Introduction To Adaptive Signal Processing

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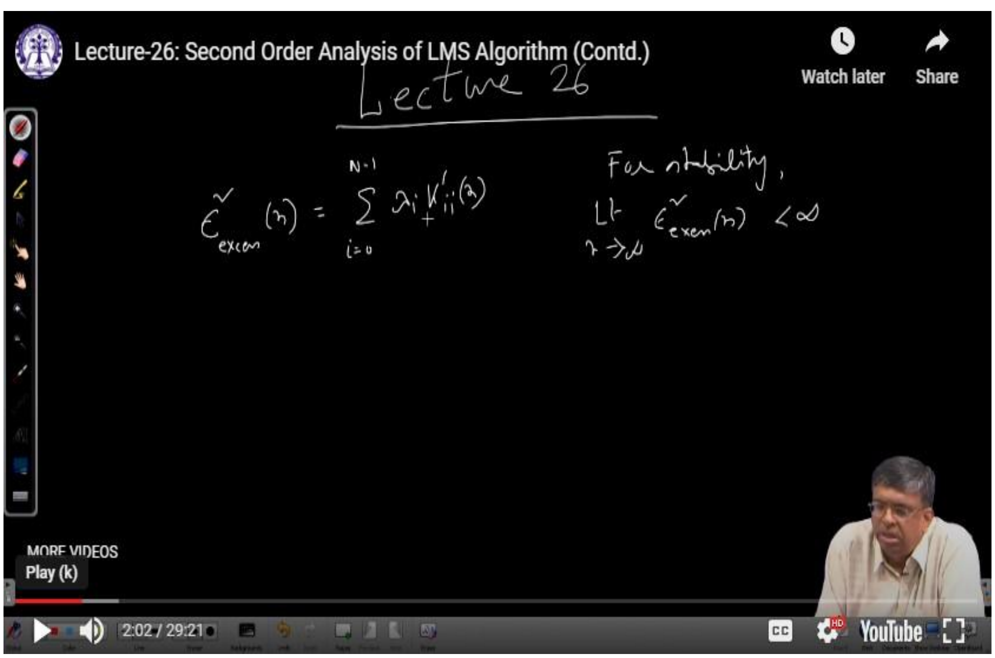
Lecture No # 26

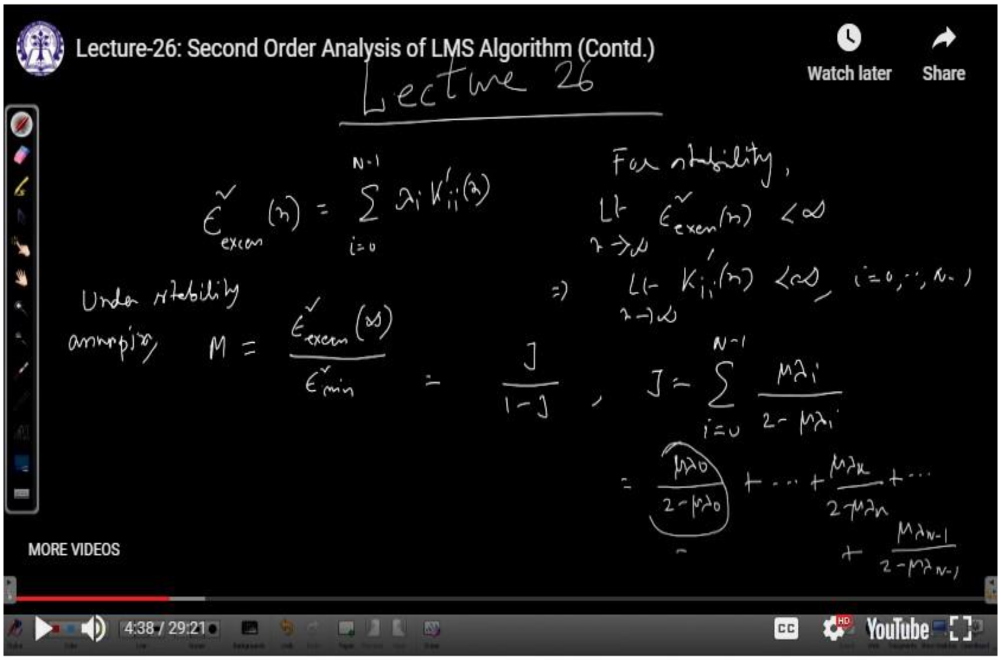
Second Order Analysis of LMS Algorithm (Contd.)

So, in the last class we already discussed this excess mean square error, there was epsilon square excess. This is what we discussed found at any index n, ok. Lambdas are the auto Eigen values of this autocorrelation matrix input autocorrelation matrix and i i means it is a transformed weighted error correlation matrix the i th diagonal entry. Their capital N number of sequences as N tends to infinity for stability this is repetition of the last part of yesterday last class should be finite. Now, remember every lambda is positive because input autocorrelation matrix is positive definite and this is transformed weighted error correlation matrix i th entry. So, which is the variance, variance also positive cannot be negative.



So, all the sequences are positive valued sequences worst case it can be 0 also, but not negative. So, it is not they can cancel each other. Therefore, if the whole thing is to be finite as N tends to infinity every sequence that is every it means lambda limit if you take limit k i i prime N this also should be finite for i equal to 0 to N minus 1. Because it cannot happen that there are two sequences which goes to infinity as N tends to infinity, but one is plus another is minus so they cancel each other.

So, summation still remains finite this is not possible because everywhere Eigen values are positive here because input autocorrelation matrix is positive definite and this is a variance which is positive these are stability. Now yesterday I defined something related to this just a normalized version called mis adjustment M which is actually it should be infinity which is nothing, but this this quantity that is limit of epsilon x epsilon square x s N as N tends to infinity that is indicated by epsilon square x s infinity just this is normalized to this. Now one thing is very clear M and it is under stability assumption under that is assuming that this sequences will converge to converge to finite value, value so that summation also convert this to some finite value under then this M which is nothing, but this a scale version is has been worked out and this is j by 1 minus j where j is a summation over this mu, mu is the step size used in LMS lambda is the ith Eigen value of the input autocorrelation matrix. So, for each Eigen value there is a term here. So, you can write like this you know I mean for mu lambda 0 I expand mu lambda 0 by 2 minus mu lambda 0 dot dot dot in general mu lambda k by 2 minus mu lambda k plus dot dot last term is mu lambda N minus 1 divided by 2 minus mu lambda N minus 1.



So, this is one term let me call it j 0 this is another term let me call it j k and let me call this j N minus 1 right. Now one thing is there that M will not exist at j equal to 1 and if you get j equal to 1 like if you sum them, you know you might have there will be actually a polynomial in terms of mu, mu into lambda 0 into product of other denominators except for these and so on and so forth. So, eventually there will be a polynomial in mu and denominator that is nu minus that is nu minus denominator also product of this factors 2 minus mu lambda 0, 2 minus mu lambda 1 dot dot dot. But a nu minus we will have things like mu into lambda 0 times 2 minus mu lambda 1, 2 minus mu lambda 2 dot dot dot then mu into lambda 1 2 minus mu lambda 0 2 minus mu lambda 2 dot dot dot and likewise simple algebra. So, if we expand and sum it will be basically a polynomial in terms of mu.

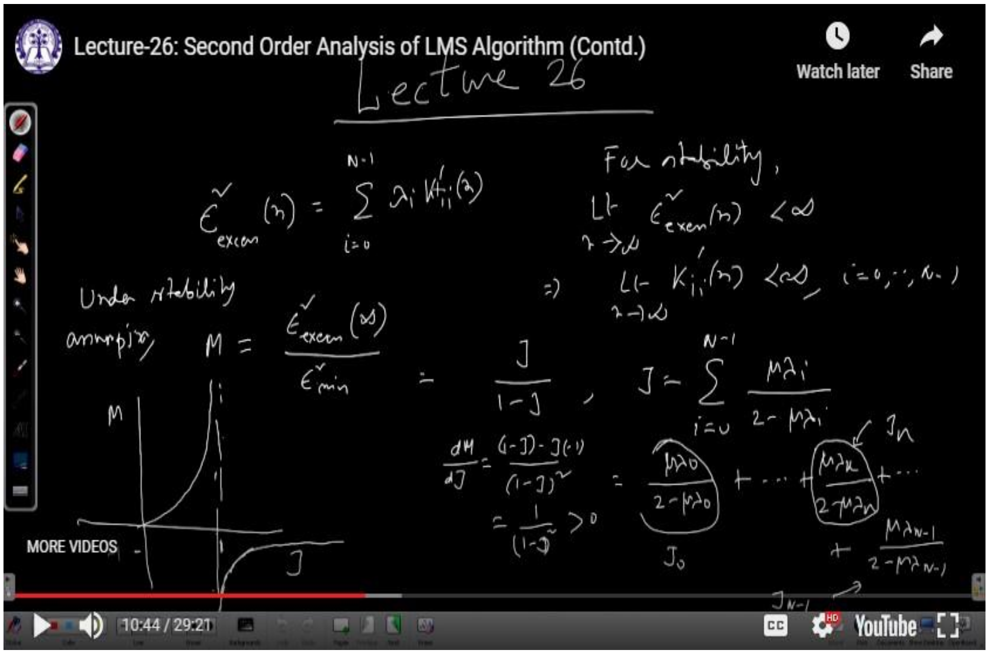
So, if you now equate j equal to 1 where M suits up to infinity you will get a denominator polynomial by denominator polynomial mu equal to 1. So, it will basically lead to a polynomial in mu equal to 0 if you take all the terms to another like a numerator by denominator equal to 1. So, numerator equal to denominator or numerator minus denominator equal to 0, denominator is a polynomial in mu, denominator is a polynomial in mu. So, denominator minus denominator is a polynomial mu that is equal to 0 you find out the roots. So, there will be capital N roots at those values of mu and those values of mu because it is a nth order j will be 1 and therefore, M will not exist will work in this line remember, but before that that is let me sum up what I am trying to say M if you take the plot M versus j we know M this numerator is non negative this is non negative say M is a non-negative quantity its value can be from 0 to infinity, but can never be negative.

Another thing you see if you take the gradient if you put j equal to 0 then M is 0 if you put j equal to 1 M suits up to infinity alright and dM dj if you take this derivative 1 minus j whole square then j you derive. So, you get 1 minus j minus j and if you derive this minus 1. So, basically 1 by 1 minus j whole square which is positive. So, gradient is always positive ok. So, its value will be like this at j equal to 1 it will suit up to infinity at j equal to 0 it is 0 and gradient is positive.

So, it can only be like this and then as j goes to infinity very large then 1 is negligible in comparison to j. So, j by minus j is minus 1. So, as j tends to infinity it is minus 1 ok and gradient is positive the moment j is little higher than 1 denominator is negative, but numerator is positive. So, value is a negative value with very high magnitude that means, you will start right after 1 you start from minus infinity and gradient is positive. So, here gradient is positive and it will go like this finally, it is minus 1 it will converge to this is a plot.

But we are interested only in the zone from for M we are only interested in the zone where M is from 0 to infinity positive values because you see this is where this is epsilon square x this is positive or non-negative this also non negative. So, M cannot be negative this is the only region ok and j cannot be less than 0 that we will see. So, this is a zone all right. So, j equal to 0 to 1 at 1 it should up to infinity. So, we have to find out when is j equal to 1 for what values of mu as I told you I repeat again equate to mu if you sum this you have a denominator polynomial in mu divided by denominator polynomial in mu numerator by denominator equal to 1 that means, denominator minus denominator equal to 0.

So, you get an effective polynomial in mu of order n that is 0. So, you factorize it in first order factors you get n number of different values of mu for which this will be you know 1. So, at those values of mu this will suit up to infinity all right. This will suit up to infinity because at those values of mu if you put summation will be j equal to 1 ok. You get n values of mu by factorizing that for resulting polynomial numerator minus denominator or numerator equal to denominator or denominator minus denominator equal to 0 you factorize it.



So, you get n roots of mu at all these roots j will be 1 and therefore, n will suit up to infinity these were I observed. Now, while doing it one interesting thing you see suppose I give you a series remember we are all series like if you take any particular ith case k prime i i 0 k prime i i 1 k prime i i 2 they are getting added after means multiplied by some you know constant the constant you can take out. So, just k prime i i n ok. So, this is one value they will get added over all the i after we multiplied by lambda i. So, this is a sequence ok.

Now, the thing that I am trying to do is this a side observation suppose I give you x is real and I give you a power series like this. We know if x less than 1 x is greater than 0 say give it x less than 1 then only it converges to what 1 by 1 minus x let us call this fx. So, when you look at the series summation that will converge to a value under x less than 1, but if I do not forget about the series, I do not know the series you give me the function fx equal to 1 by 1 minus x. Remember this exists for only if you put x equal to 1 this will suit up to infinity, but it less than 1 or greater than 1 like x is 2. So, it will be 1 by 1 minus 2.

$$f(x) = \frac{1}{1-x}$$

So, the value is minus 1. So, this function will exist. So, exist for both x greater than 1 x less than 1, but not for x equal to 1 ok. But remember this does not exist if x greater than 1 the series does not converge if x greater than 1, but the function if you take separately, it will exist it will have its value greater for greater than 1 also less than 1 also only x equal to 1 it will not exist. So, one thing is there if the function is not existing for a particular x then that x this cannot converge because if it converges it will always converge to this value that is what I have seen, but if it converges to this value and then I put the value of x it suits up to infinity that means it does not converge.

Therefore, x equal to 1 where this function does not exist it cannot be a point of convergence for this series because if the series converges it will converge to this only. But if I put x equal to 1 and this does not exist this suit up to infinity that means this series does not converge because if it is converges it has to converge to a finite value 1 by 1 minus x, but x equal to 1 this does not exist it suits up to infinity. Therefore, it does not this series does not converge there because if it converges it has to converge to this function if this function does not exist at x so for that x it does not converge. Because again I am telling you if it has to converge there is no other function it will converge this to it will have to converge to this value this value. If this value does not exist at that x then this it cannot go to any anybody else it just does not converge it cannot converge to some other function because if it converges it converges to only this function.

Therefore, at least looking at this function and finding out where it does not exist where it suits up to infinity at least one thing I can say those points converges does not take place. There may be more points where also convergence does not take place I do not know, but at least that those points convergence does not take place. Like here at x equal to 1 it does not converge, but not only at x equal to 1 x greater than 1 also it does not converge that I do not know suppose, but at least x equal to 1 it does not converge this I can speculate from the function. Because function does not exist function exist for x greater than 2 greater than 1 that is separate, but at least function if it exists then I have doubt it may converge may not converge that is it, but if it converges it will converge to this function and if this one does not function at that point does not exist then this cannot go to any other function it just does not converge.

There is a logic. So, now M from the previous page we have seen M this function will not exist will suit up to infinity for certain values of mu obtained by equating j equal to 1 and solving this essentially it will lead to a polynomial in mu equated to 0 nth order polynomial. So, factorizing first order factors you get the n different values of mu and those values of mu j equal to 1 and therefore, this suits up to infinity. So, those are the values of mu where at least this will not exist therefore, this k prime i i k prime i i n will not converge for any of the i this must be assumed, but they may not be the only values of mu where it will take place it can take place maybe there are more other values also that I do not know, but at least I can identify certain values of mu that if it is this axis mu axis and this is j suppose this is one point this is another point another point at this point in j is suits up to infinity. Here it is suits up to infinity suits up to infinity these points cannot be member of the region of convergence of M because at this point M suits up to infinity. So, M does not exist therefore, they cannot converge because if the summation has to converge it will converge only to Bm and therefore, if M does not exist this cannot go to any other function it can just cannot converge because if it converges it will always be Bm understood.

So, we will now next text will be to identify these points what are these points or all that, but that is again very difficult very difficult again because if you really equate to one this in the numerator there will be complicated numerator polynomial in mu denominator also complicated numerator polynomial in mu and this equal to 1 means numerator equal to denominator. So, eventually numerator minus denominator will be some complicated polynomial in mu then you have to factorize it will be hugely complicated function in terms of all and then that way we will identify there is the ideal way, but that is very difficult. So, we will not work that way, but before that let us try to investigate J equal to as you have seen typical Jk, J is a summation of such capital N number of factors 0 is factor, first factor, k is factor, N minus 1 is factor total capital N number of factors. So, take a particular component Jk, Jk is mu lambda k by 2 minus mu lambda k. So, Jk all right this is mu x is this is suppose Jk.

$$J_k = \frac{\mu\lambda_k}{2 - \mu\lambda_k}$$

So, at mu equal to 0 this is 0 it starts at with 0. If you take the derivative with respect to mu then lambda k mu lambda k into 2 minus sorry you derive it minus lambda k. So, minus mu lambda k square and plus mu lambda k cancels. So, it is 2 lambda k by 2 minus mu lambda k whole square. Remember lambda k is kth eigenvalue of the input autocorrelation matrix R which is positive definite and therefore, every eigenvalue is positive.

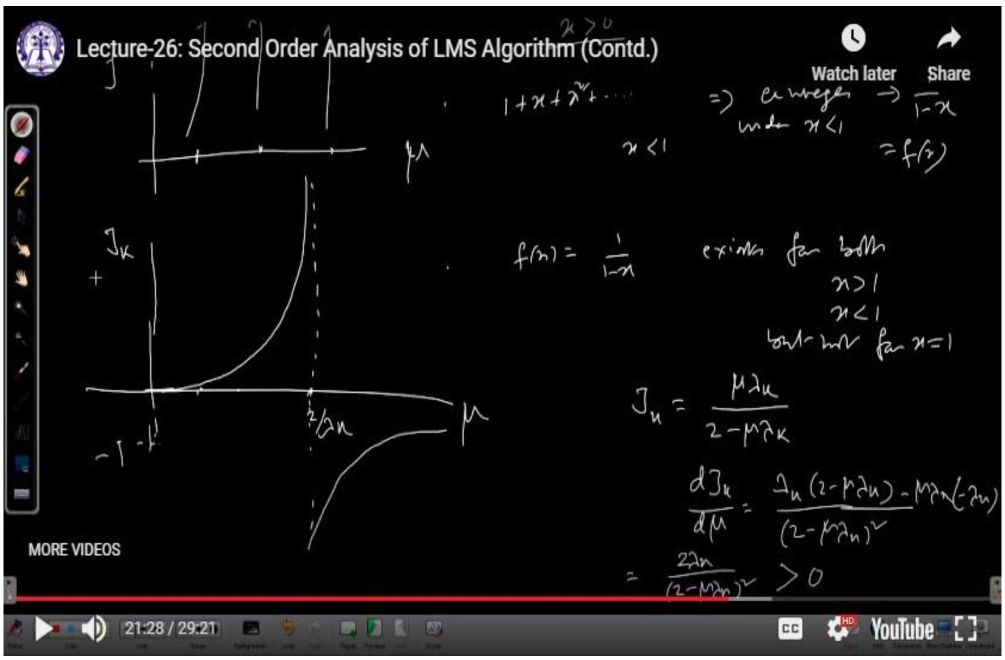
$$\frac{dJ_k}{d\mu} = \frac{J_k\,(2 - \mu\lambda_k) - \mu\lambda_k(-\lambda_k)}{(2 - \mu\lambda_k)^2}$$

$$= \frac{2\lambda_k}{(2 - \mu\lambda_k)^2} > 0$$

So, d by 2 is positive, d by 2 is already positive because it is power 2. So, this is greater than 0 which means gradient will always be positive this suits up to infinity when mu lambda k equal to 2 that is mu equal to 2 by lambda k. So, if it is 2 by lambda k here it will suit up to infinity. It starts at 0 it will go up like this and the moment mu crosses this space 2 by lambda k little bit this denominator becomes negative when the numerator is positive. So, it will be again a negative value with very high magnitude in fact, from minus infinity it will start and as mu increases to infinity 2 is negligible then.

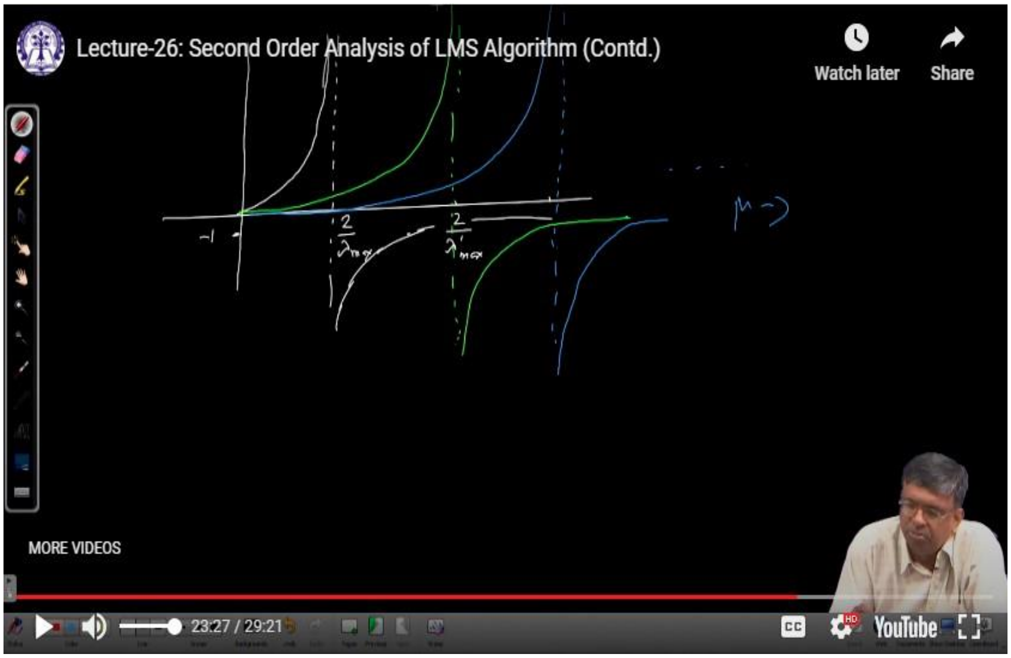
So, numerator, numerator cancels it is minus 1. So, again it will go to minus 1. So, 2 by lambda k. So, if it is 2 by I think now suppose the case for the highest eigenvalue lambda 0 to lambda capital M minus 1 one of them will be highest let me call it lambda max for that the corresponding plot will be having 2 by lambda max here. So, 2 by lambda max will be closest of all of these points to the origin because lambda max is greater than other lambdas.

So, 2 by lambda max is less than 2 by other lambdas. So, that will be closest to the origin that will come first let me call it point of instability or point of discontinuity that will come first then 2 by next lambda then 2 by next lambda like that and this function will have to be superimposed to get the total J it was only Jk.

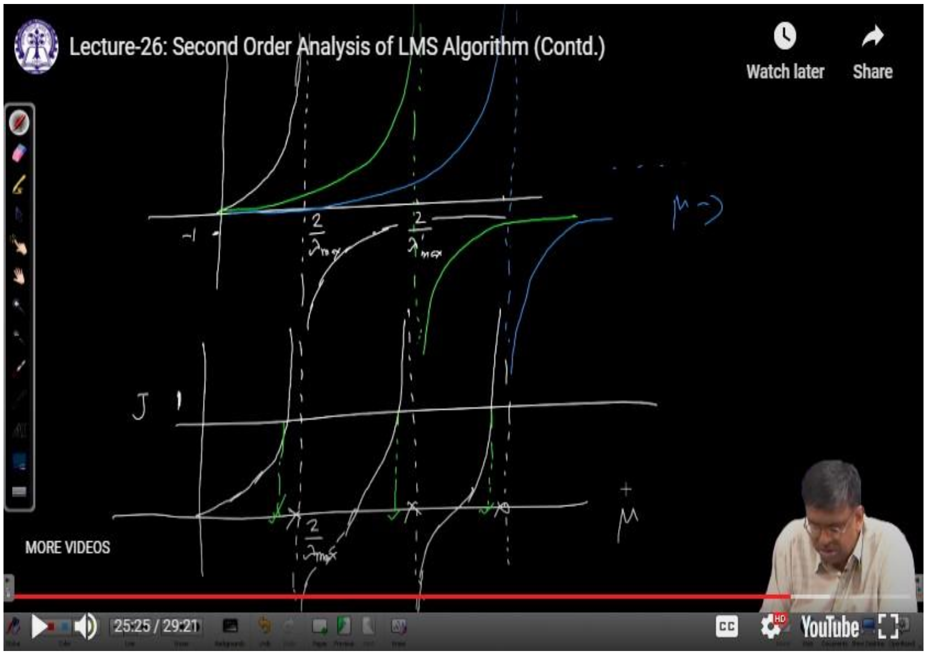


So, we do that if you do that, we will have 2 by lambda max then 2 by some lambda will be lambda prime max which is next to lambda max and dot dot dot dot. So, this one will have this is go to minus 1. Then next one another one will start like this another one here may be and dot dot dot dot is may be x is.

So, if you superimpose you see this is shooting to infinity. So, even if these guys are positive, but they are finite. So, this so superimposition will still suit to infinity because others are finite this is in getting infinity. So, it will suit up to infinity then again it starts at minus infinity. So, others are finite, but total will be minus infinity and good will go up, but it will not be like this because this guy is shooting up to infinity.



So, even if these fellows are finite this will be shooting up to infinity again like this here. So, this is 2 by lambda max this is another point. This is how the total J versus mu curve will look like and we are trying to find out this is 1. So, at what points what are these points? What are these values of mu for which J equal to 1? These are the values we obtain by solving that polynomial equation. Let mu into the order of polynomial equal to 0 and those are the distinct values of mu for which J equal to 1 and therefore, M should suffer infinity at least they cannot be you know they cannot be party to the this thing region of convergence.



We have to find them, but as I told you finding them is very difficult is very tedious. So, we will again apply some tricks. This trick is this again we will use some basic property from real numbers. Suppose a, b, a plus b is real so that they lie between 0 to 1 both a, b, a plus b. Then if I take a by 1 minus a plus b by 1 minus b what happens a into I get a plus b minus 2ab and here 1 minus a plus b plus ab.

$$0 < a, b, a + b < 1$$

$$\frac{a}{1-a} + \frac{b}{1-b} = \frac{(a+b) - 2ab}{1 - (a+b) + ab}$$

Now ab is positive if I take out this positive contribution and since a plus b is less than 1 this still remains positive, but overall will be less positive than the case when ab was present. So, if I remove ab denominator will still remain positive, but less so overall value will go up. So that means, this will be less than and the denominator this minus this. So, this negative term is removed negative term is removed. So, overall, it will go up a plus b minus you know something.

$$< \frac{a+b}{1-(a+b)}$$

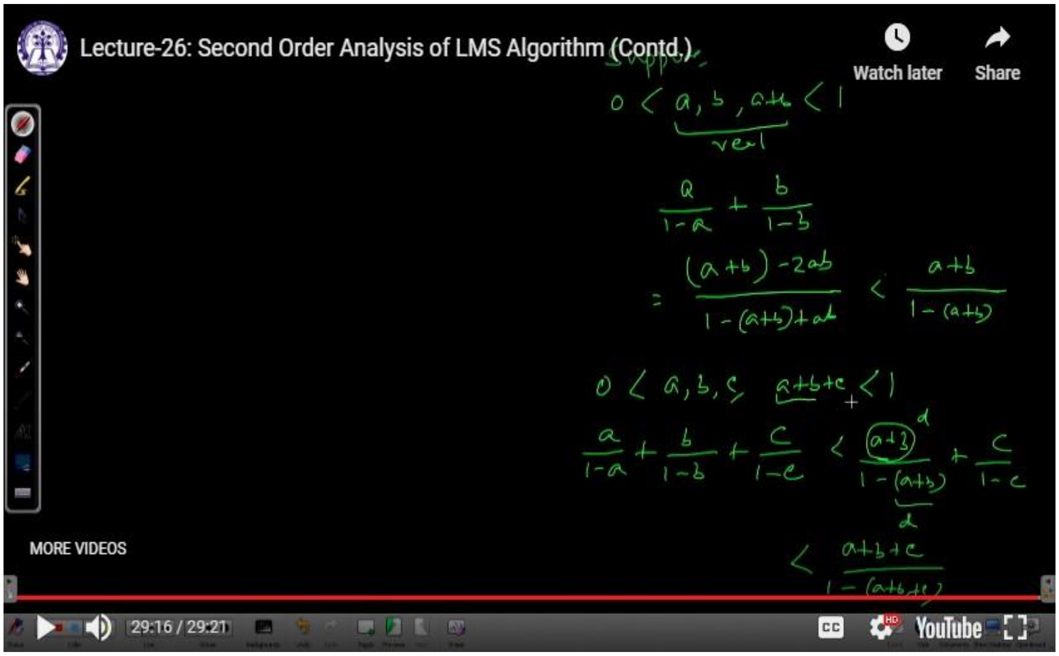
So, that part will be larger than this. So, a by this is a relation a by 1 minus a plus b by 1 minus a plus b in this case will be less than a plus b by 1 minus a plus b. Now suppose I generalize given a, b, c and a plus b plus c they are between this. See they are positive and a plus b plus c is between 1 to 0 obviously a plus b also between 1 to 0 less than 1 greater than 0 because summation if you add 3 and then still it remains between 0 to 1. So, if you add 2 of them it will obviously remain between 0 to 1. So, this is given you can easily extend from above that if I have this alright.

$$0 < a, b, c, a + b + c < 1$$

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} < \frac{a+b}{1-(a+b)} + \frac{c}{1-c}$$

So, first 2 we have already seen a plus b by 1 minus a plus b plus now a plus b if you call it say something say d. So, d also as we have seen d is less than 1 greater than 0 a plus b less than 1 greater than 0 c also and d plus c also this is d this is c this is d see d plus c less than 1 greater than 0 both d and c also less than 1 greater than 0. They apply the same logic here. So, it will be less than d plus c that is a plus b plus c by 1 minus b plus c. So, a plus b which is d plus c and you can go on extending right.

$$< \frac{a+b+c}{1-(a+b+c)}$$



So, this is what I will be using there in the next class this beautiful property. So, till then thank you very much.