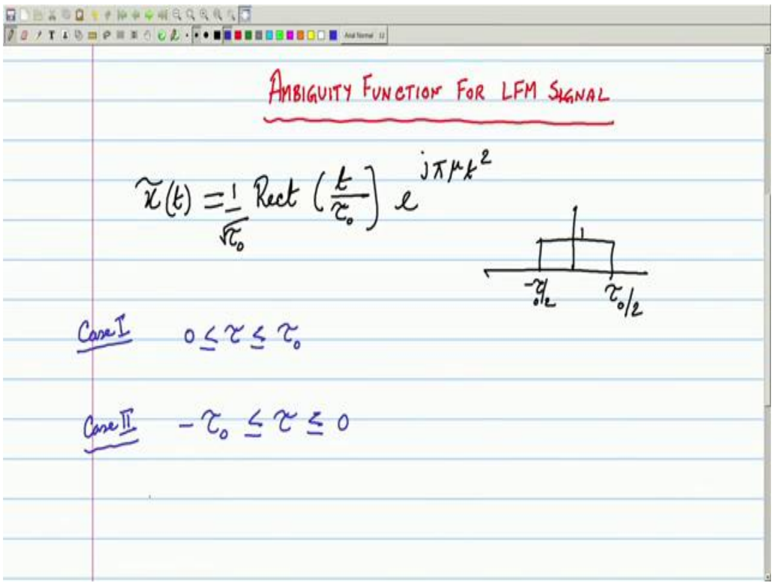
Principles And Techniques Of Modern Radar System
Prof. Amitabha Bhattacharya
Department of E & ECE
Indian Institute of Technology, Kharagpur

Lecture – 45
Detection in Radar Receiver (contd.)

Key Concepts: Range Doppler ambiguity function for LFM chirp, definition of compression gain

Welcome to the NPTEL lecture on Principles and Techniques of Modern Radar Systems. We have already seen the spectrum of the chirp signal.

(Refer Slide Time: 00:39)



Today we will see the ambiguity function so, that we will be able to understand what happens to the resolution for chirp signal or LFM signal. Already we have seen the complex envelop of a up chirp signal in the previous lecture, actually in that lecture you can refer that we have written the sorry.

Yesterday we have written the complex envelop as t by tau naught; that means, it is existing a rectangular function from minus tau naught by 2 to plus tau naught by 2.

$$\tilde{x}(t) = \mathrm{Rect}\left(\frac{t}{\tau_0}\right) e^{j\pi\mu t^2}$$

And it has a exponential second order phase function. But, this actually you see that if this is the signal and if the amplitude is 1; then if I have it from minus tau naught by 2 to plus tau naught by 2; the energy of the signal becomes tau naught 1 square into tau naught so, that it is better to normalize it.

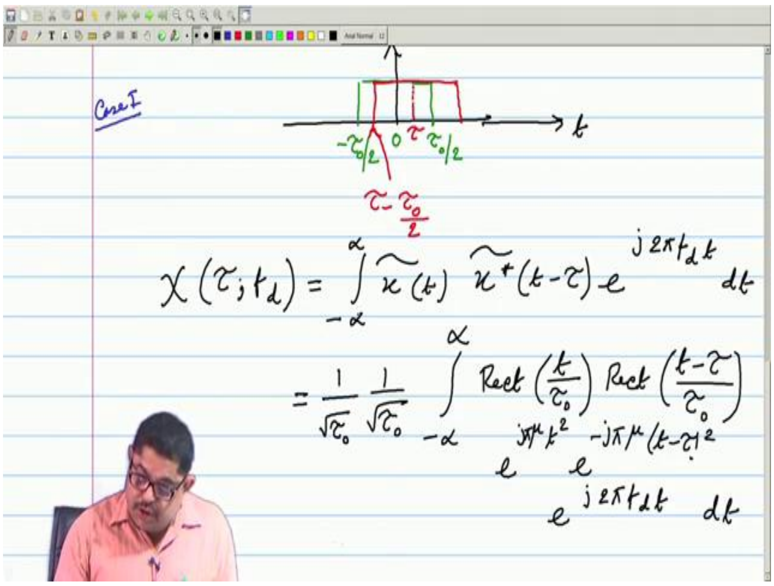
So, I can write that today I will normalize the complex envelop with respect to energy. So, if I do that this becomes 1 by root over tau naught, ok.

$$\tilde{x}(t) = \frac{1}{\sqrt{\tau_0}} \, Rect\left(\frac{t}{\tau_0}\right) e^{j\pi\mu t^2}$$

So, just this must change so, this is the our complex envelop. Now, we have already earlier derived that if I know the transmitted signal complex envelop I can find the ambiguity function. Now; obviously, ambiguity function is a correlation type of function.

So, depending on the shift or delay, we will have various cases so, first is the case I. So, where what is that case that is 0, less than equal to tau less than equal to this tau naught. So, this is the first case and there will be case II; the other case that will be minus tau naught less than equal to tau less than equal to 0. So, we will break this ambiguity function into these two intervals; first let us do for case I.

(Refer Slide Time: 04:10)



So, oh sorry this is case I. So, the picture I can draw like this, let us take this is my t axis this is my y axis. Now let me represent let us say the x star so, the signal I have taken a green name so, that ok, so this is my 0. So; obviously, this is tau naught by 2, this is minus tau naught by 2. Now, case I means I have given a positive shift. So, let me take a red colored one, so that will give me.

So, here; that means, how much what is tau? Let us say this is my tau, so these two end points. So, this one will be tau plus tau naught by 2; and this one will be tau minus tau naught by 2. And so we can see that these two functions their product will exist from tau minus tau naught by 2 to tau naught by 2. So, now I can write the ambiguity function expression.

So, that we know; I am again repeating previously we have derived this is nothing, but x t; then e to the power j 2 pi f d t d t this is the expression.

$$\chi(\tau; f_d) = \int_{-\infty}^{\infty} \tilde{x}(t)\, \tilde{x}^*(t-\tau)\, e^{j2\pi f_d t}\, dt$$

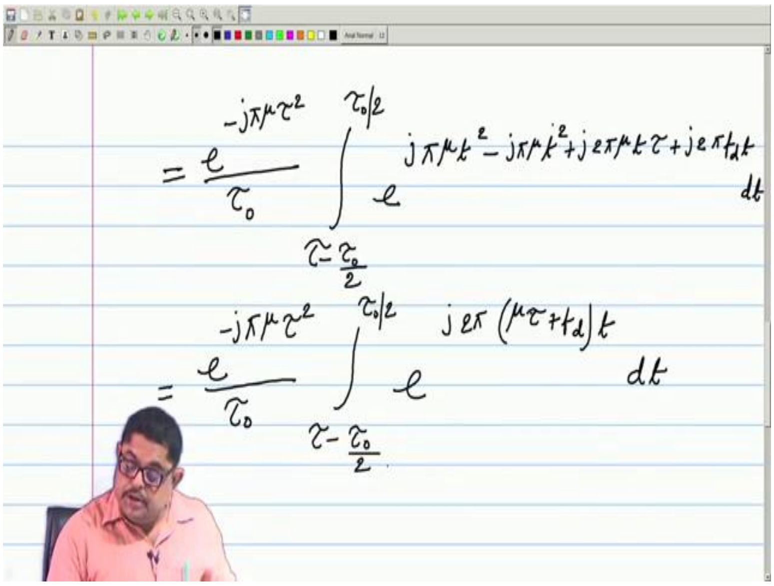
So, now, I will have to put that now x t I have already written. So, it is 1 by tau naught and these two again there will be another 1 by tau naught from the next one and this is minus infinity to infinity.

So, the first one is rectangular t by tau naught and the second one is rectangular t minus tau by tau naught. And also there is a phase term e to the power j mu t square or j pi there will be j pi mu t square. Then for this it will be e to the power minus j pi mu t minus tau whole square and then there will be e to the power j 2 pi f d t d t. So, this integration we now need to carry out.

$$= \frac{1}{\sqrt{\tau_0}} \frac{1}{\sqrt{\tau_0}} \int_{-\infty}^{\infty} \mathrm{Rect}\left(\frac{t}{\tau_0}\right) \mathrm{Rect}\left(\frac{t-\tau}{\tau_0}\right) e^{j\pi\mu t^2}\, e^{-j\pi\mu (t-\tau)^2}\, e^{j2\pi f_d t}\, dt$$

So, it is quite easy that go to the.

(Refer Slide Time: 08:52)



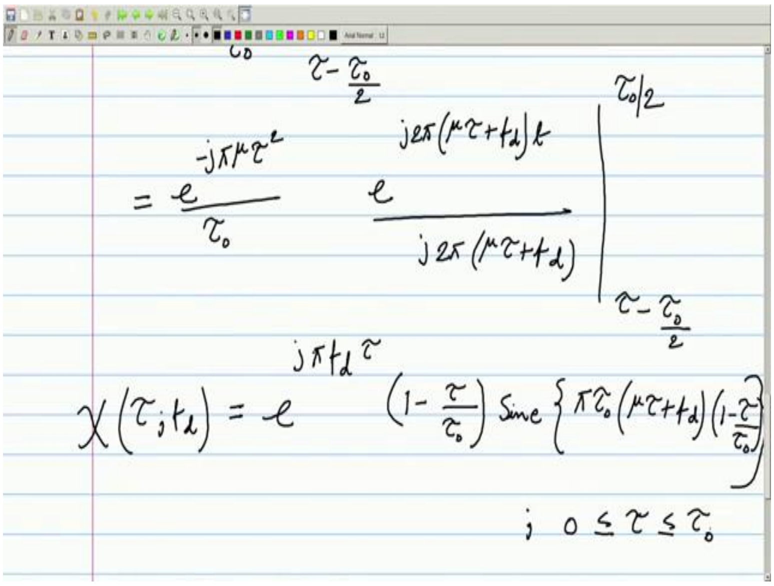
So, this will be the constant things I can take out e to the power minus; if you do the simplification e to the power minus j pi mu tau square sorry tau square by tau naught this is out of integration. And as we said the integration limit is from tau minus tau naught by 2 to tau naught by 2; and what remains there is e to the power j pi mu t square minus j pi mu t square plus j 2 pi mu t tau plus j 2 pi f d t d t.

$$= \frac{e^{-j\pi\mu\tau^2}}{\tau_0}\int_{\tau-\frac{\tau_0}{2}}^{\tau_0/2} e^{j\pi\mu t^2 - j\pi\mu t^2 + j2\pi\mu t\tau + j2\pi f_d t}\,dt$$

So, it is a whole exponential thing e to the power j integration. So, this can be easily done you know this will you have done it many times. So, I think these two terms will also get cancelled the first two terms. So, finally, we will be having e to the power minus j pi mu tau square by tau naught, then tau minus tau naught by 2 to tau naught by 2; e to the power j 2 pi also I can take common and then it becomes mu tau plus f d t d t.

$$= \frac{e^{-j\pi\mu\tau^2}}{\tau_0}\int_{\tau-\frac{\tau_0}{2}}^{\tau_0/2} e^{j2\pi(\mu\tau + f_d)t}\,dt$$

(Refer Slide Time: 11:44)



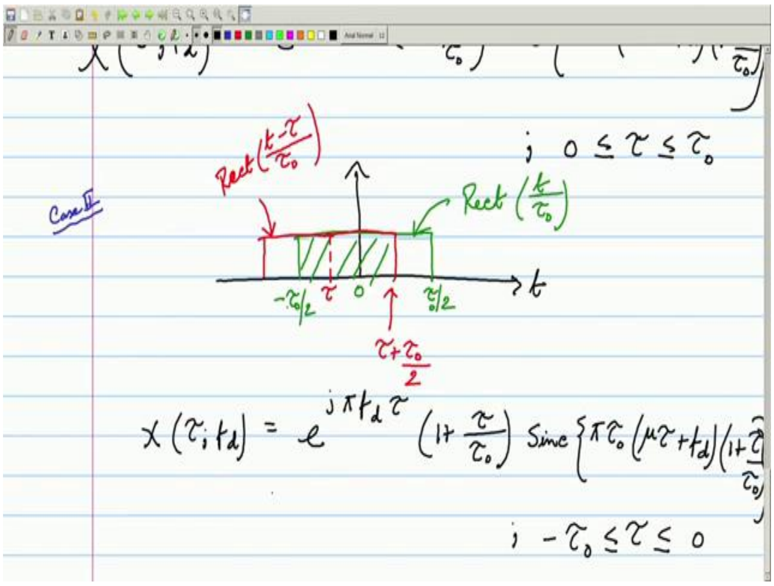
Now this can be easily integrated. And so, if I integrate this mu tau square by tau naught. This will be e to the power j 2 pi mu tau plus f d t by j 2 pi mu tau plus this is your elementary calculus. And the limits of t are tau minus tau naught by 2 to tau naught by 2.

$$= \frac{e^{-j\pi\mu\tau^2}}{\tau_0} \left. \frac{e^{j2\pi(\mu\tau+f_d)t}}{j2\pi(\mu\tau+f_d)} \right|_{\tau-\frac{\tau_0}{2}}^{\tau_0/2}$$

So, if you do that you can get the ambiguity function for the chirp signal up chirp. This will be e to the power plus j pi f d tau 1 minus tau by tau naught sinc pi tau naught mu tau plus f d 1 minus tau by tau naught for what range because this is only for this, tau less than equal to tau naught, this is the first case result.

$$\chi(\tau;f_d) = e^{j\pi f_d\tau}\left(1-\frac{\tau}{\tau_0}\right)\text{Sinc}\left\{\pi\tau_0(\mu\tau+f_d)\left(1-\frac{\tau}{\tau_0}\right)\right\} \quad ; \quad 0\le\tau\le\tau_0$$

(Refer Slide Time: 13:53)



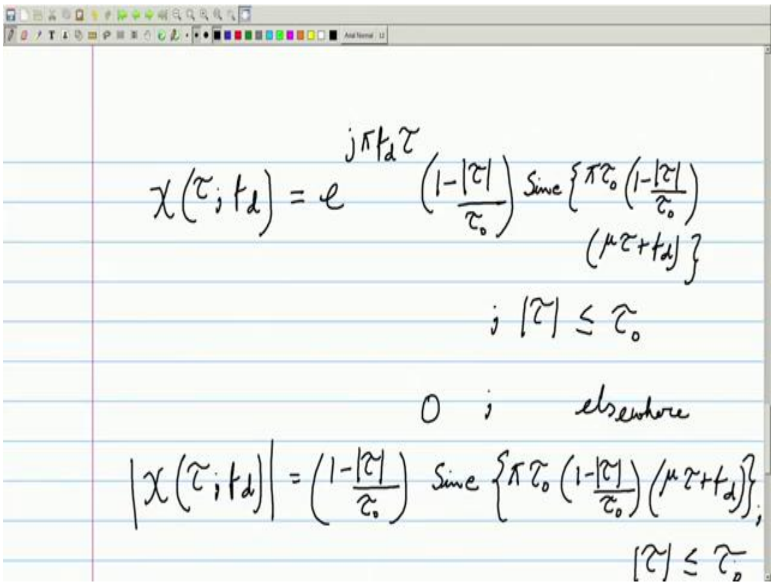
Then case II; let me first draw the case II thing again. So, this is my time axis and let me say that. So, this is my 0 tau naught by 2 minus tau naught by 2. Now here we are giving a negative shape so; that means, in general it is something like this. So, this will be our tau so; that means, this will be tau minus tau naught by 2, but more important is this one tau plus tau naught by 2.

So, actually I can say that this is rectangular t minus tau by tau naught and this one green one is rectangular function t by tau naught. So, here the integration limit will exist from minus tau by 2; minus tau naught by 2 to this much so; that means, this. So, we can now find a ambiguity function in this case. And here if you do only the changing the limit, it will come out as e to the power j pi f d tau; 1 plus tau by tau naught sinc pi tau naught mu tau plus f d 1 plus tau by tau naught. And this is for this range minus tau naught is less than equal to tau less than equal to 0.

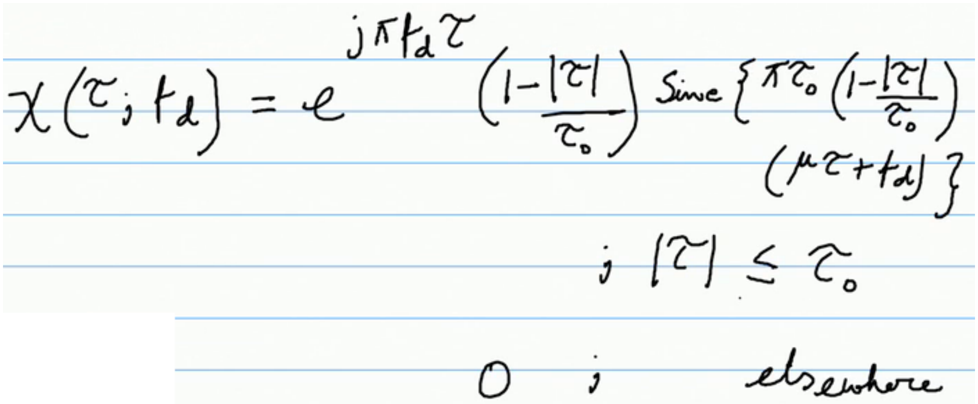
$$\chi(\tau; f_d) = e^{j\pi f_d \tau}\left(1+\frac{\tau}{\tau_0}\right) \text{Sinc}\left\{\pi\tau_0(\mu\tau+f_d)\left(1+\frac{\tau}{\tau_0}\right)\right\}; \quad -\tau_0 \le \tau \le 0$$

So, we can combine case I and case II.

(Refer Slide Time: 17:26)



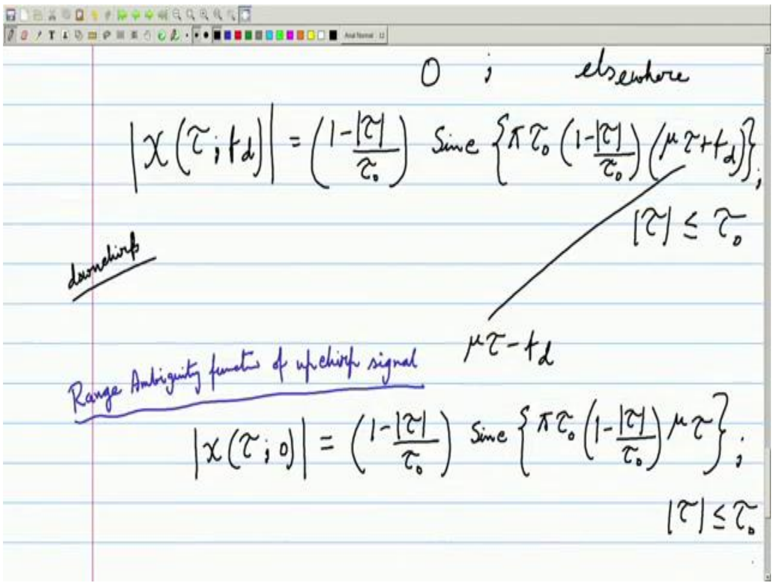
So, combining we can write that for all ranges of tau we can say that the ambiguity function will be, e to the power j pi f d tau 1 minus we can use the modulus function sinc of pi tau naught 1 minus tau by tau naught mu tau plus f d; for tau less than equal to tau naught modulus f tau and 0 elsewhere so, ambiguity function exists here.



And generally we take the so, this is the ambiguity function and its magnitude will be that time only this term will go. So; that means, I can write that the magnitude of the ambiguity function generally that is called ambiguity function. The magnitude is given by 1 minus tau by d tau naught sinc pi tau naught 1 minus modulus tau by tau naught mu tau plus f d for tau less than equal to tau naught.

$$|\chi(\tau; f_d)| = \left(1-\frac{|\tau|}{\tau_0}\right) \text{Sinc}\left\{\pi\tau_0\left(1-\frac{|\tau|}{\tau_0}\right)(\mu\tau+f_d)\right\}; \quad |\tau| \le \tau_0$$

(Refer Slide Time: 20:12)

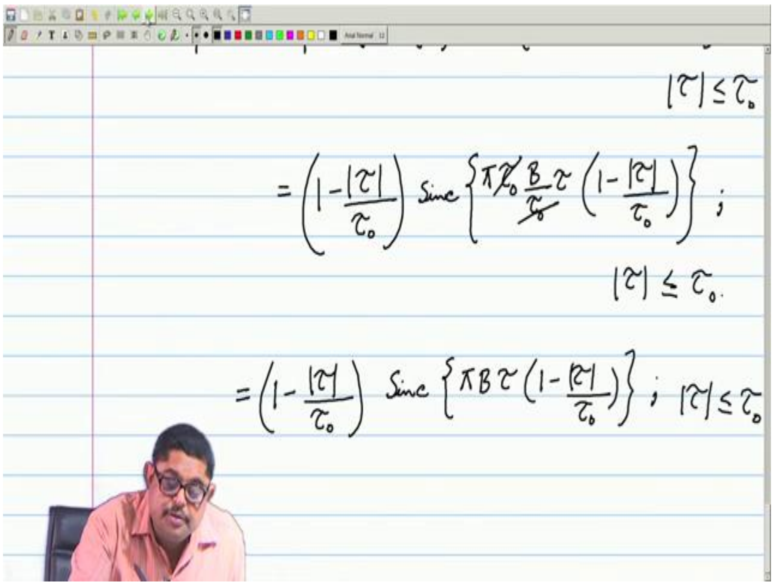


So, this is for up chirp, if I have down chirp; for down chirp this portion should be replaced by mu tau minus f d because that is the only change there so, we got this. And from here actually we can see the range ambiguity function; so, I can say that range, actually this is a 2 dimensional signals it can be plotted in a 3 d plot. But more meaningfully we will use a 2 d plot that is why we are specialized in to range ambiguity function of up chirp signal.

So, what is range ambiguity function that time f d should be put to 0; that means? So, that will be simply 1 minus tau by tau naught sinc pi tau naught 1 minus tau by tau naught mu tau otherwise 0.

$$|\chi(\tau; 0)| = \left|\left(1-\frac{|\tau|}{\tau_0}\right) \text{Sinc}\left\{\pi\tau_0\left(1-\frac{|\tau|}{\tau_0}\right)\mu\tau\right\}\right|; \quad |\tau| \le \tau_0$$

(Refer Slide Time: 22:28)



So, here you see there is this new parameter, we can put that value mu was the LFM parameter. So, that we know it is B by tau naught so, that I can put pi tau naught B by tau naught tau 1 minus tau by tau naught. So, these two cancels.

$$= \left| \left(1-\frac{|\tau|}{\tau_0}\right) \mathrm{Sinc}\left\{\pi \cancel{\tau_0} \frac{B}{\cancel{\tau_0}} \tau \left(1-\frac{|\tau|}{\tau_0}\right)\right\} \right|; \quad |\tau| \le \tau_0.$$

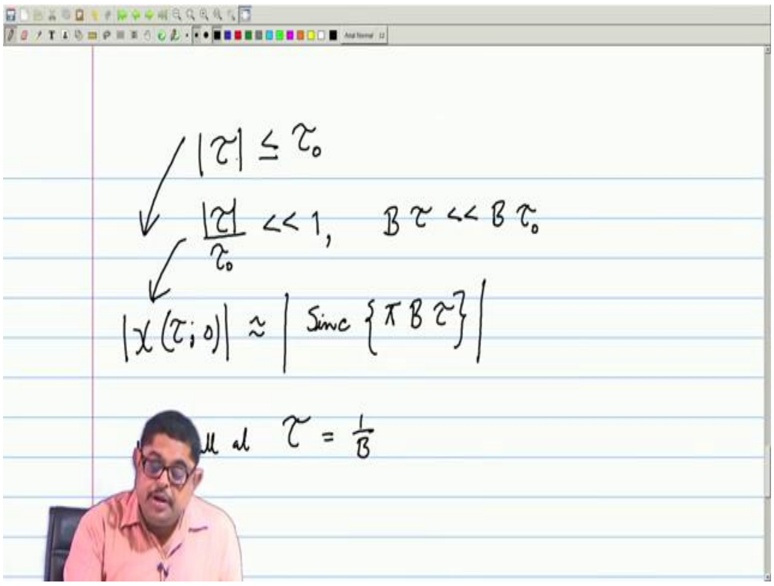
And we are left with you see that 1 minus tau by tau naught sinc pi B tau 1 minus tau by tau naught.

$$= \left| \left(1-\frac{|\tau|}{\tau_0}\right) \mathrm{Sinc}\left\{\pi B \tau \left(1-\frac{|\tau|}{\tau_0}\right)\right\} \right|; \quad |\tau| \le \tau_0$$

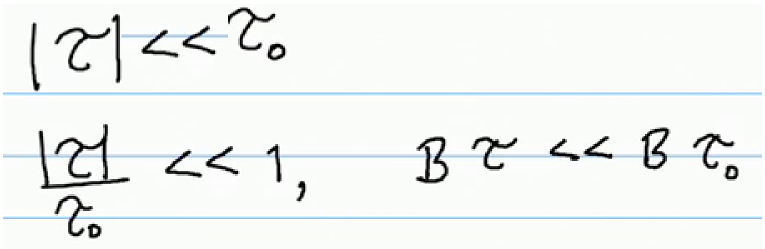
Now, what we have gained? So, to understand that you see that; obviously, if I have a when tau is equal to tau naught, this range ambiguity function goes to 0.

From here you can see whatever happens to sinc this when tau is equal to tau naught it goes to 0. So, that was true for the constant frequency pulse range ambiguity function also. If you look at that expression constant frequency pulse range ambiguity function, you will see that the expression was similar the only change is here these. So, a sinc function goes to 0.

(Refer Slide Time: 24:53)



So, but let us look carefully that what happens if I let us see for cases when tau is much much less than tau naught. That means, I am saying that the compared to the pulse width at very small delays what happens. So; that means, tau by tau naught is sorry, in this case I can say that multiplying both sides by B; B tau is much less than B tau naught.



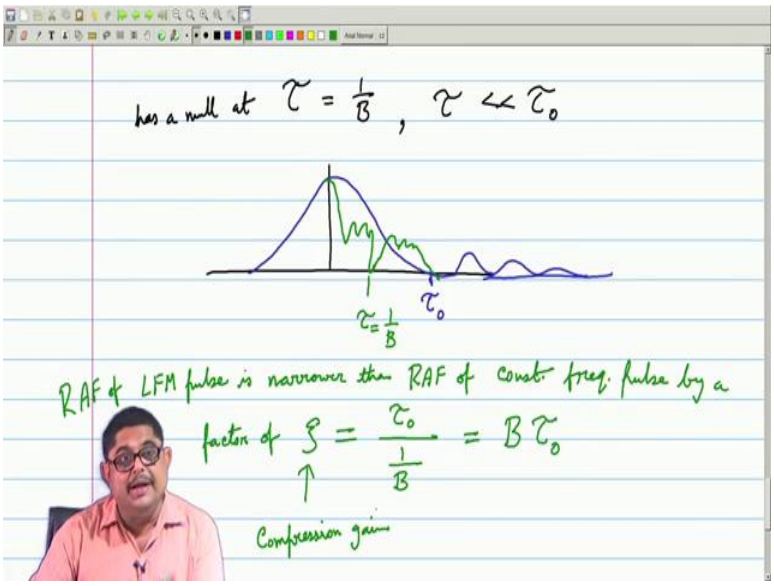
So, there this range ambiguity function approximately that is what let us see.

So, if this portion is small; that means, this thing is almost one this thing is almost one. So, it is basically a sinc type of function. So, it is given that oh, this the whole thing should have been given a modulus just correct it because, here I have taken these. So, this should have been a modulus function. So, this becomes sinc pi B tau, so near this for this condition that or for this condition; that means, smaller values these.

$$|\chi(\tau;0)| \approx \left| \mathrm{Sinc}\{\pi B \tau\} \right|$$

Now this has a null at what point where tau. So, it has a null at tau is equal to 1 by B so, but what is this tau? It is much much less than they should be much smaller than tau naught that is why this thing is going to take.

(Refer Slide Time: 28:12)



So, has a null at tau which is this tau is much much less than tau naught. So, compared to a single frequency pulse, it is happening at a much before. So, if we plot it we will see that; suppose for a constant frequency pulse let us say the sinc is like this, but here for the LFM pulse it will happen something like these. So, this is the important point actually this point was; obviously, at tau naught.

But here we are getting a null at a point tau is equal to 1 by B and it is much much smaller than this. So, you see that LFM pulse is having an ambiguity function whose width is much less that is why we can set the resolution very small here. Now how much is gain? So, let us say is that the LFM pulse is narrower than so, range ambiguity function of LFM pulse is narrower than, range ambiguity function of constant frequency pulse by a factor of let us say zeta.

So, what is zeta? So, in case of this thing it is tau naught and in case of this it is 1 by B so, that is B tau naught.

$$\zeta = \frac{\tau_0}{\frac{1}{B}} = B\tau_0$$

So, this is called the compression gain; this zeta is the compression gain so, how much we have achieved? A by giving the constant frequency in the pulse transmitted pulse, we have achieved a narrowness by a factor of B into tau naught.

I think you are recognizing B into tau naught this is the important thing, actually this is also related to the product of the two resolutions so, that B tau naught is coming. So, we will see the calculation, we will do a sample calculation because already for a constant frequency pulse we have done the resolution calculations. We will recalculate that for this thing for a LFM pulse in the next class.

Thank you.