Principles and Techniques of Modern Radar Systems
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Lecture - 54
Statistical Detection Theory
(Contd.)

Key Concepts: Rayleigh PDF, mean and variance of Rayleigh PDF, calculation of PDF of the phase of the received signal when no signal is present, determination of PDF of the envelope in presence of an echo signal, Rician distribution

Welcome to this NPTEL lecture on Principles and Techniques of Modern Radar Systems, we were discussing Statistical Detection Theory. So, yesterday in the previous class we have seen that what will be the distribution at the output of the video amplifier, what will be the probability distribution for envelop or phase.

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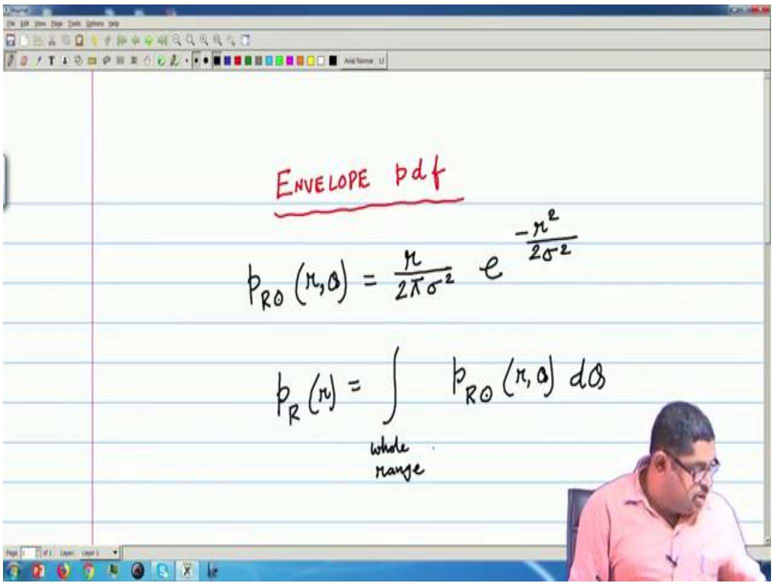
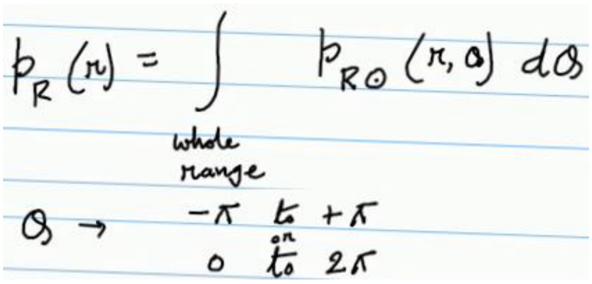


Now, we have derived up to here, the joint probability distribution of envelope and phase joint probability is given by these.

$$p_{R\theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

So, as we can see that in this expression or this joint probability is independent of the one of the variable theta.

So, that shows that r and theta are independent. So, if we want the envelop distribution r distribution for r, we can integrate over the whole range of theta. So, we can write next that p R r is nothing, but integration p R theta d theta oh sorry, this is a single integration. So, we should write that and this should be over the whole range. Now what is the whole range of theta? We know that theta is a periodic function of 2 pi.



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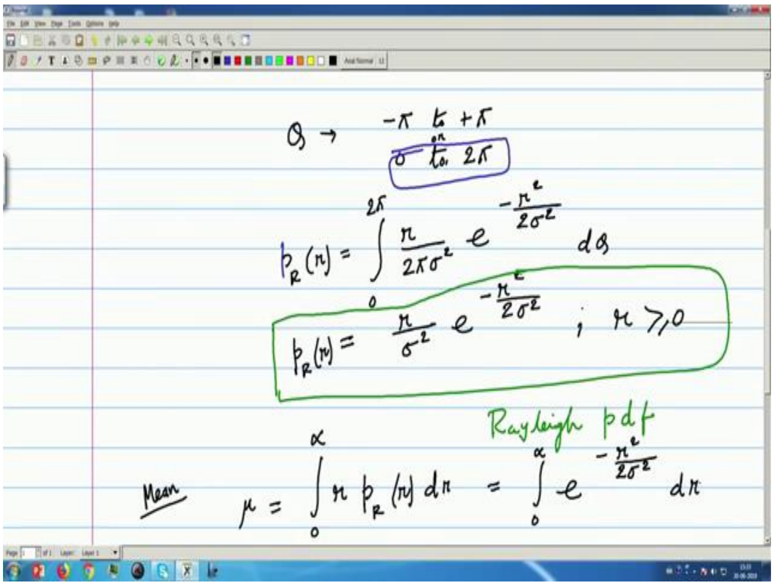

So, theta can vary from minus pi to pi or 0 to 2 pi or any other thing also you can take. So, generally without losing generality let us take these 0 to 2 pi. So, if we do that what we get p R r is equal to 0 to 2 pi r by 2 pi sigma square e to the power minus r square by 2 sigma square d theta. So, the whole thing will come out and we can integrate. So, this 2 pi will go and we will be left with r by sigma square e to the power minus r square by 2 sigma square; one thing we should remember that r is greater than equal to 0.

$$p_R(r) = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\theta$$

$$= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad ; \quad r \geq 0$$

Now this distribution p R r is a new distribution, because you see that with this r actually it is different from the Gaussian distribution. Had there been a constant term here then it would have been Gaussian; but due to this presence of r, this is a new distribution this is called Rayleigh distribution. Rayleigh p d f, basically any signal which is buried in noise; that means, when signal is almost absent, we get the envelop is this.

Now, here you see only one parameter is there sigma; what is sigma? Sigma is the constituent Gaussian p d f; that means, those two orthogonal Gaussian p d f's their standard deviation is sigma. And what is the mean of this distribution? We can easily calculate the mean, if we know p d f we can find mean.

So, mean mu of this distribution will be since r is varying from 0 to infinity, to 0 to infinity r p R r d r and if we do that by putting this value and integrating by parts you can easily do this. So, that will come out to be. So, if you do that you will get, ultimately you will come here 0 to infinity. I am jumping several steps, you do it easily you will find that this is e to the power minus r square by 2 sigma square d r.

$$\mu = \int_0^{\infty} r \, p_R(r) \, dr = \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr$$

Now once we see this, we can recognize that this is the integral that is a Gaussian p d f. So, we know Gaussian p d f with some constants and this is half of the Gaussian p d f. So, we know the area under the full Gaussian p d f is 1.

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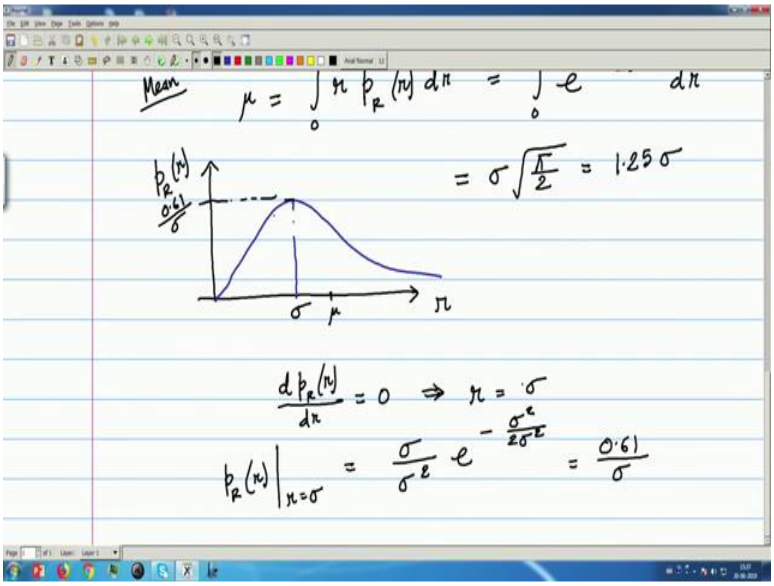

So, doing that you can find out that this is sigma pi by 2 it will come out. So, that is pi by 2 if we put the value 1.25 sigma.

$$\mu = \sigma\sqrt{\frac{\pi}{2}} = 1.25\sigma$$

So, you see that Rayleigh p d f it is mean is not 0, like the constituent Gaussian p d f's their mean was 0, but this is not, mean is not 0; that means, if we draw this p d f. So, I am drawing the Rayleigh p d f p R r versus r. So, unlike Gaussian which is centered about the mean, this is not that.

So, this will be something like and. So, first for any p d f we need to find out what is the point of the peak. So, for that what we can do that what is this point; that means, this point if I project what is this. So, for that I can differentiate this that d p R r d r is equal to 0, if I put that you will find that the answer is sigma.

So, that shows that this point is nothing, but the sigma point so; that means, it is peak is at r is equal to sigma.

$$\frac{d\, p_R(r)}{dr} = 0 \Rightarrow r = \sigma$$

So, its mean will be somewhere here, 1.25 of the mean. So, mu is here and what is this value? So, if in the p d f we put that. So, p R r r is equal to sigma, if we put we will get the value as. So, we can do that sigma by sigma square e to the power minus sigma square by 2 sigma square. So, that will give us 0.61 by sigma.

$$p_R(r)\Big|_{r=\sigma} = \frac{\sigma}{\sigma^2} e^{-\frac{\sigma^2}{2\sigma^2}} = \frac{0.61}{\sigma}$$

So that means, this value will be 0.61 by sigma that is the value of the peak of this distribution and so, after this we have found the mean. So, it shows that the peak is not at mean, peak is at sigma and that value is this peak value. And also you know that variance is an important quantity, because for our electrical type cases with stationary, quasi stationary assumption.

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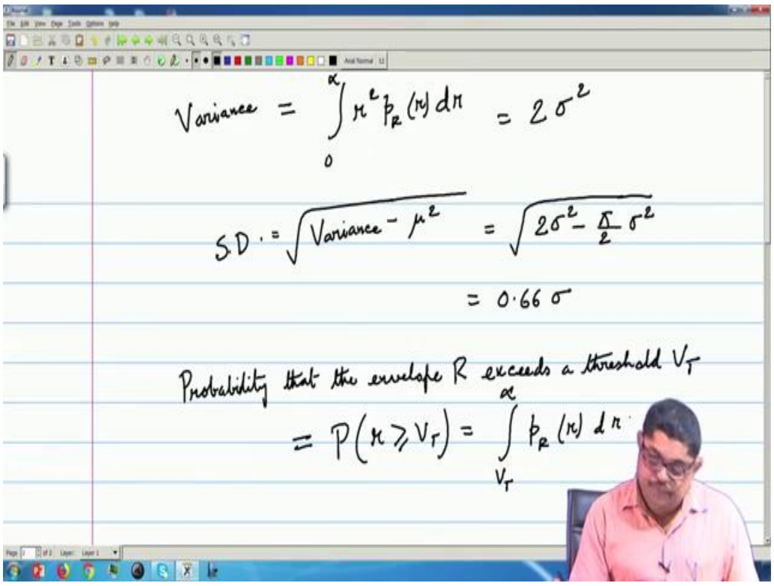

We can get the power etcetera all come from there. So, variance of this distribution will be 0 to infinity r square p R r d r. So, again you put the value in by integration by parts you can find that. So, if you do this we will get the value as 2 sigma square.

$$\text{Variance} = \int_0^\infty r^2 p_R(r)\,dr = 2\sigma^2$$

So, now it is a non-zero mean distribution, so its standard deviation will be root over variance minus mean square minus mu square so; that means, root over 2 sigma square minus mu; that is mu square is pi by 2 sigma square. So, if we do it will be 0.66 sigma.

$$S.D. = \sqrt{\text{Variance} - \mu^2} = \sqrt{2\sigma^2 - \frac{\pi}{2}\sigma^2}$$
$$= 0.66\,\sigma$$

So, once we have the variance standard deviation we know everything about the distribution. Now one thing will be requiring later that probability that the envelope R exceeds a threshold, let us say that threshold is V T, we are calling V threshold. So, what is that probability that will be nothing, but p that r is greater than equal to V and since we know the p d f we know, this will be V T to infinity p R r d r.

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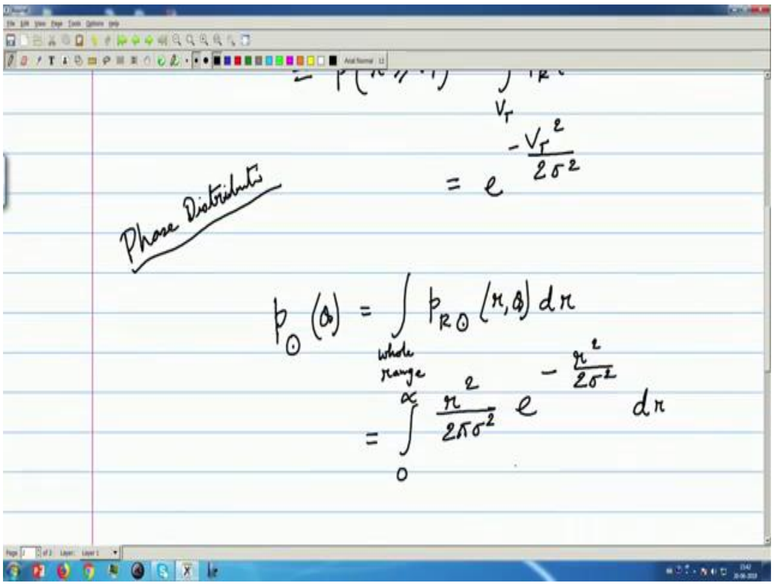

So, if we put the values, it will be we can easily put the value we know everything. So, integrating, this integration is tractable, it will come as e to the power minus V T square by 2 sigma square.

Probability that the envelope R exceeds a threshold V_T

$$= P(r \geq V_T) = \int_{V_T}^{\infty} p_R(r)\, dr$$

$$= e^{-\frac{V_T^2}{2\sigma^2}}$$

So, probability of exceeding the threshold is given by these ok. So, this is about envelope detection. Now let us find the phase if the phase is there the phase detector will detect the phase between the two quadrature Gaussian random variables x and y.

So, that. So, now, it is phase distribution let us calculate again; we will start from the joint distribution. So, what will be phase distribution, it will be nothing but again whole range p R r, p R theta sorry, p R theta r theta d r. So, if we put the value and whole range means for r whole range is 0 to infinity and this will be joint distribution, so r square by 2 pi sigma square e to the power minus r square by 2 sigma square d r.

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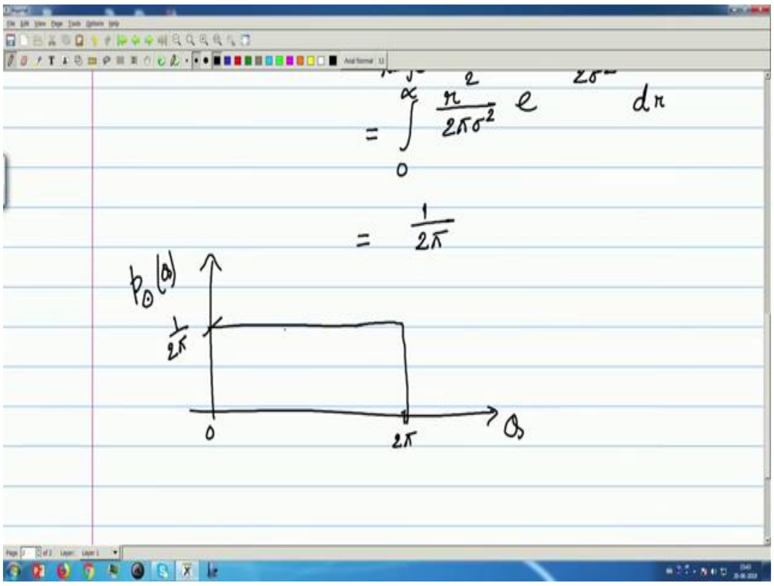

So, again this integration can be easily done and you will see that the value is 1 by 2 pi.

$$p_{\Theta}(\theta) = \int_{\substack{\text{whole}\\ \text{range}}} p_{R\Theta}(r,\theta)\, dr$$

$$= \int_0^{\infty} \frac{r^2}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

$$= \frac{1}{2\pi}$$

So, you see that the phase distribution is a constant. So, if we plot the p theta versus theta we are taking 0 to 2 pi. So, it is a constant and we know that 0 to 2 pi is the whole range. So, the area under these phase is this p d f should be 1; that means, we know these value, these value is again 1 by 2 pi. So, these value is 1 by 2 pi and this is 2 pi you see area is 1. So, we are correct. So, phase distribution you see it is an uniform distribution.

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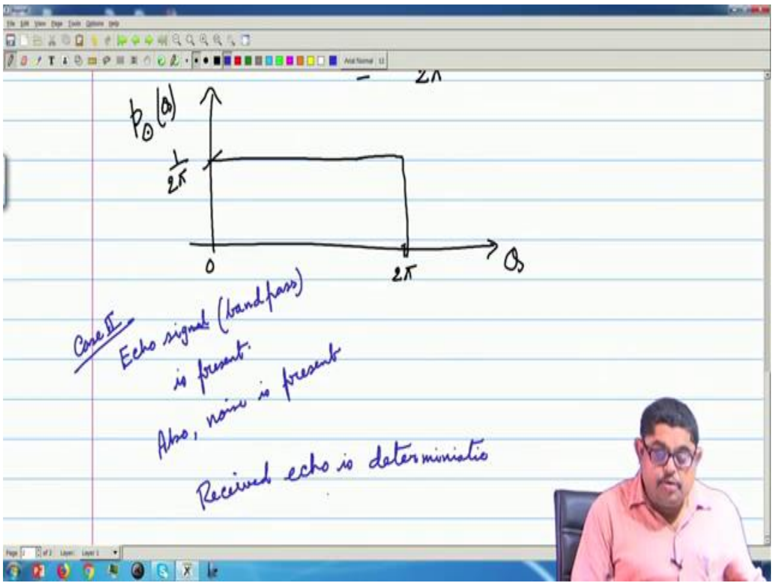

Now, next let us go to the another case that, previously all these were for when the signal was absent; now case two is echo signal; obviously, band pass is present, also those noise is present, noise with the previous same thing that band pass noise, Gaussian random variable etcetera. Now, the question is this echo signal it is not a random signal, it is a deterministic signal. Here someone may ask that, I have myself said that the target RCS is a random variable it fluctuates.

Yes, I still say that the RCS, target RCS sigma is random variable; but that this target RCS fluctuation that has a distribution, that distribution has a mean that is not a zero mean thing that is not a Gaussian thing. So, that distribution has a mean so; that means, any signal it goes then scatters back that has a definite value. So, received echo that is deterministic, please remember this. So, received echo is deterministic. So, now, we have then….

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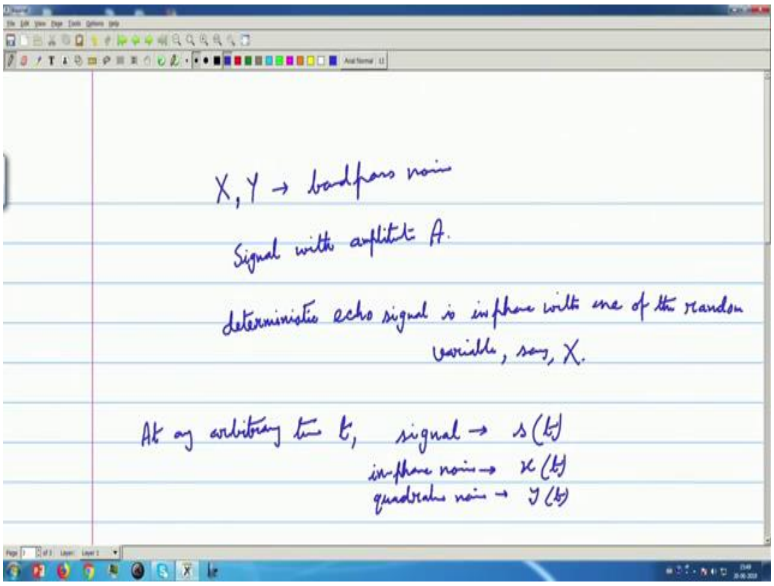

So, as before we have X and Y band pass noise with all those same parameters; also we have an deterministic signal with let us say amplitude A. Now here this X and Y they are orthogonal. Now what we can do without losing generality, we can break this band pass noise in X Y such a manner that X and A they are in phase; that means, the in phase part X and A deterministic signal, they are at same phase whatever remaining that is at Y. So, we are saying that deterministic echo signal is in phase with one of the random variable say X.

We will deliberately make that always we can do that whatever the amplitude in that space in that 2 D space, the signal we are breaking the in phase part of noise collinear with that. And now here I want to give a caution again I am saying X and Y they are random variable. So, they cannot be you cannot say what is their expression. So, they cannot be represented by a function.

But I can always say that any arbitrary time t, any arbitrary time t we can say that. So, at any arbitrary time t, we can say that signal has some value, let that value I am representing by s t. And in phase noise and quadrature noise also has the value. So, in phase noise let I write it as x t and quadrature noise y t. You say I am not saying that it is a function of time etcetera, what I am saying this is a sample value; actually in random variable parlance this is called sample value up in phase noise, sample value of the quadrature noise at time t. So, x t, y t are sample functions of random variable X and Y.

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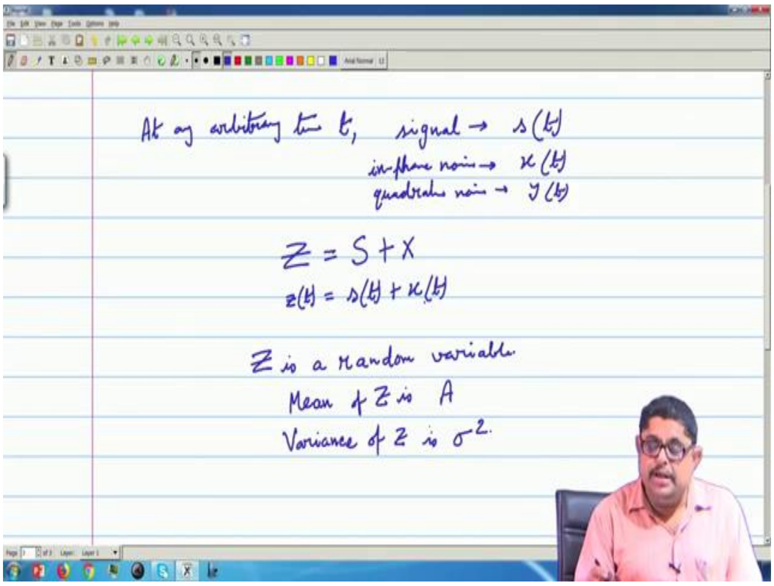

And so, we assume that we write like these Z is equal to S plus X so; that means, small z t can also be written as this s t plus x t.

$$Z = S + X$$
$$z(t) = s(t) + x(t)$$

Again here this is the sample value, this is the function value maybe. Now the question is what is then Z? Obviously, Z is equal to S plus X, S is a deterministic thing, but X is a random variable. So, definitely Z is a random variable, this is a obvious thing. So, I can write Z is a random variable, but Z mean is not 0 though X's mean is 0, Z's mean is not 0.

So, what is the mean of Z? If X's mean is 0 that is our assumption that the noise, zero mean noise. So, if this mean is 0; what is Z's mean? Mean of z is, then it will be the amplitude of this signal that is A. And what will be the, these are all simple things for additive thing we know. So, variance of Z that will be, the variance of signal is 0, but variance of X is there. So, Z is sigma square; also you see that Z and Y they are in quadrature.

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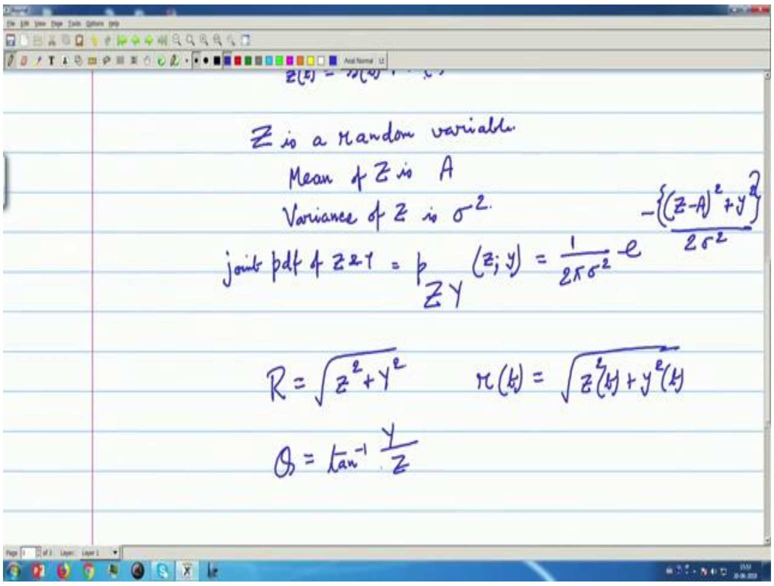

So, Z and Y are independent. So, their joint p d f we can write, joint p d f of two random variables Z and Y that we can write as p Z Y small z y is equal to 1 by 2 pi sigma square e to the power minus here you see it is a nonzero thing. So, z minus A whole square plus I should write like this by 2 sigma square.

$$p_{ZY}(z;y) = \frac{1}{2\pi\sigma^2} e^{-\frac{\{(z-A)^2+y^2\}}{2\sigma^2}}$$

And again as before let us say that envelope; what is the envelope R? R will be root over z square plus y square, or if you want to write in terms of sample values r t will be z square t plus y square t like that. And what is theta? Tan inverse Y by Z.

$$R = \sqrt{z^2 + y^2} \qquad r(t) = \sqrt{z^2(t) + y^2(t)}$$

$$\theta = \tan^{-1}\frac{y}{z}$$

Again we can have that same thing converting to polar coordinate just like Rayleigh case.

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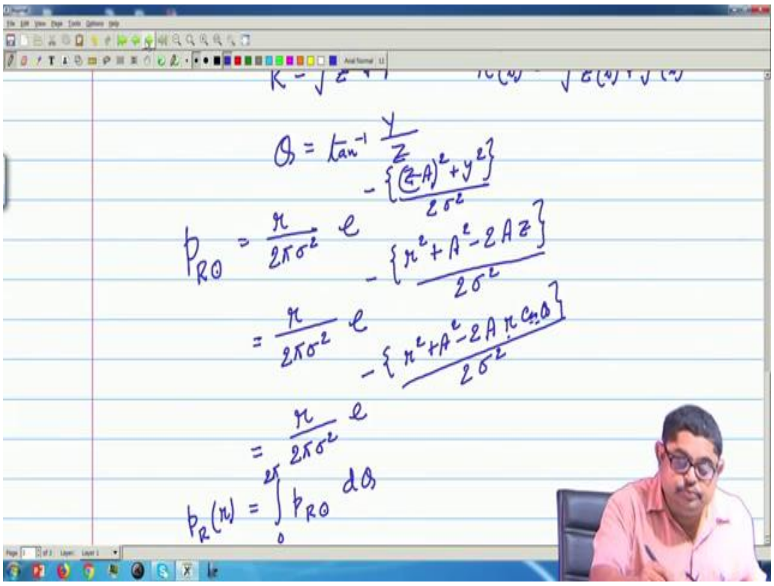

So, we can write the joint distribution p R theta is equal to r by 2 pi sigma square e to the power minus z minus A whole square plus y square by 2 sigma square and that is r by 2 pi sigma square e to the power, I can break this z, z I know what it is. So, I can break and if I do that, this will be r square plus A square minus 2 A z. You see here I have taken this, that if I break this I will get z square plus y square that I am writing as r square that is the only thing, divided by 2 sigma square ok.

And I need to bring theta. So, I can write this as r square plus A square minus 2 A; what is z? Z is nothing where r cos theta by 2 sigma square.

$$p_{R\Theta} = \frac{r}{2\pi\sigma^2} e^{-\frac{\{(z-A)^2+y^2\}}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{\{r^2+A^2-2Az\}}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{\{r^2+A^2-2Ar\cos\theta\}}{2\sigma^2}}$$

Now, here you see this joint distribution that is; that means r and theta they are not independent, because of this last term that r and theta they are here. So, we cannot say that thing now, but still I can have that what is the envelope distribution p R r will be 0 to 2 pi, p R theta then d theta.

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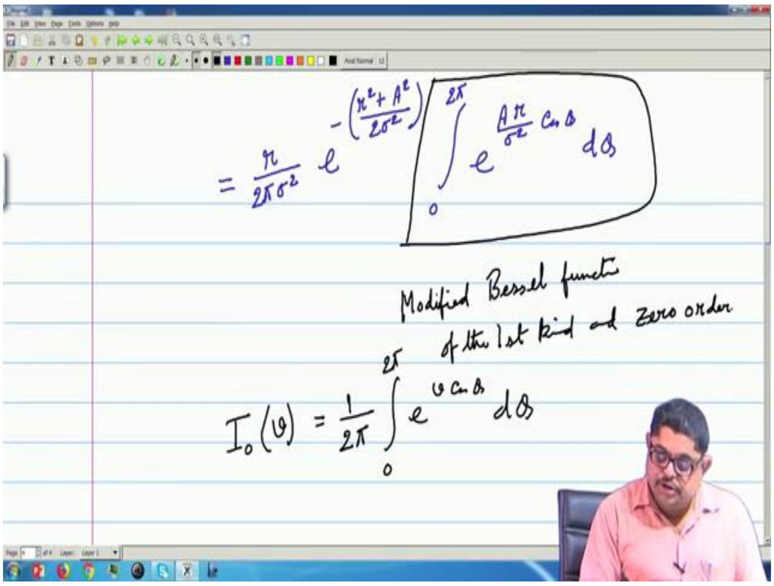

So, if we go on doing we can take out the constant terms r by 2 pi sigma square e to the power minus r square plus A square by 2 sigma square and 0 to 2 theta integration there will have e to the power A r by sigma square cos theta d theta ok.

$$p_R(r) = \int_0^{2\pi} p_{R\Theta}\, d\theta$$

$$= \frac{r}{2\pi\sigma^2} e^{-\left(\frac{r^2+A^2}{2\sigma^2}\right)} \int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos\theta}\, d\theta$$

So, this integration actually I think in engineering, sciences comes regularly that, e to the power x cos theta d theta basically with some multiplier.

So, this is actually 0 to 2 pi, this regularly comes in various cases of a thing. And actually the answer to this integration is a special function which is called Bessel function; many of you have recognized that. So, to be stricter sense it is called this, this thing you should say that this part is modified Bessel function of the 1st kind. And 1st kind means for that z generally is the thing and it is zero order.

So, zero order Bessel function the first kind this is something like cosine, only thing is at 0 it is not exactly cosine that one thing, it is some other thing. So, this is generally written like this I or J. So, I naught v is 1 by 2 pi 0 to 2 pi e to the power v cos theta d theta. So, this is the generally modified Bessel function of first kind in zero order.

$$I_0(v) = \frac{1}{2\pi}\int_0^{2\pi} e^{v\cos\theta}\, d\theta$$

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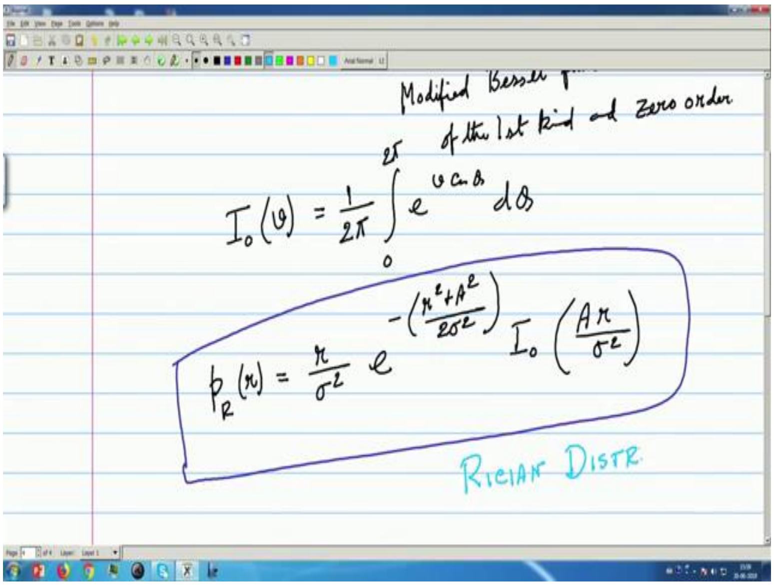

So, in this term I can write my p R r that will be r by sigma square e to the power minus r square plus A square by 2 sigma square I naught A r by sigma square.

$$p_R(r) = \frac{r}{\sigma^2} e^{-\left(\frac{r^2+A^2}{2\sigma^2}\right)} I_0\left(\frac{Ar}{\sigma^2}\right)$$

So, this is another distribution we have come; that signal is present in the deterministic signal is present, noise is present. Now it is no longer Gaussian or Rayleigh, it is another distribution; this distribution is again familiar this is called the Rician distribution, under the name of the scientist eyes Rician distribution.

So, we will discuss about this distribution more in the next class, but we have found this thing. So, initially we have said that when the echo signal is absent, the envelop detector, the envelop distribution is Rayleigh. Now we are saying it is a new way thing, it is a Rician thing and it is something like Bessel function with some exponential part. We will discuss about the behavior of this distribution in the next class.

Thank you.