Introduction To Adaptive Signal Processing

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Lecture No # 12

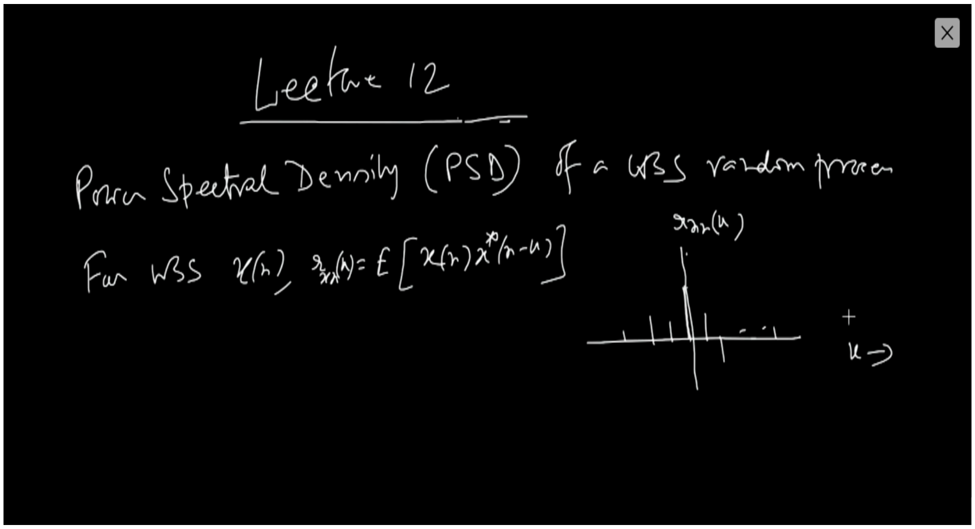
Power Spectral Density (PSD)

We have been considering discrete time random processes. And then wide sense stationary random process their correlation structure. In fact, we have shown in the last class that correlation and covariance matrices. Assume a special structure called Toe-Plitz, where the diagonal entries are same sub diagonal entries are same and like that ok. Those matrices have been studied extensively many algorithms have been devised to invert these matrices or you know solve special equations involved with these matrices like that ok. That we will not cover in this course, but Toe-Plitz matrices had been very popular at you know have been widely studied extensively studied in 80s 70s and also 90s ok.

We will cover something called Power Spectral Density today. This is valid only for a wide sense stationary process. We have seen for WSS xn, rxx what a correlation rxx k is xn x star n minus k right. So, it depends only on the gap between the two indices n and n minus k.

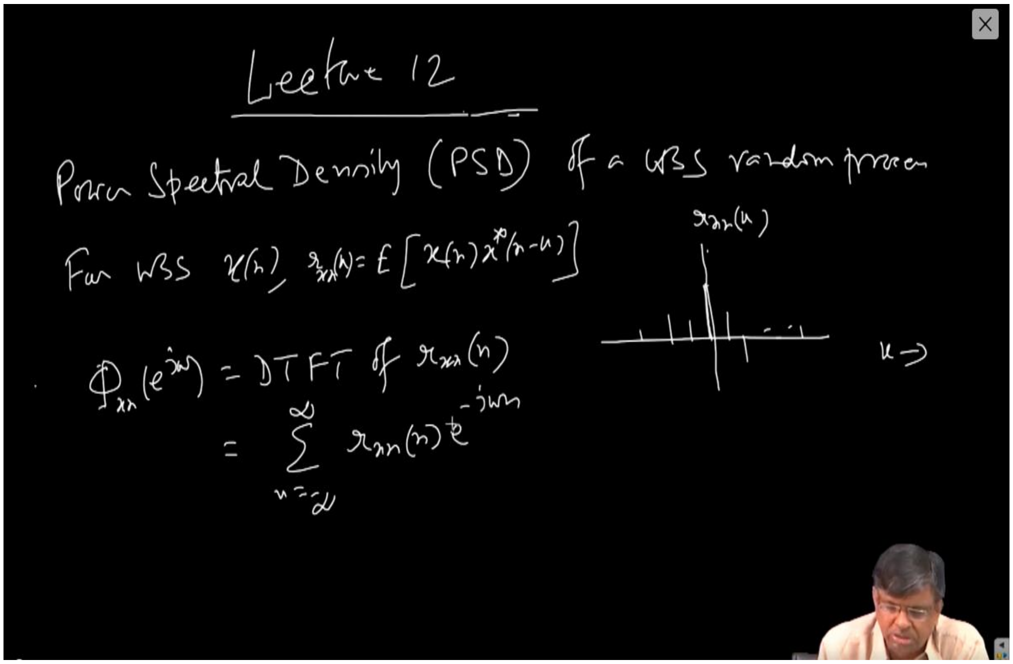
This is how close or how far they are usually. So, that gap is k. So, it is a function of k. But then rxxk then becomes a sequence because k is 0, k can be 1, k can be 2, k can be minus 1, minus 2 like that. So, here rxx0, rxx1, rxx2, rxx minus 1, rxx minus 2 like that.

So, it is a sequence. rxxk can be you know I mean you can plot rxxk. So, this axis is k you can have various values rxx0 dot dot dot like that.



The moment you have a sequence then it comes to my mind well if I take the discrete time Fourier transform DTFT of this what do I get what is the meaning of that. So, let phi xx e to the power j omega be the DTFT you know when you write DTFT discrete time Fourier transform we write in the form of e to the power j omega, where omega is the variable e is a constant, j is a constant, but still you write like this.

Because DTFT is a power series summation where powers of this e to the power j omega comes we will see it. So, DTFT of rxxk or maybe rxxn does not matter which index you put like this is the formula which you are familiar with in DSP rxxn e to the power minus j omega n, n from minus infinity to infinity. So, all powers of e to the power j omega positive and negative powers.



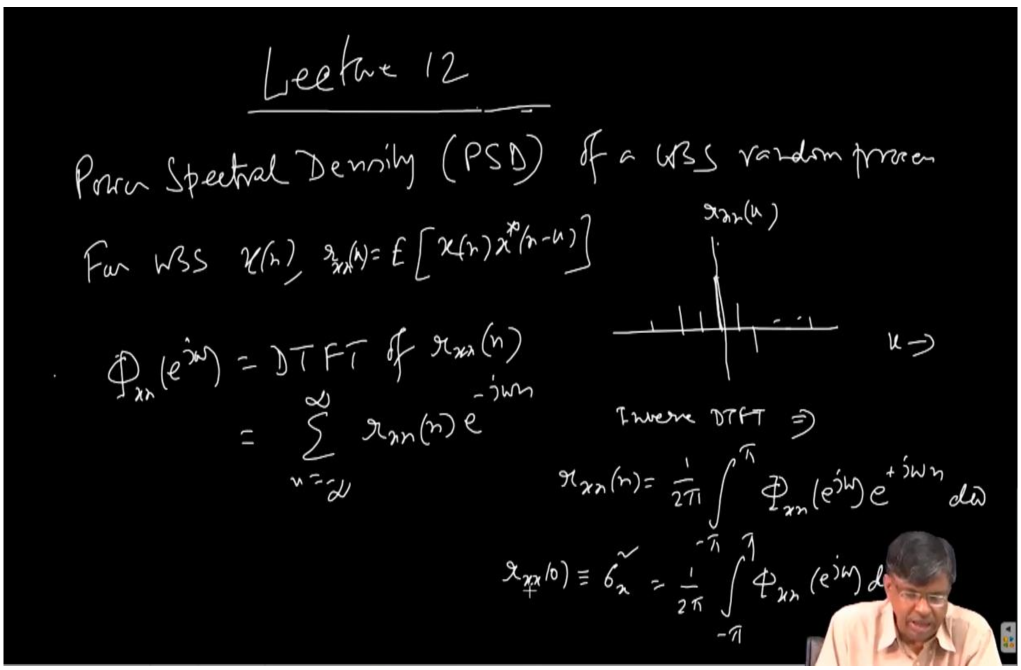
So, it is a power series that is why we write in the form of phi x e to the power j omega. Though, actual variable is omega it is a function of omega because e is a constant, j is a constant all right.

Now, what is the meaning of this, that is what we have to see. what is the meaning of this? If this is the DTFT inverse DTFT will give me this guy rxx at a particular n of your choice, n is fixed here, n is of your choice. For that n you have to carry out this integral, inverse DTFT integral. You have studied in DSP this quantity e to the power plus j omega n. This n is fixed here n comes from outside.

For that n we carry out the integral with respect to omega all right. This is rxxn if this is the DTFT this is inverse DTFT. Then what is rxx0 and rxx0 remember is equivalent to what e of xn x star n k is 0. So, e of mod xn square and if xn is 0 mean it is the variance. Variance is average power ok.

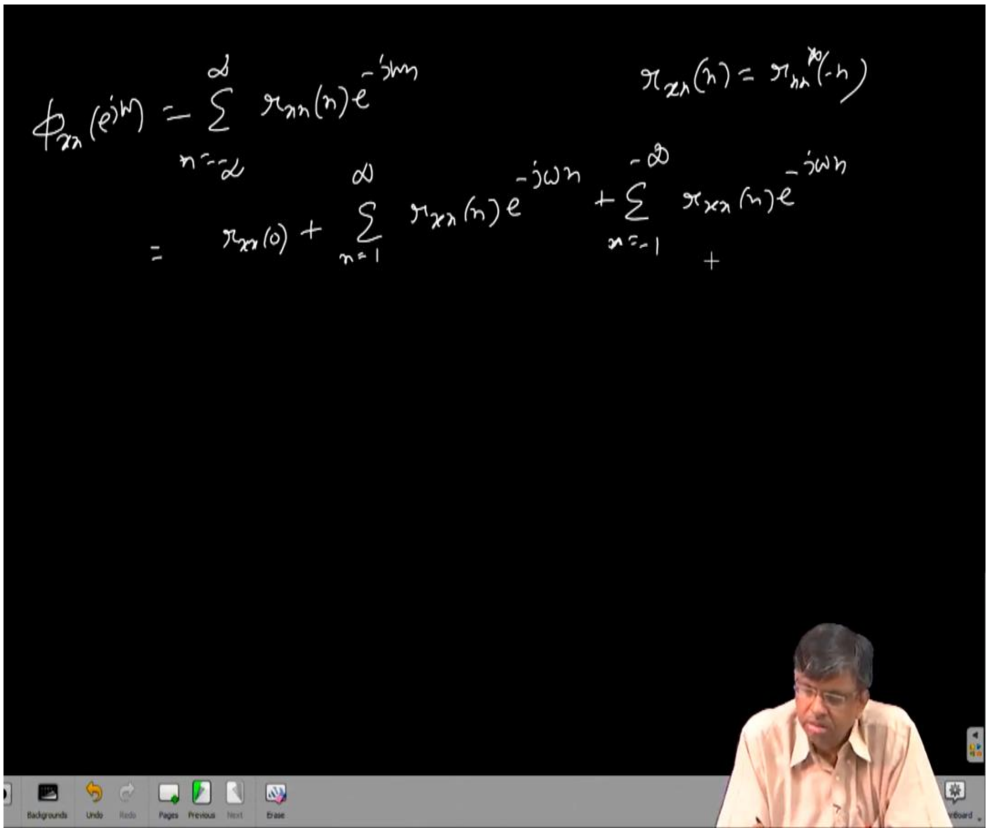
If k is not 0 mean it will be your mod xn square means expected value of the total power at any index n, n does not matter because of yn stationarity. So, sigma xn square when this is 0 mean. So, xn is itself is the increment or decrement and therefore, it is expected value of the power of the increment. So, this variance, but that by this integral will be you put n equal to 0. So, e to the power 0 1 that means, this is such a function which when integrated over the entire range minus pi to pi you remember DTFT is periodic over a range minus pi to pi, whatever you see in minus pi to pi that gets repeated.

Both in magnitude sense and phase sense. So, this function is just the integral. So, total area under this gives you the average power ok. It is like and we will soon see that this is actually non negative that also we will see. This is real non negative ok.



Before I say something about its physical interpretation let me first prove them then I will come to this expression again. Let us see pi xs into the power j omega I am repeating that same statement. Remember the other day I told you about one property Hermitian property of this thing auto correlation function Rxxn is same as Rxx star minus n. That is why the auto correlation matrix is or correlation matrix is conjugate symmetric to uplift and conjugate symmetric ok. So, this summation I write I divide into three parts one is n equal to 0 e to the power 0.

So, Rxx0 remember Rxx0 is variance which is always real and non negative. Variance e to the power mod xn whole square mod xn square makes it non negative real and expected value. So, it is a real stuff. Then one summation I take over the positive range 1 to infinity and another summation I take over the negative range minus infinity to the minus 1 or maybe better if I write this minus 1 to minus infinity. It does not matter whether you start at minus 1 and go up to minus infinity or start at minus infinity and go up to minus 1 you cover the same points.

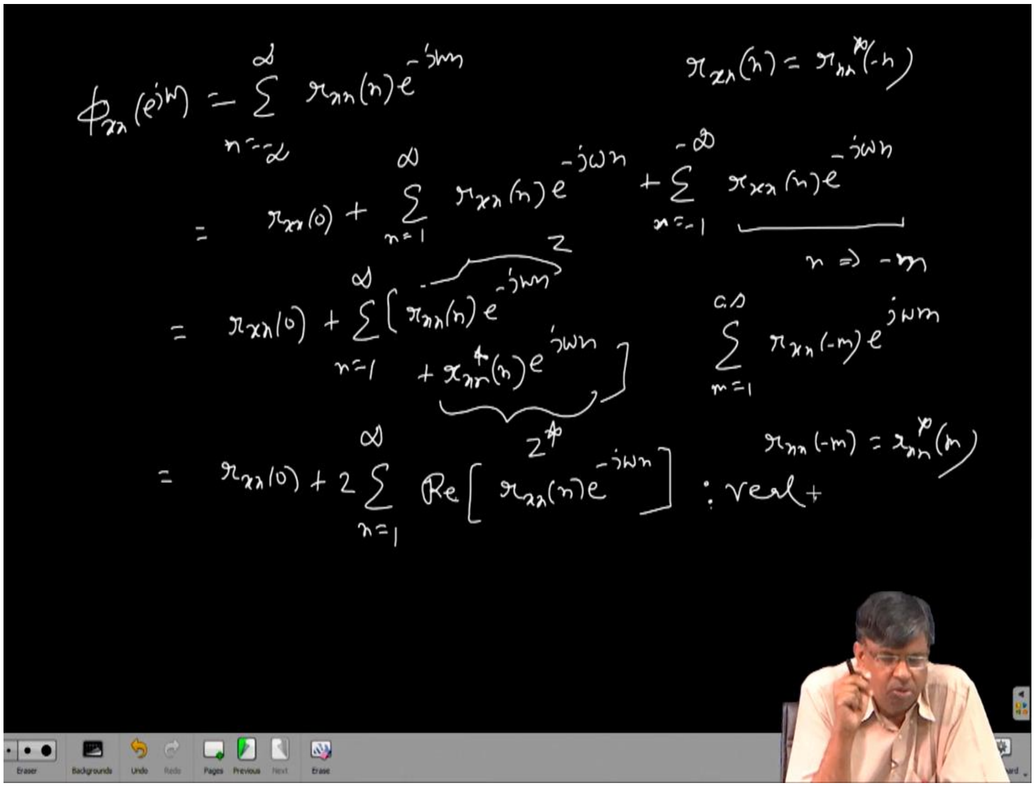


But I prefer these and then same expression Rxxn e to the power minus j omega n. Here n if I replace by minus n then what will happen I have a Rxx minus n e to the power plus j omega n and if it is n equal to my n if I replace minus n then n will be from 1 to infinity. So, minus n equal to minus 1 means n will be 1, minus n equal to minus infinity means n will be infinity or if you still do not understand you can call it minus m say. So, it will be Rxx minus m it will be e to the power plus j omega m and summation when n is minus 1 m is plus 1 and when n is minus infinity m is plus infinity and it does not matter whether you write I mean whether the index is m or l or r or k anything. So, I can as well write n here n.

So, if I do that and remember Rxx minus m is Rxx star minus n is Rxxn. So, if I apply star on both sides star goes from here. So, Rxx minus m will be Rxx star m. So, if I put it here then n equal to 1 to infinity this summation is there Rxxn e to the power minus j omega n and another one same range now 1 to infinity instead of m I again bring n and so, Rxx minus n means Rxx star n e to the power j omega n instead of m again I am writing in terms of n. So, these two things same range of summation if you call this z you can see this is z star Rxxn e to the power minus j omega n if you apply conjugate on this, conjugate on a product of complex numbers is product of conjugated complex numbers.

So, it will be Rxx star n which I have here e to the power plus j omega n which I have here. So, if it is z it is z star and any z plus z star is twice real part of z which is real. So, this will be twice 2 will go outside the summation real part. So, this is your z z plus z star there is twice real part of z ok. So, this is real this is real.

So, this whole thing is real that is what I said 5xx e to the power j omega is a real function ok.



We will also show little later that it is actually non negative. Now coming to the previous page this was Rxx0 Rxx0 was the average power ok. And this real function is such when integrated what the entire range of the DTFT from minus pi to pi because DTFT is periodic from minus pi to pi over a period of 2 pi. So, if I integrate this function this area I mean function over that range then this scale factor it gives me power.

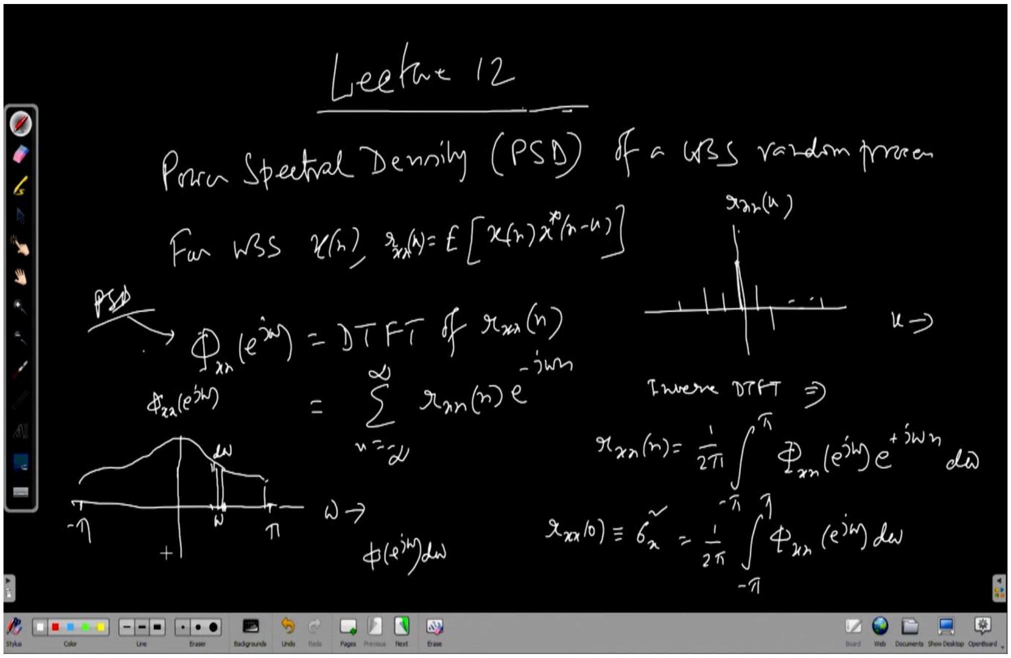
So, if I have something like this then area under this will give me the total power ok. And if I have if I take any point omega and very small width d omega ok. So, if d omega is very small then function does not change appreciably within that. So, height is almost like constant it will be phi e to the power j omega ok. So, area under this will be this into d omega ok.

And total area will be just that is total power will be, I will forget about this 1 by 2 pi it is just a scale factor. It will be just summation of this over all such small small slots, there is a definite integral it will be integral. So, that is why this is called power spectral density this times d omega gives you the distribution of Rxx 0 that lies in the small band that omega, omega 2 omega plus d omega. There power this is after all if it is very small then we can say that the power, the fraction of the power average power lying in the band will be linearly proportional to this width. Because larger the width more will be the power and linearly proportional because you know it is very small.

So, in the while keeping it very small if d omega increases in the band fraction of the power in that band will increase and vice versa. So, it is linearly proportional to d omega. So, proportionally constant is this which is a function of where this slot d omega is located ok, there is a omega. So, this function value at that point ok. So, this is called actually power spectral density because this times d omega gives you, I mean this is easier to understand this way this time d omega gives you the fraction of the power at the small band located omega 2 omega plus d omega assuming the function does not change.

So, it is just the rectangle kind of stuff area under the rectangle is this. That will give you the fraction of the power in that small band of width d w d omega at the point omega ok. So, that is why this is called density this times d omega gives you that power this times d omega gives you the power ok. So, it is density ok. So, this is a power spectral density and when you integrate it and multiply by 1 by 2 pi integrate from minus pi to pi you get the total power I mean average power all right.

So, this is the power spectral density which one this one this is pst. D density because you have to multiply by d omega width then you get the area of this which is the power within this band and then integrate you get the total power.



Now we have shown that this is real we will also show that this is non negative. For that some other result is required which we will do now. Suppose I have one xn which is 0 mean not 0 mean 0 mean may not be necessary.

So, let me not take that make that assumption which is just WSS it is going into LTI linear time invariant system which is characterized by impulse response or unit sample response and output yn you know is convolution between this and this. This convolution you can write in two ways it is same as hn star Axn. So, HR Axn is your choice it is fixed R is a summation index in the summation. So, R is just local variable HR xn minus R, but this is also actually xn we have seen xn hn this is also equivalent to in fact, n minus R if you replace by m R is n minus m. So, this becomes n minus m this becomes m and range from plus infinity to minus infinity which will give you the same result if it is from minus infinity to plus infinity ok.

So, you get the same thing. So, here these things you have studied in DSP that is why I am not speaking much on this because these are not really focus of the lecture x comes first. So, x will have the index with only R like h was first here hn star xn. So, h got a R this got this guy x got n minus R. So, here x got R h got gets n minus R alright, but between the two it is this form which is always very useful not this ok. If this is the form which is very useful I mean originally convolution comes this way then you show we show by taking n minus R to be m or R is n minus m HR xn minus hm xn minus m and m from plus infinity to minus infinity which is same as m minus infinity to plus infinity because the range is same and then it becomes this summation m you can replace by R you get this summation.

So, both are same, but this is in practice more useful it I mean if you choose this form. It is easier to solve problems anyway my point is input is random. Every sample is a random variable in this convolution you are summing you are multiplying every sample by an h value and adding them. So, this is this entire thing which is yn that is also random, because h0 xn, h1 xn minus 1, h2 xn minus 2, h minus 1, xn plus 1 dot dot dot. So, we are linearly combining that is multiplying every random sample of x by some constant and adding.

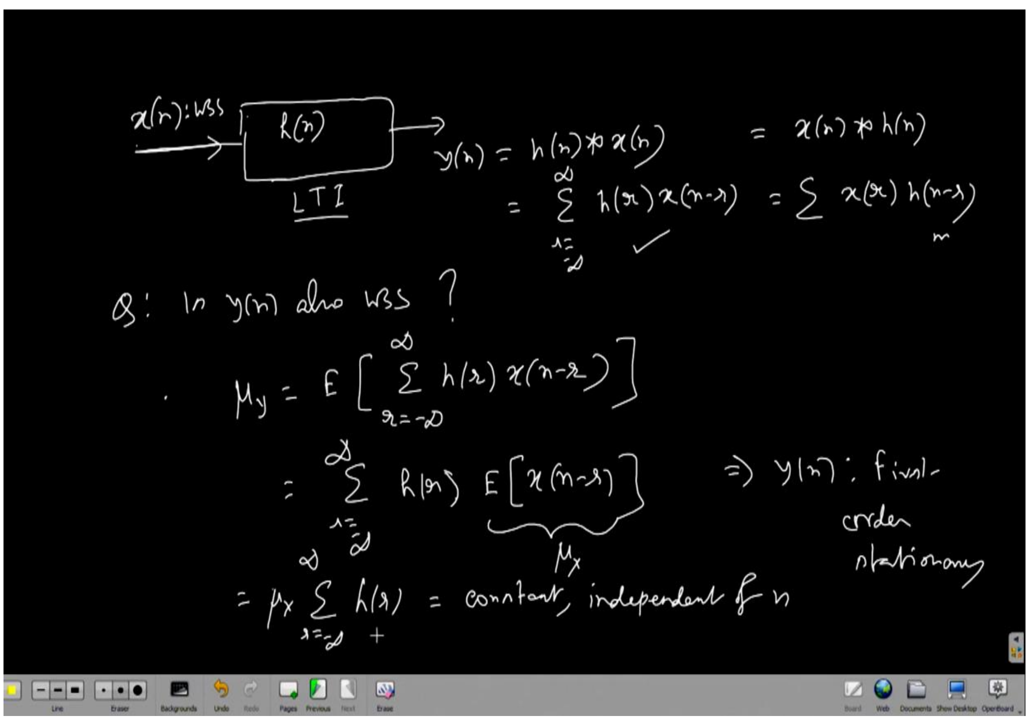
So, if samples of x are random the summation which is yn that is also random. If that be then the question is if input is Wss is yn also Wss question. We will see answer is yes first Wss means mean mu y It will be expected value of this summation is a summation like h0 xn, h1 xn minus 1, h2 xn minus dot dot dot. E over all of them we have seen E is linear that is E of the summation will be same as you can push E on each random variable.

So, it will be E on each sorry. E will work there will be on this this is not random. So, this is deterministic constant that times a random variable E on that and E on a summation means you have to apply E on every component in the summation and what this component I mean this is a way just scalar non random. So, E will not come over E will come on the random part and we know this is Wss. Its mean is mu this is a expected value of xn minus r, but xn is a Wss process. So, its mean is mu independent of what is this index it does not depend on the index.

So, E of this will be mu say mu x. So, it will not be a function of a nor r because it is Wss wide sense stationary. So, mean does not depend on the index its same everywhere and then the mu x you can take out as common. Since Hr H is a given sequence unit sample response or impulse response sequence that we are adding there is nothing random here and it does not depend on the index ok n. It does not depend on the index n.

does not depend on the index n. So, you get a constant here and this is a constant. So, constant independent of n means Yn is first order stationary. That is its mean mean E of Yn does not depend on n where is n because n has gone from here. mu x is a constant the mean for all the random variable same, and this is just a summation of the impulse response, if impulse response is known summation is can be computed once for all this is a scalar number this into mean. So, it does not depend on the index n and that is why it is stationary in the first order that is in the sense of mean.

That is here also for Yn that is mean will not depend on what index you are choosing n, it will be same everywhere and that is mu y which is given by this constant.



And then we come to the second order there is a correlation of Yn or Yy k. There is at a gap of k. So, Yn Y star n minus k alright and now Yn go to the previous page. We chose this HR Xn minus R summation over R HR Xn minus R this form.

H gets does the index at X gets n minus that index this H convolved with X that kind of form I have put n here. So, this is by Yn at Y star n minus k another summation H will come first and X will go second, no need to choose the same index R, but this is independent of this. So, choose L HL X n minus R minus L and then star. So, n minus R n minus k, this is n minus k.

So, n minus k and Yn. So, that is a n here n minus k. So, this is this is n minus k this you have to work out the star on a summation, summation of the complex numbers as I have been telling again and again conjugation of a summation of complex number means inside the summation every such complex number has to be conjugated and but this is a product of two numbers. So, conjugate of a product we know like Z1 Z2 star is Z1 star Z2 star. So, essentially it will be H star L X star n minus k minus L alright. Then this is the random part this is the random part ok.

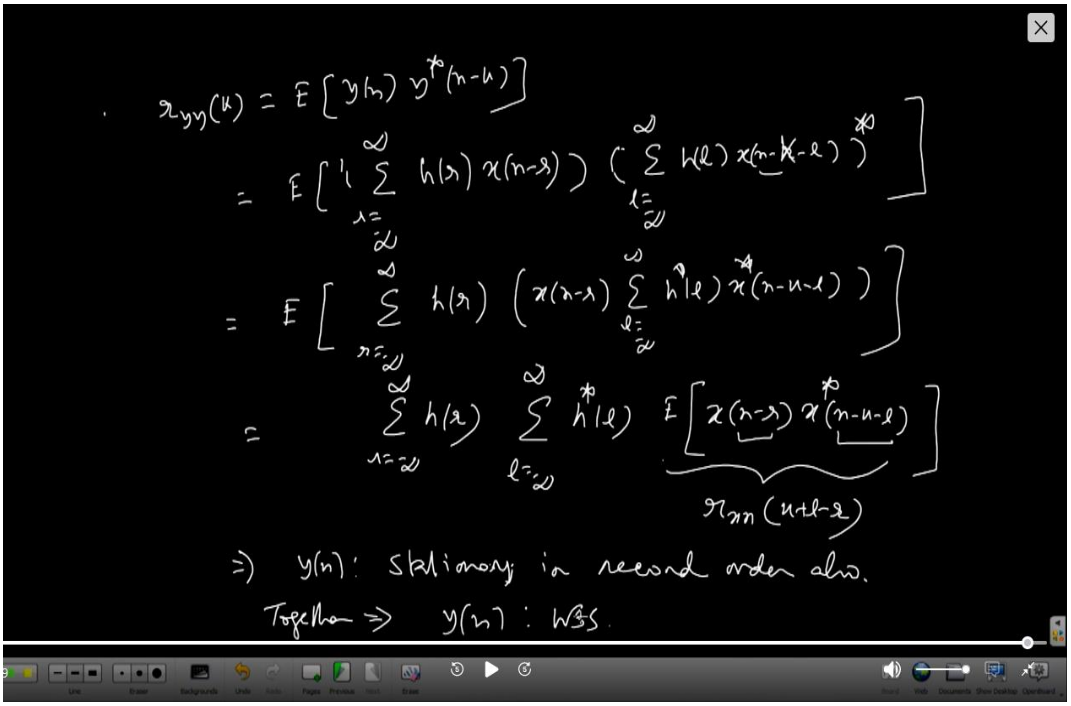
Suppose this is a this is Y star n minus k as we know. Now, this I can push inside the summation this is over R this is Y star n minus k, it does not depend on R I can push inside the summation multiply with this instead of that means, instead of carrying out this first and then multiplying with by this I can push inside this summation for every such product I further multiply by this and then add. I will get the same thing. But that means, this into this Y star n minus k that has n and X n minus R there are n. I club them together that product then multiplied by HR then this. If I do that it is this part Y star n minus k and Y star n minus k I write like this and again here instead of carrying out the summation first and then multiplying by this I can push this inside.

So, every such product I further multiply by X n minus R and then add and then this and this I club together under one bracket. So, this will be E of with the summation HL and X n minus R I have pushed in n minus R and X n minus k minus L star. So, there will be star on this also because star was over the product. So, Z 1 Z 2 Z 1 star Z 2 star instead of putting the star here let me ok. Now, what does it mean every time you take an R put an HR then take all values of L for each L carry out the product and then you have to apply and then do again another R and for that R again within bracket carry out the sum.

So, you got a huge sum and expected value of each means expected value you can push So, you got a huge sum and expected value of each means expected value you can push inside the sum and only these are random. These are not random. So, E will finally, come here ok. This is a huge sum of product of two random variables multiplied by some scalars and summed and summed. So, E outside so, you will be working on and every such term only the product of the random part ok.

These are not random. So, E comes directly on this part and you see this is nothing, but correlation what a correlation or correlation this is one index of X this is another index. So, gap between the two indices is this index minus this index n minus R and n minus k minus L. So, if you subtract this from this will be k plus L minus R. So, this is nothing, but auto correlation RX X k plus L minus R all right. Now you see this from this entire sum n has gone there is no n k was the gap and indeed it is a function of k L you sum over L.

So, L goes after the summation R also another summation index you sum over R. So, R also goes ok because you are putting all values of R and adding you are putting all values of L and adding. So, in the end in L and R will go it will be only a function of k. So, it is also I mean this correlation also depends only on the gap between the two. Therefore, this is a function of k means Y n stationary in second order also, second order means correlation.



So, together first order and second order stationary implies Y n weight strain stationary. So, I stop here now and from here I just answer this question in the next lecture that ok. Y n also Wss, what will be power spectral density of Y n, in terms of the power spectral density of the input and also the impulse response that is what I will derive in the next class. Thank you very much.