VLSI Signal Processing

Professor Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

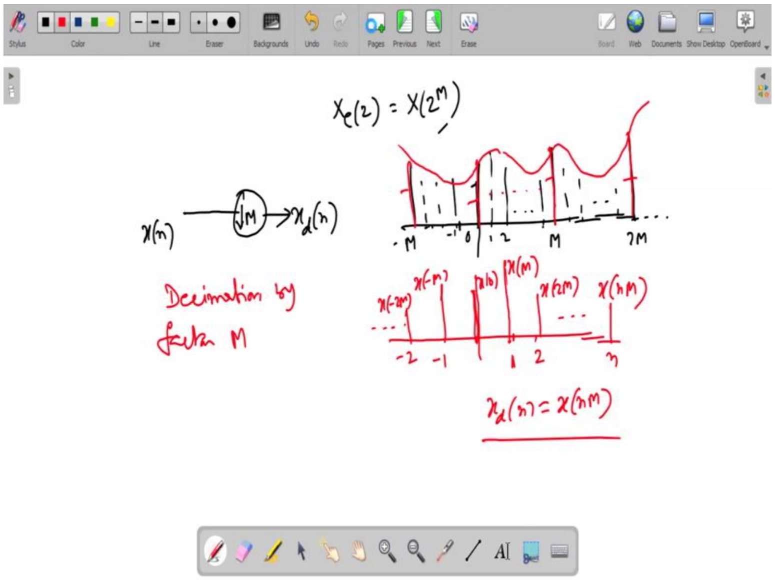
Lecture – 29

Polyphase Decomposition of Sequences

Okay, in the last class I started with a little bit of a multi-rate signal processing stuff because I said that when as such you do unfolding, then if the original DMD had some complexity, hardware complexity. Then if you unfold by j, integer factor j; or you gain, you are gaining speed by a factor j but the hardware cost goes up. Your hardware gets repeated, replicated j times; so hardware cost goes up, power consumption goes up, in general j times. By j times. But, then I said also that in certain cases, like an FIR filter, if you parallelize it, and so unfold it by a factor j, we can employ some tricks by which hardware cost will not go up by the same factor j. It will go up of course by a factor much less than j by some DSP tricks.

That is what, this is very interesting thing and that is what brought into this topic. And that uses concepts from very basic things about multi-rate signal processing. Last class I considered that signal expansion, there is, between every pair of samples you insert new sampling points and put zeros there. May be capital M minus 1 number of zeros between every pair of samples; that is called expanded sequence. Expanded sequence has a Z-transform equal to original Z-transform, with z replaced by z to the power M, capital M, if you are plugging in M minus 1 zeros between every pair of samples. That is, if M is the factor of expansion, then capital X z to the power M is Xe z.

(Refer Slide Time: 02:00)

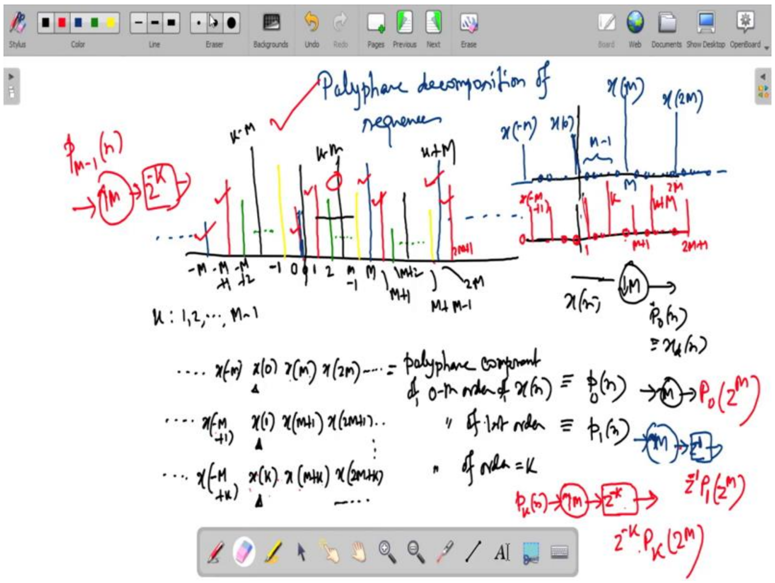


There is expanded sequence Z-transform is, this is, Xe z is the expanded sequence, this was done. There is one more issue called decimation, down sampling by an integer factor M, d for decimation. It means that, suppose you had sequence of samples; this is an Mth sample, 0th, 1, 2 dot, dot, dot.

Then again many samples, this is 2Mth, 0, minus 1, all that. Now, if I select this sample, this, this, this, like that; there is 0th, Mth, 2Mth, 3Mth, minus Mth, and bring them here. Then I have got x0. Then Mth sample occurring now at position number 1, first sample which x M. Then at position number 2, x 2M, dot, dot, dot, dot. On this side, at position number minus 1, x minus M; at position number minus 2, x minus 2M, dot, dot, dot, dot. This is called decimated sequence, decimation by factor M. M is an integer either 2 or 3 or 4. M1 has no meaning, you will get by the same thing.

So, actually what it means, if it were actually obtained by sampling an analogue signal like this. I am throwing away these samples, so I am basically sampling here, then sampling here, sampling here, sampling here. So, I am bringing down the sampling rate, increasing the sampling period by a factor capital M or bringing down the sampling rate by a factor capital M. It is called decimation, decimation by a factor M. The sequence that is generated xd n is related to x n by this; that is xd n, its any nth sample is nothing but original sequence at n into M. Like x is 0, x 0 into M, that is 0; xd 1, that is x 1 into M, x M, this is the xd 1; xd 2 here, 2 into M; x 2M so and so forth. Alright, these two basic operations.

(Refer Slide Time: 05:13)



Then I go for, let me erase this. So, I have a sequence, the sampling points are this is 0, sorry. This is 0, 1, 2, this is M minus 1, this is M, this is M plus 1, M plus 2, this is M plus M minus 1, this is 2M, this is minus 1, this is minus M, this is minus M plus 1, this is minus M plus 2, dot, dot, dot, dot. So, I just colored the samples. Suppose from this I take the 0th, Mth, 2Mth, minus Mth and like that.

That is, I just decimate it, down sample it by a factor capital M as I just told you a little while ago. Decimation, if I just decimate it by a factor capital M, then what I get is this. I get x0, instead of drawing samples, I am writing the sample values like x0 and I put a notch below this showing that this origin. Then I take the next sample, this I decimate means, I throw away the intermediate samples. My sample number 1 will be original x M. So it will be x M. Then will be, again I throw away this M minus 1 samples; next I take x 2M that will be at sample number two, so x 2M, dot, dot, dot, dot.

On this side I throw away the intermediate samples. I take Mth, so minus 1th sample will be x minus M, dot, dot, dot, dot. This is called the polyphase component of 0th order of x n it is denoted by p for polyphase 0 n. And p0 n obviously is nothing very simple, if you take x n, decimate by M, what do you get? This p0 n, which is nothing but decimated version of x n, very simple. Then what I do? You look at this red colored samples, it is at 1, say form another sequence by making the red color 1 x1, but I take origin below this. I am constructing this subsequence, so x of 1, red color 1; I take and let the origin be here. Then next red colored 1 is, at if it is 1, it is 1 plus M; M plus 1 is same as 1 plus M, so M plus 1. Next will be x 2M plus 1 here, on the left hand side this thing, sorry. This is polyphase component of x n of first order. It is indicated by p1 n.

In general, dot, dot, dot, if I take a general term here. Suppose this is kth, this is k plus Mth, this is k minus Mth. k, k plus, like 1, 1 plus M. Similarly k, k plus M, k plus 2M, and all that; k, k minus M. So, I take x of k and put the origin here like I did. So next will be x M plus k, x 2M plus k, dot, dot, dot, dot, dot; x minus M plus k and dot, dot, dot, dot, dot. This is general thing, this is polyphase component of x n of order equal to k. Obviously k has to be either I mean 1, 2 up to M minus 1; this is the range, this is a range 1 to M minus 1, so general thing.

Now, you see look at this p1 n; in p1 n I shift it to the right by 1, so then x1 will come to this point. Let me draw it here, first I plot this p0 n and I expand, suppose I expand p0 n by M. That means after x0, there are M minus 1 zeros; then after x M there are M minus 1 zeros, so on and so forth. So, what I get is n zeros, in that Mth place x M comes is x M, between them M minus 1 zeros. So x0 here, M minus 1 zeros, so x M comes to its original place Mth. Then again zeros, M minus 1 zeros, x 2M, so on and so forth.

Between them M minus 1 zeros, so x minus M. For p1 n what I do, I send, pass it through a delay first, then expand. Expand. Then hold up and if I had pass it through a delay first that means, origin will move here, origin will move here, or no, let me correct. There is some mistake I am making. First I expand and then delay by 1. Then roll up in, if I expand this fast between x1 and x M plus 1, M minus 1 zeros will come.

Between x M plus 1 to x 2M plus 1, M minus 1 zeros will come. Here also, if you push it to a right by 1; x1 will come here, x1 will come here, x1. Then there will be zeros because already expanded. At the Mth position also 0. Then will come the next point, M plus 1. Then again there will be zeros, at 2M also it will be 0. Then will come similarly here, here this 0, 0, dot, dot, dot, 0 below this 0. But here there is a point, this will shift here because originally I had zeros between them and I am shifting it to right. So, this will be come here. If I add the 2, if I add the 2 you will see this x0 will come in place, that time it will get added with 0 so no interference.

Then x1 added with 0 so this will come up, this will come up, then zeros and zeros they will add so I will not get anything. Then x M plus 0, so x of M this will come up; because x of M plus 0 here. And here x of M plus 1 and 0 from here, so x of M plus 1 this will come, so these two will come. Likewise, there is again zeros, then x 2M sample has come here with a 0 here, so x 2M will come up. Okay, x 2M here, 2M his will come up and again 2M plus 1 this will come up. Similarly, here also this will come up minus M and this will come up; otherwise it between zeros.

So, if there now you take p2 n, again expand by M and then pass it through two delay, p2 n. What is p2 n or may be say pk n? Take a general case. What will you get? pk n; kth sample x k at x M plus k; I had M minus 1 zeros. Again between x M plus k and 2M plus k, I had M minus 1 zero, same here. Now, after inserting the zeros there in expansion I am delaying it by k. So, x k from origin this sample will you move to kth location. So, x k this fellow will come back here at some kth position. Because some shifting I have inserted zeros and zeros and zeros, M minus 1 zeros, M minus 1 zeros, M minus 1 zeros and then it is shifting to the right.

So, x k shifting to the right by k; so x k from 0 position will go to kth position. So, x k come backs to its original position. M plus k goes to what? I am shifting to the right by k; so M plus k earlier was at 1, so it will be going to at what? M plus k. This is, this was 0, this was 1, M plus k. Now, all the zeros will come up. So M plus k will be at Mth position. Then I shift it to the right by k further.

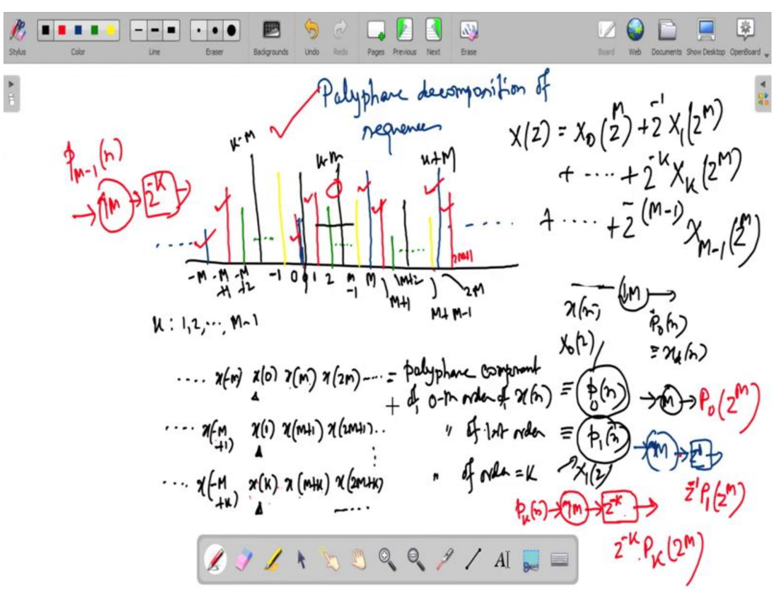
When I expand I have M minus 1 zeros, so x k at 0th; x M plus k at Mth; x 2M plus k at 2Mth; so on and so forth. Then this shift by k so x k goes to kth; x M plus k from Mth goes to M plus kth; x 2M plus k from 2Mth goes to 2M plus kth, likewise. So, that is why these fellows come back, here also.

When you add them they find zeros from top and nobody else. So when you add, you know, you get there is no interference, x0 remains x0, x1 remains x1, xk remains x1 like that. So you can understand that if I take p0 n just expand, p1 n expand then delay once, p2 n expand then delay twice, pk n expand then delay k times. And lastly expand and then add them you will get back this. There is very easily just you take various such figures p0, p1, expand and then insert 0. Expand, that is insert zeros, M minus 1 number of zeros between every pair of samples, shift it to the right by appropriate number of indices like if it is k, pk n, like this sequence.

Expand, that is between insert new zeros, M minus 1 zeros between every pair of samples here. Then shift it to the right by k, same you do here, same you do her,e and then add, you will get back easily this one. That is, every sample will come back to its original place. One will not you know, overlap or intersect or interfere into the other, whole sequence will be constructed. Now, if that be, so this is called as 0th order polyphase factor, first order polyphase factor, kth order polyphase factor, and M minus 1th order.

Now, Z-transform here will be what. If you expand, that is insert zeros, M minus 1 zeros, here it will be original P0 z to the power M, this we have done earlier. If there is a sequence and that is expanded, that is between every pair of samples M minus 1 zeros are brought in. Then output Z-transform will be input Z-transform but z replaced by z to the power M; we did it last time. Here what it will be? Z inverse times, z inverse times Z-transform here that is, P1, P1 z to the power M. Here what will it be? Z inverse k times original Pk z to the power M. That means X z will be what? In Z-Domain X z, X n sequence is summation of them, summation of them. So, X z will be what? This plus this plus this, like that.

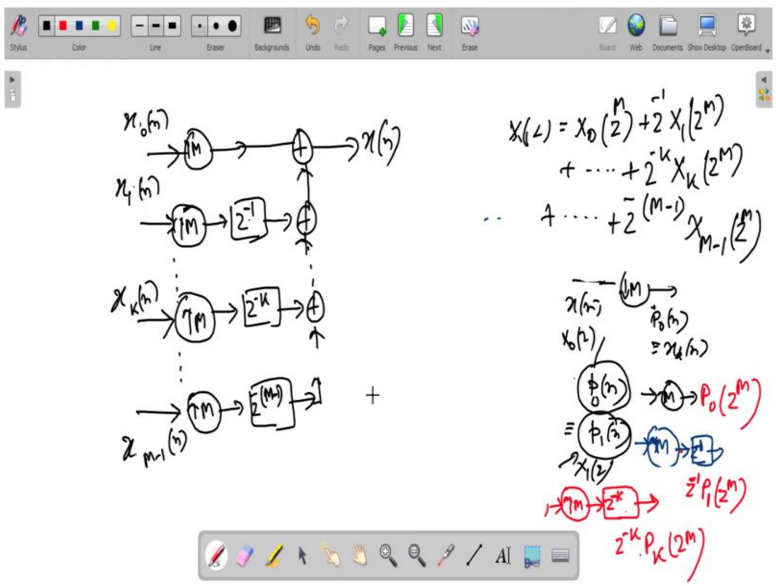
(Refer Slide Time: 25:09)



That means X z, now instead of p0 z, I am calling it as X0 z. This I am calling X0 z. Similarly, this I am calling X1 z, in Z-Domain of course and so on and so forth. There I kept p just to emphasize the fact they are called polyphase factors. That is why I used the letter p. But now I have come back to X0, X1 notation because I am starting with sequence X n Z-transform capital X z; so let the components be written in terms of X, capital X0, capital X.

So, this is X0 z to the power M from this one, then z inverse X1 from next one, dot, dot, dot, dot. Z inverse k, Xk this fellow, just capital P replaced by capital X, dot, dot, dot. Lastly, this is called the polyphase decomposition of a given sequence X n. This in time-domain, this in Z-Domain.

(Refer Slide Time: 27:06)



Time-domain we have seen that is, in time-domain what we saw? P0 n expanded P1 n, then z inverse, dot, dot, dot. If I add them in time-domain, then I get back x n. If you take Z-transform x z is P0, capital P0 which is, okay instead of P let be again change it, change it to X. I am making this mistake. So, capital X z, is capital X0 z to the power M because expanded by M, then class z inverse. Capital X1 z to the power M and so on and so forth. This is in time-domain that is in Z-Domain. This is called polyphase decomposition.

Using these, in the next class we will show, how we can you know, bring down the increase in hardware complexity in the FIR digital filter when you unfold it by say, factor 2, factor 3 like that. Okay, nicely okay, that we will do in the next class. Thank you very much.