Principles And Techniques Of Modern Radar Systems
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Lecture – 44
Detection in Radar Receiver
(Contd.)

Key Concepts: Introduction to the LFM coefficient, derivation of Fourier transform of upchirp, delay resolution time constant and Doppler resolution constant for LFM waveforms

Welcome to this NPTEL lecture on Techniques and Principles of Modern Radar Systems. We have seen in the previous class the we have started finding the a waveform which can give us a handle to design both the range and Doppler for velocity resolutions choose independently. So, that waveform is linear frequency modulation. We have seen the how what is a linear frequency modulated waveform.

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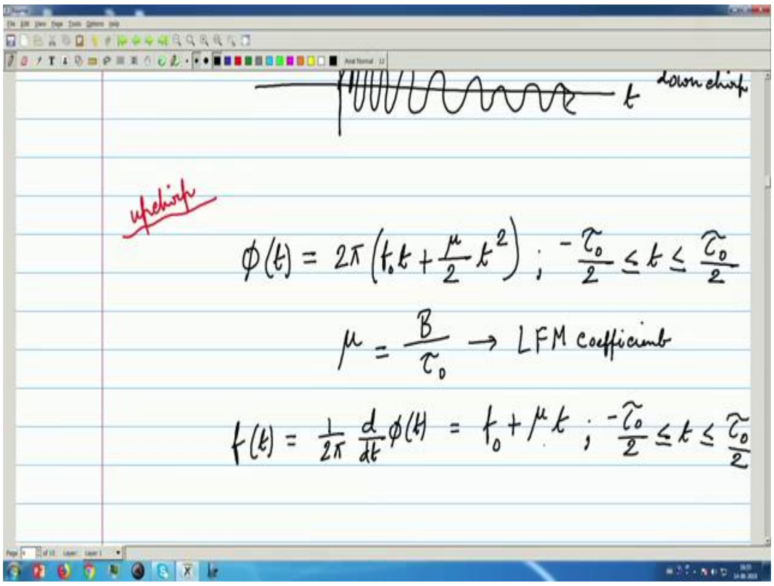

So, consider the up chirp waveform. So, for up chirp, so the instantaneous phase can be written as

$$\phi(t) = 2\pi\left(f_0 t + \frac{\mu}{2} t^2\right) ; \; -\frac{\tau_0}{2} \le t \le \frac{\tau_0}{2}$$

$$\mu = \frac{B}{\tau_0} \rightarrow \text{LFM coefficient}$$

So, this is the way it is given for minus tau naught by 2; where we have introduced a parameter mu, actually this is nothing but this slope, for up chirp this is the slope. So, mu is will be B by tau naught and for down chirp it will be minus B by tau naught. So, since we are considering up chirp I can write that mu is equal to B by tau naught and it is sometimes called LFM coefficient, actually the frequency slope of the waveform.

So, if this is the phase expression, we can find what is the instantaneous frequency f t that is nothing, but 1 by 2 pi d d t of phi t. So, that gives us f naught plus mu into t. So, you see that this is again in that interval.

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_0 + \mu t ; \; -\frac{\tau_0}{2} \le t \le \frac{\tau_0}{2}$$

So, this is this thing that gives us the frequency, you see the instantaneous frequency is linearly varying that is why the name linear frequency modulation. So, unlike our communication signal in communication this linear variation is absent, there is a frequency deviation term there comes.

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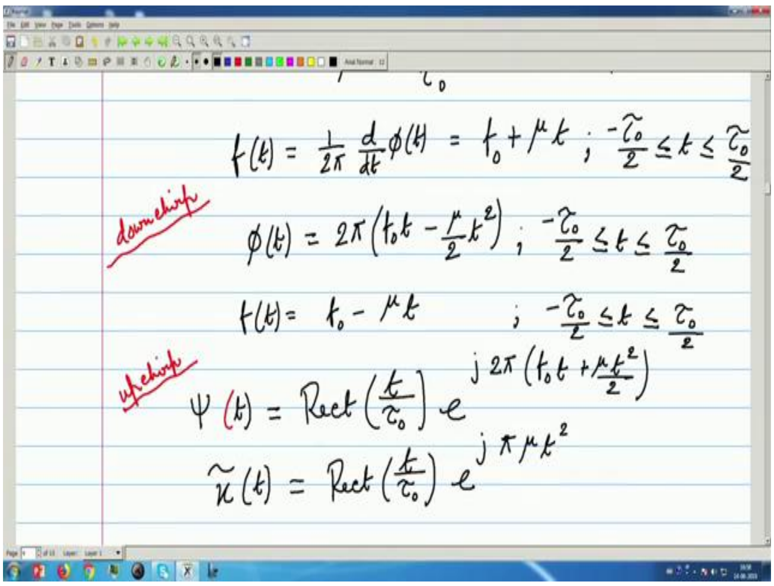

Similarly, I am also writing the down chirp parameters, because both type of things are possible. So, for down chirp from f naught it will go down by minus mu by 2 t square. Again the and f t will be similarly f naught minus mu t.

$$\phi(t) = 2\pi\left(f_0 t - \frac{\mu}{2}t^2\right); \quad -\frac{\tau_0}{2} \leq t \leq \frac{\tau_0}{2}$$

$$f(t) = f_0 - \mu t \quad ; \quad -\frac{\tau_0}{2} \leq t \leq \frac{\tau_0}{2}$$

So, we can write what is the pre-envelope; again let us come to up chirp. If a signal has a phase like this what is it is pre-envelop.

So, that pre-envelope I can write as rectangular t by tau naught e to the power j 2 pi f naught t plus mu t square by 2. So, if I know this I can easily write what is the analytic signal, complex envelop. So, that will be rectangular t by tau naught e to the power j pi mu t square.

$$\psi(t) = \mathrm{Rect}\left(\frac{t}{\tau_0}\right) e^{j2\pi\left(f_0 t + \frac{\mu t^2}{2}\right)}$$

$$\tilde{x}(t) = \mathrm{Rect}\left(\frac{t}{\tau_0}\right) e^{j\pi\mu t^2}$$

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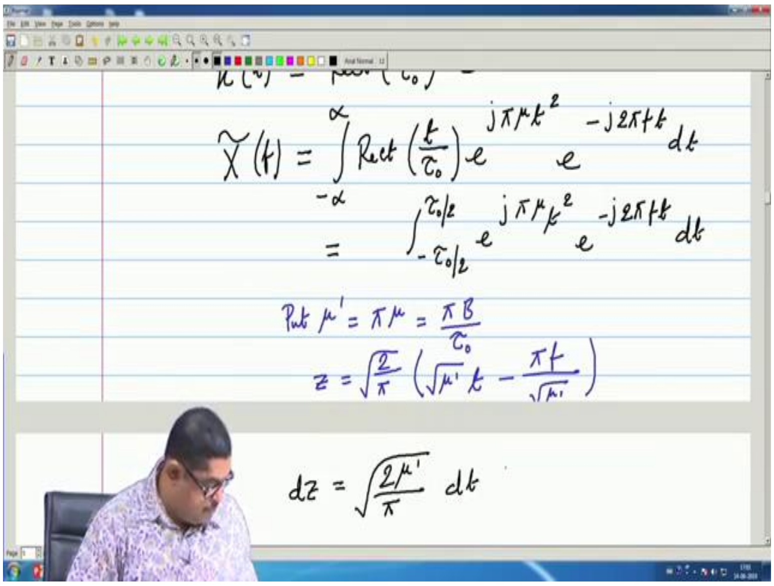

So, once I get this I can find what is the Fourier transform X f x tilde f that will be minus infinity to infinity rectangular t by tau naught e to the power j pi mu tau square e to the power minus j 2 pi f t d t is equal to, now due to presence of this I can take that the thing will be minus tau naught by 2, to tau naught by 2 e to the power j pi mu t square e to the power minus j 2 pi f t d t as simple as that.

$$\tilde{X}(f) = \int_{-\infty}^{\infty} Rect\left(\frac{t}{\tau_0}\right) e^{j\pi\mu t^2} e^{-j2\pi f t} dt$$
$$= \int_{-\tau_0/2}^{\tau_0/2} e^{j\pi\mu t^2} e^{-j2\pi f t} dt$$

Now, we require some putting some variable. So, I can say put mu dash is equal to pi into mu; that means, pi B by tau naught and also put z is equal to root over 2 by pi root over mu dashed t minus pi f by root over mu dashed. So, if you do this, then d z will become root over 2 mu dashed by pi d t.

$$\text{Put } \mu' = \pi\mu = \frac{\pi B}{\tau_0}$$
$$z = \sqrt{\frac{2}{\pi}}\left(\sqrt{\mu'}\, t - \frac{\pi f}{\sqrt{\mu'}}\right)$$
$$dz = \sqrt{\frac{2\mu'}{\pi}}\, dt$$

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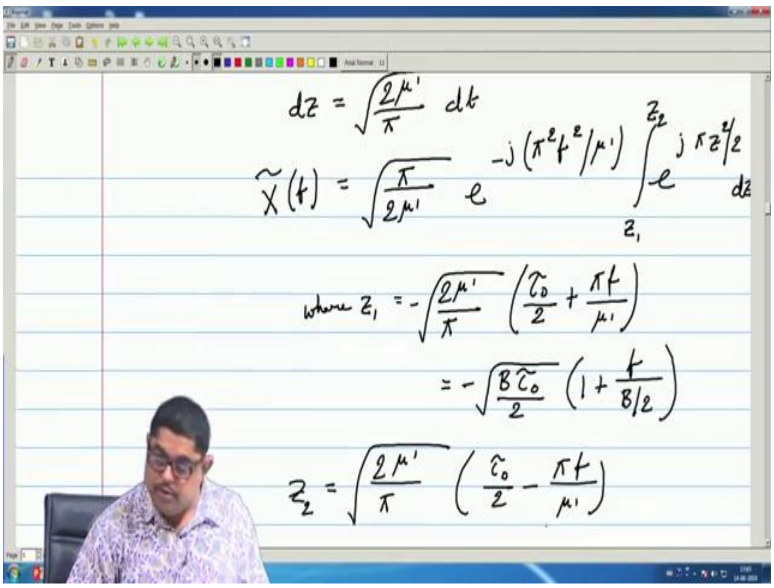

And putting this, you can get the X f that will be root over pi by 2 mu dashed e to the power minus j pi square f square by mu dashed, then z 1 to z 2 e to the power j pi z square by 2 d z.

$$\tilde{X}(f) = \sqrt{\frac{\pi}{2\mu'}}\, e^{-j(\pi^2 f^2/\mu')} \int_{z_1}^{z_2} e^{j\pi z^2/2}\, dz$$

Where these two limits we need to specify z 1 is minus root over 2 mu dashed by pi tau naught by 2 plus pi f by mu dash is equal to minus; if you put this mu dash value it comes out to be B tau naught by 2 1 plus f by B by 2.

$$\text{where } z_1 = -\sqrt{\frac{2\mu'}{\pi}}\left(\frac{\tau_0}{2} + \frac{\pi f}{\mu'}\right)$$
$$= -\sqrt{\frac{B\tau_0}{2}}\left(1 + \frac{f}{B/2}\right)$$

And z 2 will be 2 mu dash by pi tau naught by 2 minus pi f by mu dashed.

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$$= -\sqrt{\frac{B\tau_0}{2}}\left(1 + \frac{f}{B/2}\right)$$
$$z_2 = \sqrt{\frac{2\mu'}{\pi}}\left(\frac{\tau_0}{2} - \frac{\pi f}{\mu'}\right)$$
$$= \sqrt{\frac{B\tau_0}{2}}\left(1 - \frac{f}{B/2}\right)$$
$$\tilde{X}(f) = \sqrt{\frac{\pi}{2\mu'}}\, e^{-j(\pi f^2)/\mu'}\left[\int_0^{z_2} e^{j\frac{\pi z^2}{2}}\, dz - \int_0^{-z_1} e^{j\frac{\pi z^2}{2}}\, dz\right]$$

Which if you put the value, this will be root over B tau naught by 2 1 minus f by B by 2.

$$z_2 = \sqrt{\frac{2\mu'}{\pi}}\left(\frac{\tau_0}{2} - \frac{\pi f}{\mu'}\right)$$

$$= \sqrt{\frac{B\tau_0}{2}}\left(1 - \frac{f}{B/2}\right)$$

So, the this integration now can be written like this, root over pi by 2 mu dashed e to the power minus j pi f square by mu dashed; then that integration can be easily definite integral can be written like this e to the power j pi z square by 2 d z minus 0 to minus z 1 e to the power j pi z square by 2 d z.

$$\tilde{X}(f) = \sqrt{\frac{\pi}{2\mu'}}\, e^{-j(\pi f^2)/\mu'}\left[\int_0^{z_2} e^{j\frac{\pi z^2}{2}}\, dz - \int_0^{-z_1} e^{j\frac{\pi z^2}{2}}\, dz\right]$$

Now this integral e to the power j pi z square by 2 d z that is a known integral, it also come in case of antenna. If you want to find the field from antenna for assumed a sinusoidal current distribution you get this type of thing these are called Fresnel integrals. They are given in books, they are tabulated, they are well known integrals.

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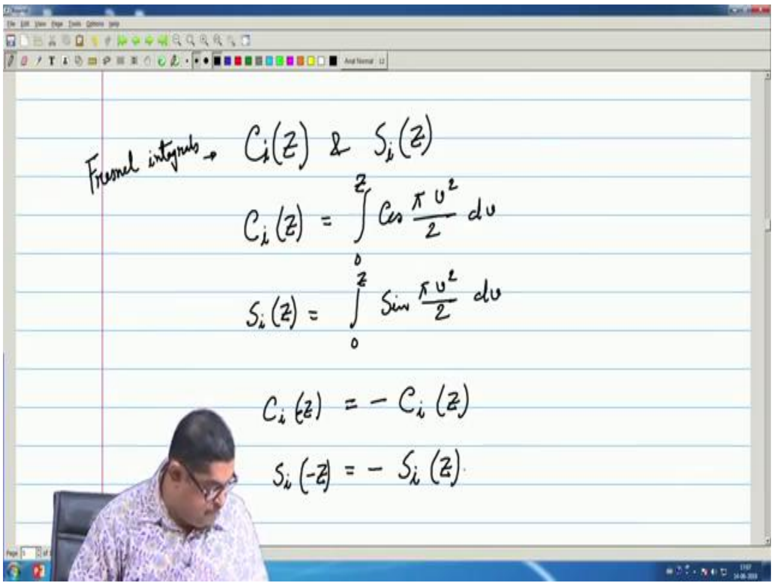

So, the Fresnel integrals are this C z and S z C for cosine and S for sine they are Fresnel integrals. So, they are defined C i, we should call them C i and S i z integral. So, C i z they are 0 to z cos pi v square by 2 d v and 0 to z sin pi v square by 2 d v.

$C_i(z)$ & $S_i(z)$

$$C_i(z) = \int_0^z \cos\frac{\pi v^2}{2}\,dv$$

$$S_i(z) = \int_0^z \sin\frac{\pi v^2}{2}\,dv$$

And also they have a nice property that C i z or C i of minus z is equal to minus C i z and S i minus z that is minus S i z.

$$C_i(-z) = -C_i(z)$$

$$S_i(-z) = -S_i(z)$$

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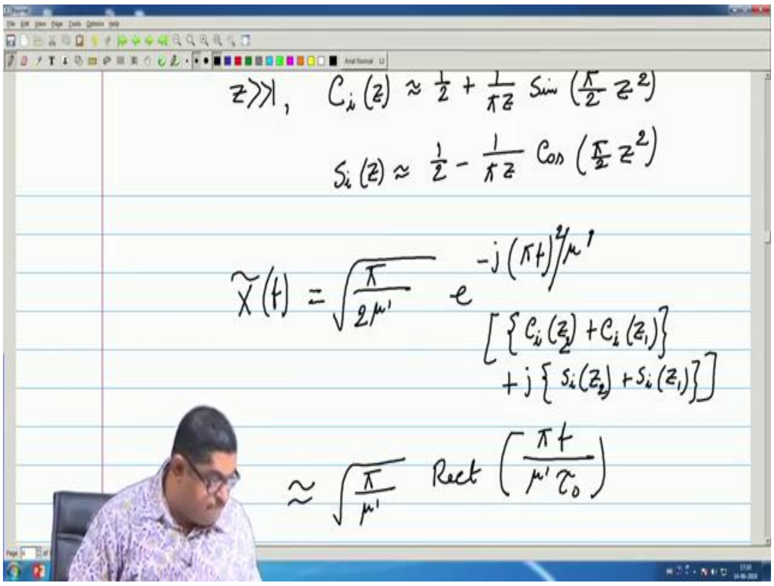

And for z greater than 1, they can be given by some analytic expressions. So, C i z they almost become half plus 1 by pi z sin pi by 2 z square and S i z for larger domains they become half minus 1 by pi z. You can refer to Abramowitz and Stegun a very good book where these things are all given.

$$z \gg 1, \quad C_i(z) \approx \frac{1}{2} + \frac{1}{\pi z} \sin\left(\frac{\pi}{2} z^2\right)$$
$$S_i(z) \approx \frac{1}{2} - \frac{1}{\pi z} \cos\left(\frac{\pi}{2} z^2\right)$$

So, utilizing that actually it is easy to write this Fourier transform of the chirp signal X f as pi by 2 mu dashed e to the power minus j pi f whole square by mu dashed into C i z 2 plus C i z 1 plus j S i z 2 plus S i z 1.

$$\tilde{X}(f) = \sqrt{\frac{\pi}{2\mu'}}\, e^{-j(\pi f)^2/\mu'} \left[\left\{C_i(z_2) + C_i(z_1)\right\} + j\left\{S_i(z_2) + S_i(z_1)\right\}\right]$$

So, if you put these approximations, then these above approximations for the large z that will give it as pi by mu dashed rectangular pi f by mu dashed tau naught.

$$\tilde{X}(f) \approx \sqrt{\frac{\pi}{\mu'}}\, \mathrm{Rect}\left(\frac{\pi f}{\mu' \tau_0}\right)$$
$$= \sqrt{\frac{\tau_0}{B}}\, \mathrm{Rect}\left(\frac{f}{B}\right)$$

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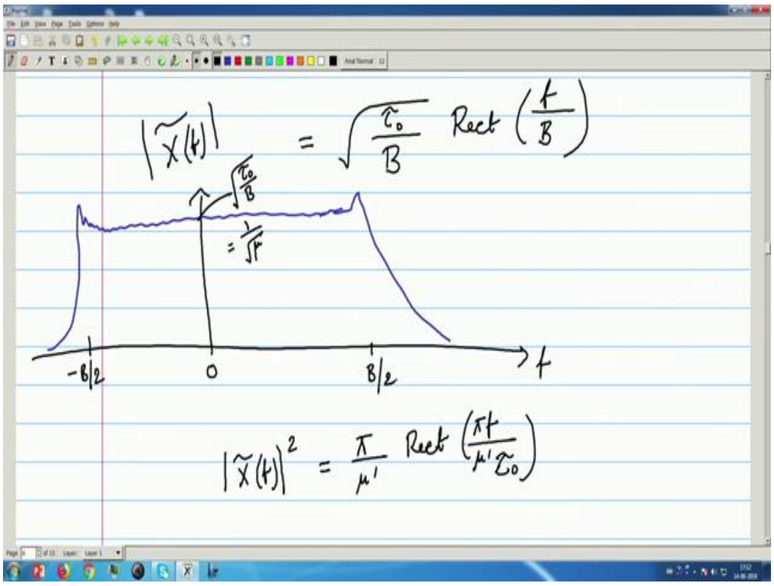

Now, unraveling this value of mu naught this is nothing, but root over tau naught by B rectangular f by B. So, if we plot this magnitude. So, X f magnitude this is 0, this is f, you say it is a rectangular function from minus B by 2 to plus B by 2. So, actually this becomes peaky here.

So, this value is root over tau naught by B or you can say this is 1 by root mu. So, at last we got a spectrum which is fully a rectangular function, it is a remarkable thing. So, we can find out what is the energy. So, energy will be pi X f square, first let me find pi by rectangular pi f by mu tau naught.

$$|\tilde{X}(f)|^2 = \frac{\pi}{\mu'}\ \mathrm{Rect}\left(\frac{\pi f}{\mu' \tau_0}\right)$$

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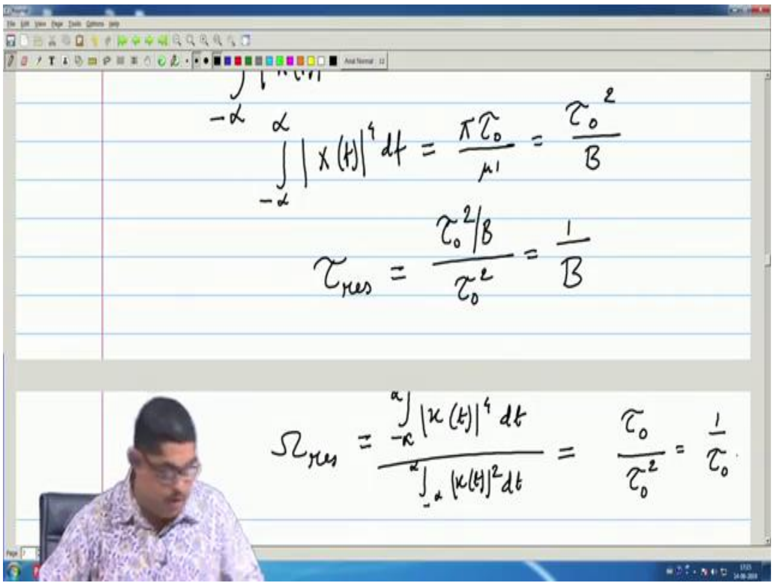

And so, we will have to as before evaluate the energy of this signal, so X f square d f that if we do, it will be a rectangular function, so tau naught. Also you see this thing we can put or these are same because these are we are taking magnitudes. So, I can drop these also. So, that let us continue here. So, that will be pi tau naught by mu dash is equal to tau square by B.

$$\int_{-\alpha}^{\alpha} |x(t)|^2 dt = \tau_0$$

$$\int_{-\alpha}^{\alpha} |x(t)|^4 dt = \frac{\pi \tau_0}{\mu'} = \frac{\tau_0^2}{B}$$

So, what is our delay resolution constant; that will be tau naught square by B by tau naught square. So, it is 1 by B ok.

$$\tau_{res} = \frac{\tau_0^2/B}{\tau_0^2} = \frac{1}{B}$$

And similarly if we find the, this resolution that will be again the, we know minus infinity to x t 4 d t by. So, we have already got that expression, energy is also there, so we can find that it will be tau naught by tau naught square; so it is 1 by tau naught.

$$\Omega_{res} = \frac{\int_{-\infty}^{\infty} |x(t)|^4\, dt}{\int_{-\infty}^{\infty} |x(t)|^2\, dt} = \frac{\tau_0}{\tau_0^2} = \frac{1}{\tau_0}$$

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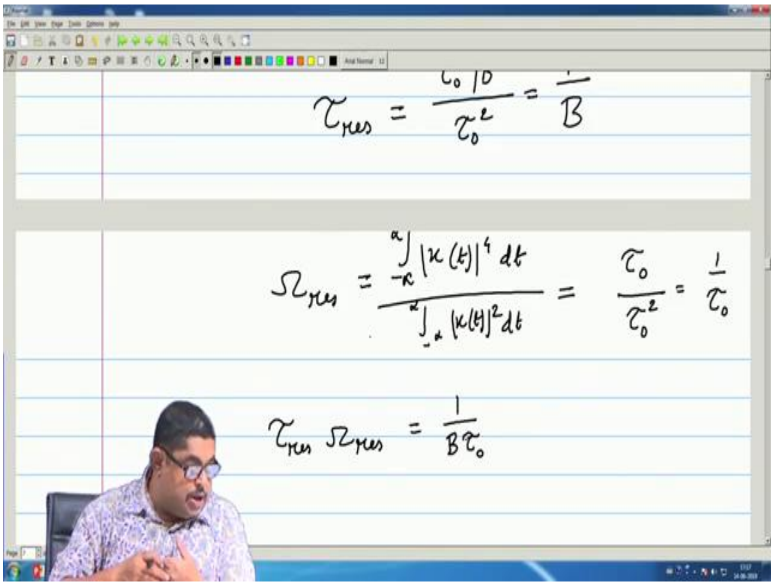

So, you see that this product is now 1 by B tau naught.

$$\tau_{res}\,\Omega_{res} = \frac{1}{B\tau_0}$$

Now what is B and what is tau naught. Let us go back, when we defined our chirp pulses yes, we have B we have tau naught B means how much the modulator can change in this time; this depends on the present day electronics similarly. So, B and t tau naught they are independent things and the product is given by that. So, this thing we can go on increasing and that will change and moreover the two terms, the resolutions they are independent.

So, if I want to make the delay resolution constant lower, I will have to increase the bandwidth. If I want to reduce the value of Doppler frequency constant, I will have to increase the time; the time in which the modulator will sweep. So, this thing is in my hand independently I can do. And so this we can say that at last we got a signal whose match filter that gain, that SNR gain can be made as large as possible by making B tau naught of the signal.

So, what is the effect on that on range resolution and Doppler resolution? So, for that what we will do, in the next class we will we have the expression for the signal. So, we can find the ambiguity function and if I can find the ambiguity function, we can easily find out what is the resolutions, Range solution and delay resolution that will do. And already we have seen the complex envelop of this signal. So, that from that we can easily find the ambiguity function, because we know what is the technique of finding that; so that we will do in the next class, ok.

Thank you.