Advanced Microwave Guided – Structures and Analysis
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Lecture 2
Scattering Matrix Concepts

Unitary property of the scattering matrix

In this section we are going to investigate the unitary property of the scattering matrix, for that we make use of the power conservation theory. Now, the power conservation theorem states that,

$$\sum_{n=1}^{N} |V_n^-|^2 = \sum_{n=1}^{N} |V_n^+|^2$$

which physically means that the sum of the incident power which is appearing on the right hand side is equal to the total reflected power which is appearing on the left hand side.

The outgoing wave at the n^{th} port is related to all the incident waves at other ports through the relationship

$$V_n^- = \sum_{i=1}^{N} S_{ni} V_i^+$$

which we can obtain from the scattering matrix definition, from the definition of the scattering metrics parameters. So, therefore, the power conservation can be expressed as,

$$\sum_{n=1}^{N} \left| \sum_{i=1}^{N} S_{ni} V_i^+ \right|^2 = \sum_{n=1}^{N} |V_n^+|^2 \quad \text{— } 3$$

Since we have no restriction on the choice of V_n^+, we will choose all V_n^+ to be 0 except at the i^{th} port.

So, then choose, all $V_n^+ = 0$ except V_i^+

From (3) we obtain

$$\sum_{n=1}^{N} |S_{ni} V_i^+|^2 = |V_i^+|^2 \quad — (4)$$

Canceling $|V_i^+|^2$ on both sides, we obtain

$$\sum_{n=1}^{N} |S_{ni}|^2 = 1$$

And that can be also written as,

$$\sum_{n=1}^{N} |S_{ni}|^2 = 1 = \sum_{n=1}^{N} S_{ni} S_{ni}^* \quad — (5)$$

where S_{ni}^* is the complex conjugate of S_{ni}. This equation (5) holds good for any *i*. So, what does equation (5) state? It states that the product of the scattering matrix parameters of one particular column times the complex conjugate of the same column, if I add up the entire column like that, the column multiplied by the complex conjugate of the element of that particular column and add that up the sum is going to be equal to *one*.

In order to go to the next phase of the relationship particularly to find out what happens if we do not choose one particular column if we choose two different columns, what is going to happen?

We go to the next proof and for that we will choose

$V_n^+ = 0$ for all *n* except *n* = *s* and *r*.

If we apply the same relationship (3), we can obtain.

$$\sum_{n=1}^{N} |S_{ns} V_s^+ + S_{nr} V_r^+|^2 = |V_s^+|^2 + |V_r^+|^2$$

$$\sum_{n=1}^{N} (S_{ns} V_s^+ + S_{nr} V_r^+)(S_{ns} V_s^+ + S_{nr} V_r^+)^* = |V_s^+|^2 + |V_r^+|^2$$

$$\sum_{n=1}^{N} |S_{ns} V_s^+|^2 + \sum_{n=1}^{N} |S_{nr} V_r^+|^2 + \sum_{n=1}^{N} S_{ns} V_s^+ S_{nr}^* (V_r^+)^* + \sum_{n=1}^{N} S_{nr} V_r^+ S_{ns}^* (V_s^+)^* = |V_s^+|^2 + |V_r^+|^2 \quad - (6)$$

Now, we know from equation (4) that

$$\sum_{n=1}^{N} |S_{ns} V_s^+|^2 = |V_s^+|^2$$

$$\sum_{n=1}^{N} |S_{nr} V_r^+|^2 = |V_r^+|^2$$

Now (6) can be written as

$$\sum_{n=1}^{N} S_{ns} V_s^+ S_{nr}^* (V_r^+)^* + \sum_{n=1}^{N} S_{nr} V_r^+ S_{ns}^* (V_s^+)^* = 0$$

$$\text{or} \quad \sum_{n=1}^{N} \left[S_{ns} S_{nr}^* V_s^+ (V_r^+)^* + S_{ns}^* S_{nr} V_r^+ (V_s^+)^* \right] = 0 \quad - (7)$$

So, now these two quantities V_s^+ V_r^+ can be chosen according to my free will because they are general. This equation will hold good regardless of the values of V_s^+ and V_r^+.

So, let us choose $V_s^+ = V_r^+$. So, when we do that equation number (7) can be rephrased as

$$|V_s^+|^2 \sum_{n=1}^{N} (S_{ns} S_{nr}^* + S_{ns}^* S_{nr}) = 0 \quad - (8)$$

If we make the choice $V_s^+ = j\, V_r^+$, then from equation (7) we get

$$j|V_r^+|^2 \sum_{n=1}^{N} \left(S_{ns} S_{nr}^* - S_{ns}^* S_{nr}\right) = 0 \quad — (9)$$

So, because neither of these two quantities V_s^+ or V_r^+ are 0. The only way both these equations (8) and (9) can be satisfied is given by the following condition and that is

$$\sum_{n=1}^{N} S_{ns} S_{nr}^* = 0 \quad (s \neq r) \quad — (10)$$

Because V_s^+ and V_r^+ none of them are 0. The only way equation (8) and (9) can both be satisfied is by setting

$$\sum_{n=1}^{N} S_{ns} S_{ns}^* = 0 \qquad (s \neq r)$$

So this constitutes the second condition as to what would happen if we multiply different columns of the scattering matrix.

So, it says that the product of a particular column and the complex conjugate of any other different column if we add those products together that will be 0. So, this is the second important property of the scattering matrix. A matrix which satisfies these conditions is called a unitary matrix.

If we write the power conservation theorem in the matrix form,

$$\begin{aligned} [V^-]_t [V^-]^* &= [V^+]_t [V^+]^* \\ &= ([S][V^+])_t ([S][V^+])^* \\ &= [V^+]_t [S]_t [S]^* [V^+]^* \end{aligned}$$

$$[V^+]_t \left([u] - [S]_t [S]^*\right) [V^+]^* = 0$$

Now, because the V^+ matrix is not 0, the only way this can be satisfied is $([u] - [S]_t[S]^*)$ becomes 0 or

$$[S]_t [S]^* = [U] \quad \text{---} \quad (11a)$$

$$\text{or} \quad [S]^* = [S]_t^{-1} \quad \text{---} \quad (11b)$$

So, basically, we can obtain the same unitary characteristics from these expressions. (11b) is the definition of a unitary matrix and the conditions (5) and (10) we discussed before, can be obtained by performing the matrix multiplications in equation (11a).