take only one eigenvalue and all are positive then obviously, half mu lambda 0 less than 1 half mu lambda 1 less than 1 half mu lambda 2 less than 1 and like that.

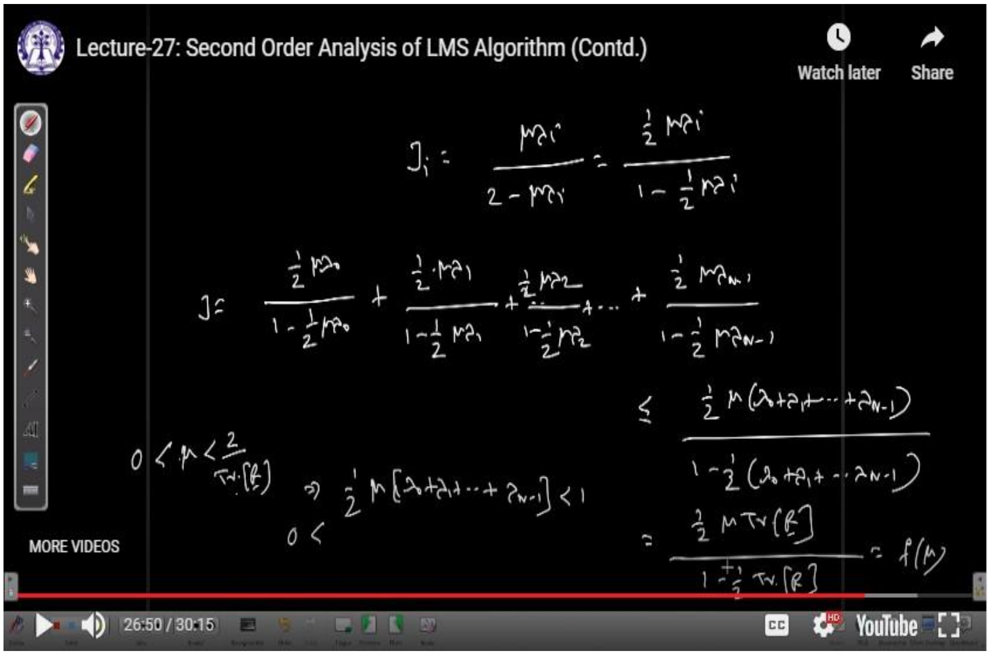
So, these two ok maybe I can take one more term here. If I take these two terms if we call this A numerator A A by 1 minus A plus B by 1 minus B A is definitely between 1 to 0 A is half mu lambda 0 because with half mu and all the eigenvalues summed and all are positive eigenvalues if that is less than 1. So, if you have only one eigenvalue or two eigenvalue multiplied by mu and half that is obviously, between 1 to 0. So, this is between

1 to 0 if you call it A A by 1 minus A and B by 1 minus B. So, this part will be less than A plus B by 1 minus A plus B.



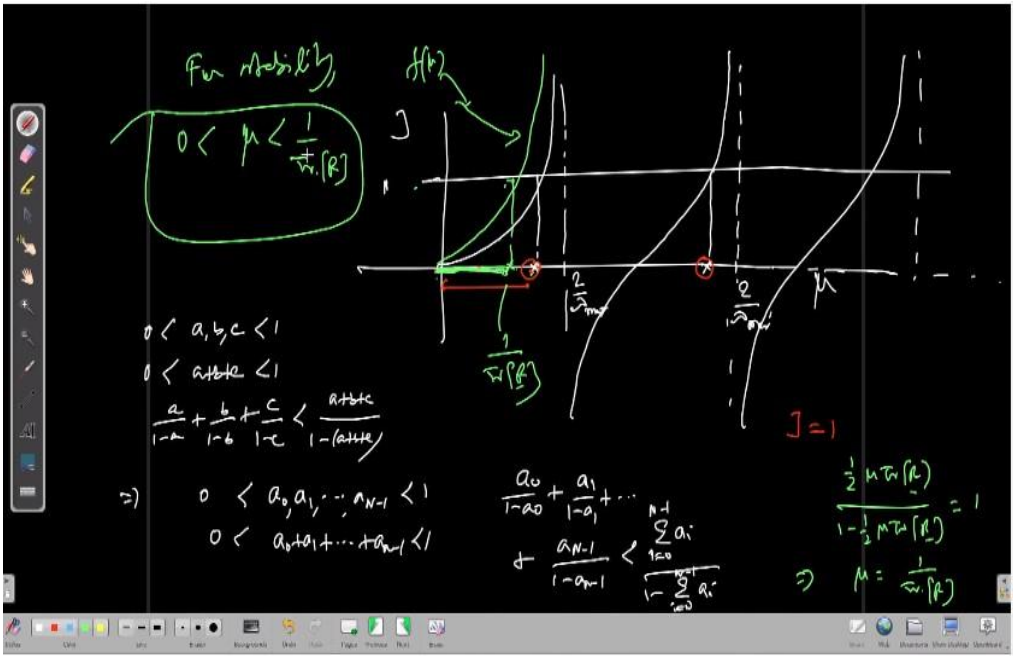
Then again A plus B means half mu within bracket lambda 0 plus lambda 1. Again, if you take half mu just lambda 0 plus lambda 1 only remember other eigenvalues. Obviously, it will still be less than 1 greater than 0 because with all eigenvalues which are positive added it is still less than 1. So, if you take just two of them obviously, it will be less than 1. So, half mu bracket lambda 0 plus lambda 1 if you call it B.

So, D by 1 minus D, D is also between 1 to 0 and this guy D is also between 1 to 0 and their summation half mu lambda 0 plus lambda 1 plus lambda 2 that is also between 1 to 0. So, that will be this will be less than D plus if you call it C D plus C by 1 minus D plus C and go on doing it. So, then we finally, that is half mu trace of R because summation of eigenvalue is same as trace of the autocorrelation matrix that is what I have used here ok. So, let me call it a function of mu. So, this function is above j for any value of mu j is less than f of mu.



That means, if I plot f of mu it will at any value of mu it will be above this. So, it might be like this f of mu because at any point of mu f of mu will be above this which means it will cut this a equal to 1 line earlier here before this point and there is within the region of stability this much there is to the left of this point of instability or divergence there is non convergence this point is to the left of this. So, now, what I will do I will take only this segment as my region of stability I know some points to the right will be missing, but I cannot find them at least this much if I take I will guaranteed that m will not suit up to infinity and therefore, algorithm will be stable you know that x is mean square error variance will not go up all that. So, what is this point? This point is when f mu this curve equal to 1 and f mu was what was f mu half mu trace R by half mu trace R if that is equal to 1. So, you take to the then this will basically mu equal to 1 by tracer.

So, this point is alright 1 by tracer. So, for convergence for stability we need mu less than 1 by tracer this zone. Alright if you choose mu here your algorithm will be stable m will be under control ok we will find n value. So, the excess mean square error variance will not suit up to infinity.



So, I stop here today we have done enough analysis of the LMSI algorithm both first order and second order this convergence in mean and that weight error correlation matrix or covariance matrix or variance of that whether it suits up to infinity or not output mean square error you know and it is a square mean or that rather excess mean square error normalized which is misadjustment in the steady state whether that remains finite or not all these things we have analyzed. I did not give proof of some other results because that is beyond us now given the paucity of time, but with this you can work out this is your region of stability alright.

So, this much for today and I will join you next time. Thank you very much.