Advanced Microwave Guided – Structures and Analysis
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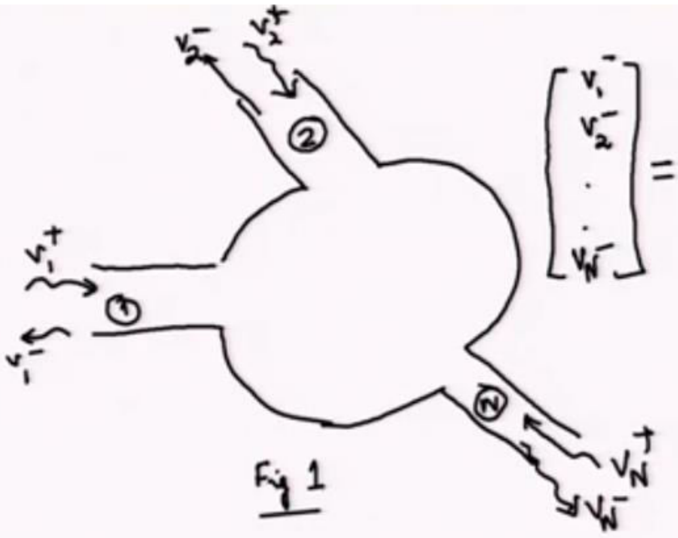
Lecture 1
Scattering Matrix Concepts

Scattering Matrix Concepts

We all know at microwave frequencies the concept of voltage and current is no longer good. We have to instead use the concept of voltage and current waves. So, we talk of reflection, we talk of transmission of waves and in order to characterize this reflection or transmission of waves, we formulate this mathematical language of scattering matrix.

So, this concept of scattering matrix enables us unlike at low frequencies to characterize the phenomenon of reflection and transmission of waves in a microwave network. So, let us look at a typical an N port microwave network.



$$\begin{bmatrix} V_1^- \\ V_2^- \\ \cdot \\ \cdot \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ - & - & \cdot & -- & \cdot \\ - & & & - & \cdot \\ S_{N1} & S_{N2} & S_{N3} & & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \cdot \\ \cdot \\ V_N^+ \end{bmatrix}$$

Fig 1

So, here V_1^+, V_2^+, V_N^+ are respectively the incident voltage waves on port 1, on port 2 and port N respectively. So, their intermediate ports, port three, port four whatever maybe. And similarly, the reflected voltage waves at port one is V_1^-, the reflected voltage wave at port 2 is V_2^- and the reflected voltage wave at port N is V_N^-. So, in order to characterize this environment, we use the scattering matrix and that relates the incident and the reflected voltage waves in the input network.

So, the left hand vector constitutes the reflected voltage waves at port 1 through port N and then we have the scattering matrix. So, the left hand side denotes the reflected voltage wave that is equal to the scattering matrix times the incident voltage waves. So, this is the mathematical description of this circuit.

Now, we attempt to find out the properties of such a scattering matrix, what are the relationship between the elements of the scattering matrix? Towards this end, we first developed the formulation towards that.

So, for that we first define under a normalized environment the voltage and current waves which is the total voltage wave is given by the incident voltage plus the reflective voltage.

$$V = V^{+} + V^{-}$$

The total current wave is given by the incident current minus the reflected current

$$I = I^{+} - I^{-}$$

and because of the normalization, the normalized voltages and currents are related by

$$\frac{V^{+}}{I^{+}} = \frac{V^{-}}{I^{-}} = Z_0$$

This is the characteristic impedance, it must however we remembered that

$$\frac{V}{I} \neq Z_0$$

So, it is only the forward voltage by the forward current and the backward voltage and the backward current they are related by this expression, but not the total voltage and total current waves and because of the normalization.

So, this Z_0 becomes 1; so, that we can write it as

$$I = I^{+} - I^{-} = V^{+} - V^{-}$$

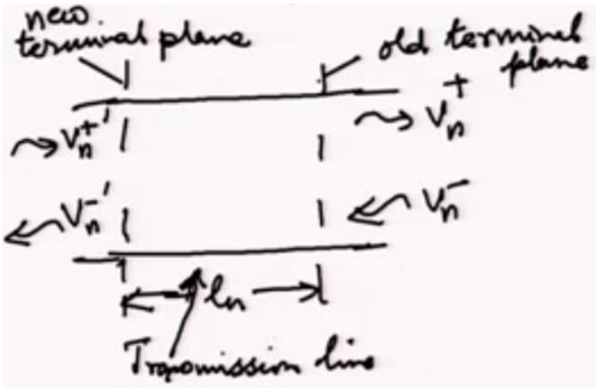
and from these two equations we get,

$$V^{+} = \frac{1}{2}(V + I)$$

$$V^{-} = \frac{1}{2}(V - I)$$

So, therefore, there is no necessity of including the current wave inside the formulation because the current is given in terms of the voltage waves V^+ and V^- and therefore, the total circuit of the total network characteristics of the total network can be represented in terms of V^+ and V^-.

Now, consider a situation where there is a transmission line and there is an old terminal plane in which I have to measured my scattering matrix and I want to translate this to this new terminal plane.



So, in this scenario we can relate the voltage waves at the old terminal plane with that in the new terminal plane and this frequently arises because S_{11} or any kind of scattering matrix parameters can be measured at a particular terminal plane and I may want to translate it, I want to derive the equivalent scattering parameters at another transmission at another terminal plane.

So, this is very simple to understand. So, θ_n is the electrical length of the line and l_n is the physical length. So, θ_n is the electrical length of the line corresponding to the physical length of l_n.

$$\theta_n = \beta_n l_n$$

So, these two relationships

$$V_n^+ = V_n^{+'} e^{-j\theta_n}$$

$$V_n^{-'} = V_n^- e^{-j\theta_n}$$

can be easily written because V_n and V_n^+ they differ by a phase of $e^{-j\theta_n}$.

So, S'_{nn} is the scattering matrix at the new terminal plane. So, it is related to $V_n^{-\prime}$ / $V_n^{+\prime}$ as

$$S'_{nn} = \frac{V_n^{-\prime}}{V_n^{+\prime}}$$

So, that can be related to the V_n^- and V_n^+ at the old terminal plane.

$$S'_{nn} = \frac{V_n^{-\prime}}{V_n^{+\prime}} = \frac{V_n^- e^{-j\theta_n}}{V_n^+ e^{j\theta_n}} = S_{nn} e^{-j2\theta_n}$$

So, this gives the relationship between the S matrix, the S'_{nn} at the new terminal plane and the S matrix or the S_{nn} at the old terminal plane. So, this is a frequently used relationship everywhere.

Symmetry of the Scattering matrix

Then we come to the symmetry of the scattering matrix. We prove that the scattering matrix is symmetrical. The total voltage wave is

$$V_n = V_n^+ + V_n^-$$

And the total current wave is

$$I_n = I_n^+ - I_n^- = V_n^+ - V_n^-$$

$I_n^+ - I_n^-$ that is equal to V_n^+ - V_n^- because we are dealing with a normalized system as previously.

So, therefore, I can write the voltage matrix as

$$[V] = [V^+] + [V^-] = [z][I] = [z][V^+] - [z][V^-]$$

So, from which the relationship between V^- and V^+ can be written as

$$[V^-] = [S]\ [V^+]$$

where S is given by

$$[S] = ([Z]+[U])^{-1}([Z]-[U]) \quad \text{— (1)}$$

where [U] is the unit matrix.

We also have,

$$[V^+] = \frac{1}{2}([V]+[I]) = \frac{1}{2}([Z]+[U])[I]$$

$$[V^-] = \frac{1}{2}([V]-[I]) = \frac{1}{2}([Z]-[U])[I]$$

$$[V^-] = ([Z]-[U])([Z]+[U])^{-1}[V^+].$$

$$[S] = ([Z]-[U])([Z]+[U])^{-1} \quad \text{— (2)}$$

So then from (1) we can get the transpose of S,

$$[S]_t = ([Z]-[U])_t([Z]+[U])_t^{-1}$$

So, the Z matrix and the U matrix are symmetrical. So, therefore, the transpose of this matrix are equal to the original matrix. So, it can be written as

$$[S]_t = ([Z]-[U])([Z]+[U])^{-1}$$

Comparing this with (2), we get

$$[S]_t = [S]$$

which means that the S matrix is a symmetrical matrix. This is the first characteristics of the S matrix.