Principles and Techniques of Modern Radar Systems
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Lecture - 35
Detection in Radar Receiver (Contd.)

Key Concepts: Mathematical expression of correlation, representation of matched filter as correlator, complex envelope analysis of input signal to the matched filter, output of a matched filter for a stationary target

Welcome to this NPTEL lecture on Principles and Techniques of Modern Radar Systems.

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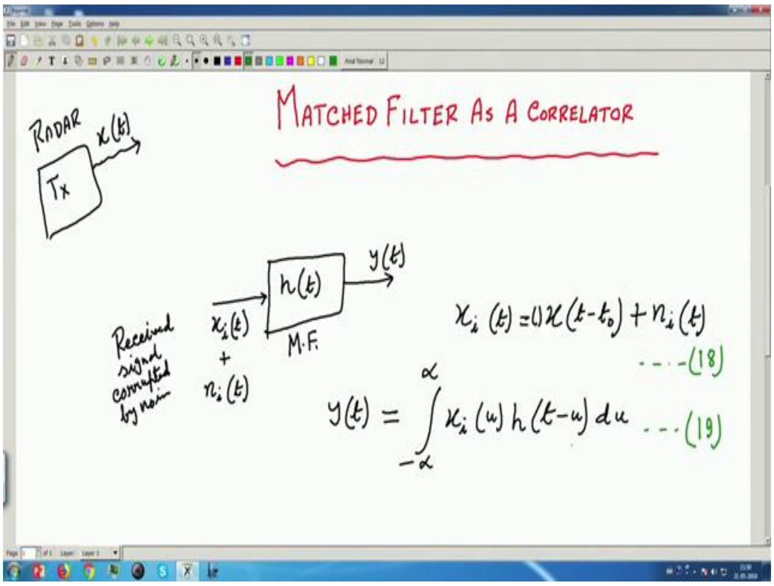

We were discussing match filter in the previous class, we have seen the match filter it output maximizes SNR. Today we will see what else match filter can do; actually today we will see that it is not only maximizes SNR but, its output is nothing but a correlation of the received signal with the transmitted signal. So, that is the beauty and that we will explore today.

So, again we are considering the previous things, that there is a radar, radar is transmitting a band pass signal x t. Now we have received it this delay round trip delay is t naught and so, we are calling the received signal corrupted by noise. So, that we are writing x i t minus t naught, because it is delayed and also n i t is the noise, that has been added to input.

So, if this is the match filter so; obviously, we can write what will be the output? Previous day we have seen match filter and we have seen the match filter output. So, we can definitely write that or let me call let me explicitly define it mathematically what is first, what is x i t? So, x i t is nothing but, x t minus t naught; actually it will be obviously, have an amplitude factor but, also may have a phase factor that for simplicity I am now not taking, so this plus the n i t.

$$x_i(t) = x(t - t_0) + n_i(t)$$

Actually there is something here that I should have, but it is immaterial; if you assume also no problem. Now what will be the output of the match filter y t, that we learnt that this is given by minus infinity to infinity x i u h t minus u d u; where u is a running variable.

$$y(t) = \int_{-\infty}^{\infty} x_i(u)\, h(t-u)\, du$$

So, the output since it is a integration, so, the output will be function of time that is y t ok. So, I can write give the equation names that the last time we ended at equation 17 so, this equation let me call equation 18 and this one is equation 19.

So, now we know the match filter. So, please refer to equation 15 where match filter was there I can see that what was the match filter that was the casual match filter so, if we put that.

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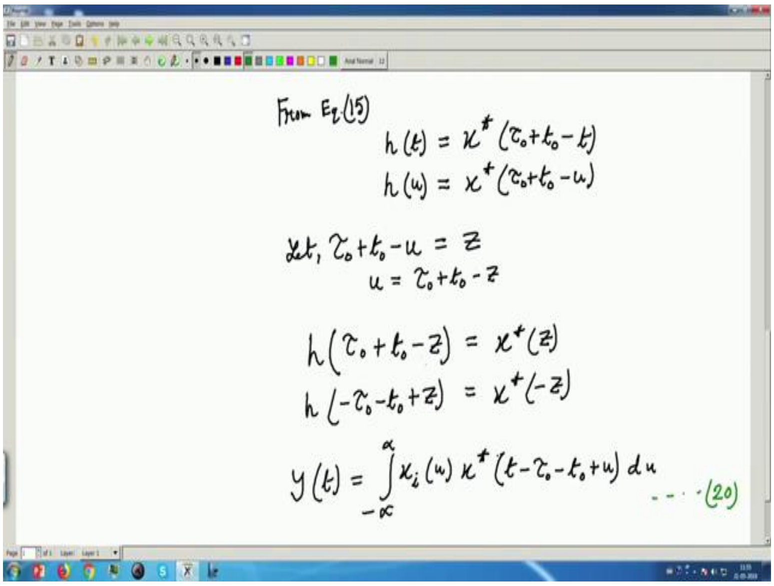

So, from equation 15 we can put that, what was that, that was h t was nothing but x star tau naught plus t naught minus t.

$$h(t) = x^*(\tau_0+t_0-t)$$

I am recalling that tau naught is the roundtrip delay for the signal from transmitter to coming back to the receiver and that is t naught and tau naught is the extra delay added for make the impulse make this match filter causal ok.

So, h t was this, but I need h you see that in case of thing this is h t minus u. So, can I write if I write it as h u this will be x star tau naught plus t naught minus u okay as simple.

$$h(u) = x^*(\tau_0+t_0-u)$$

Then let us make a substitution because, this is tau naught. So, let us make that let tau naught plus t naught plus t I am sorry, this tau naught plus t naught minus u, let us substitute is for understanding otherwise those who can do for them. So, if you do this then we know what is u, u will be, tau naught plus t naught plus z; sorry minus z and then d u will be equal to d z, where d u will be equal to minus d z so, that thing you can do.

$$\text{Let, } \tau_0 + t_0 - u = z$$
$$u = \tau_0 + t_0 - z$$

So, what we can write that h of I will have to finally, make this thing h of tau naught plus t naught minus z that will be x star z, and so h of minus tau naught minus t naught plus z, that will be x star minus z.

$$h(\tau_0 + t_0 - z) = x^*(z)$$
$$h(-\tau_0 - t_0 + z) = x^*(-z)$$

So, putting this that h u so, actually I need h of minus u thing so, that if we do you can easily do this, this is simple math. So, you will see that, now your y t that will be minus infinity to infinity, x i u x star t minus tau naught minus t naught plus u d u ok. So, let us call this equation 20.

$$y(t) = \int_{-\infty}^{\infty} x_i(u)\, x^*(t - \tau_0 - t_0 + u)\, du \quad \text{---- (20)}$$

I think I am clear that, if you just do this substitution it will come in terms of z; then it will be easier for you to do. So, after that again putting the back the value of this you are getting this y t. Now let us recall what is the definition of cross correlation between two complex function.

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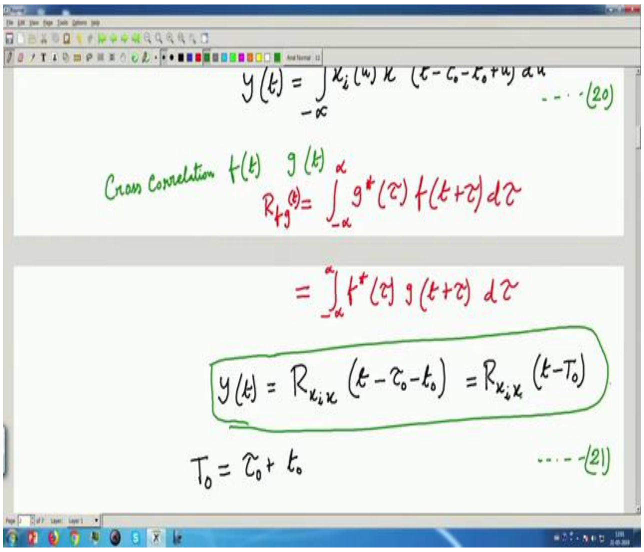

So, cross correlation between two complex function; let one function is f t and other let us say g t. So, what is the cross correlation? This definition I am now writing that cross correlation is generally denoted by this R f g their cross correlation this all of you have done in your basic network class or etcetera. So, this we know is can be written in many ways. So, I can write like this, g star tau now this R f g that is a function of time, so that is why the running variable we are changing g star tau f of t plus tau d tau.

Or if you want conjugating the other, you can also write previously what we wrote. So, this is g conjugation or I can make the conjugation, here also that you can make f star tau g t plus tau as I think this also is true.

$$R_{fg}(t) = \int_{-\alpha}^{\alpha} g^*(\tau)\, f(t+\tau)\, d\tau$$
$$= \int_{-\alpha}^{\alpha} f^*(\tau)\, g(t+\tau)\, d\tau$$

So, comparing these we can now easily understand that, what is our y t? So, y t in terms of these I can write that it is cross correlation between x i and x you see. So, and what is a variable, that t minus tau naught, minus t naught.

So, this now we have previously seen that, we can combine these two and give that time we have given a name. Actually if I put that because, t naught is the delay and tau naught is the causal thing and t naught is the round trip time delay. So, let us give it a name capital T naught. So, if I substitute this, then this can be written as R the cross correlation between the receive signal and the transmitted signal and this can be written as t minus T naught.

$$y(t) = R_{x_i x}(t - \tau_0 - t_0) = R_{x_i x}(t - T_0)$$

That means, I can say that what is match filter output? So, let us first give it a number because I think now is the time for giving it a value 21. So, what equation 21 is very important, equation 21 says that let me show the what is equation 21. That it says that match filter output is the cross correlation between the received echo signal x i and a delayed replica of the transmitted signal because, this is delayed by how much capital T naught. So, a delayed replica and these so, this is an important thing that is why this topic name is that match filter output is nothing but a correlator. What it correlates? The received signal it correlates with the transmitted signal delayed by t naught.

So, it is great that it is basically this correlation. So, if you really have that you have got a received echo from a transmitted signal then the correlation value will be high so, this is a now. Now let us with this knowledge, let us put that what happens when the range we know the range information of a closing target. So, we will do that, so that what is the range of a closing target, we know it is given by R t range and that is we know R naught minus v into so, do not think is a function that is why I am writing into t minus t naught.

$$R(t) = R_0 - v \times (t - t_0)$$

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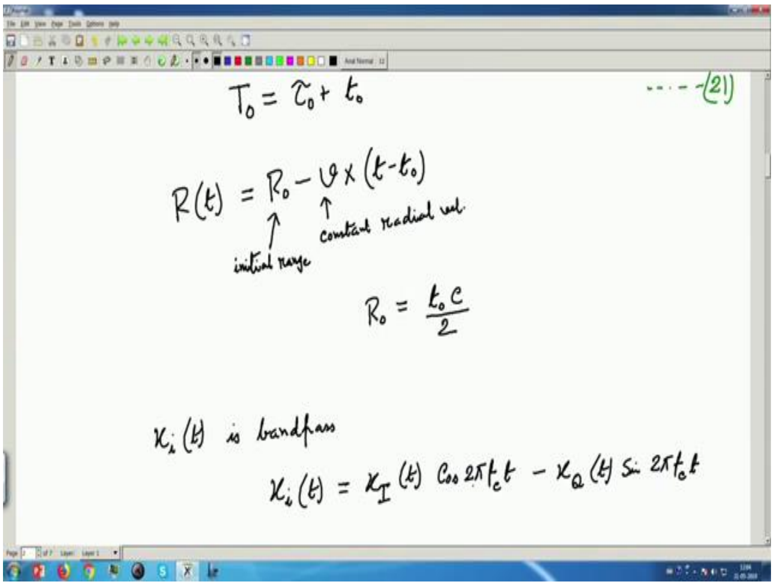

So, where v is the constant radial velocity constant, we are assuming that the target has a constant radial velocity and what is R naught? That time we had said, again I am writing it is nothing, but the initial range. So, a moving target that then we will have this range at any as a function of time. And we know also what is the expression of R naught from our if the target moves this that what is R naught? R naught is nothing but, t naught c by 2 t naught c by 2.

$$R_0 = \frac{t_0 c}{2}$$

So, that we also know because it is the half delay so, you see that what is the range, we know these.

Now, actually the radar echo signal that is band pass signal. So, x i t is band pass signal, transmitted signal was also band pass. So, band pass signal we know actually we will when analyzing we express it in terms of complex envelopes so, that will do that will simplify our thing. So, we can write that x i t is what? It will be, if suppose we take it as a in phase and quadrature because, some receivers works on that principle I-Q receivers. So, in that case it will be x I t cos 2 pi f c t minus, x Q t sin 2 pi f c t.

$$x_i(t) = x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t$$

Instead of writing e to the power, I am writing like this that this is x i t and in terms of this I can write.

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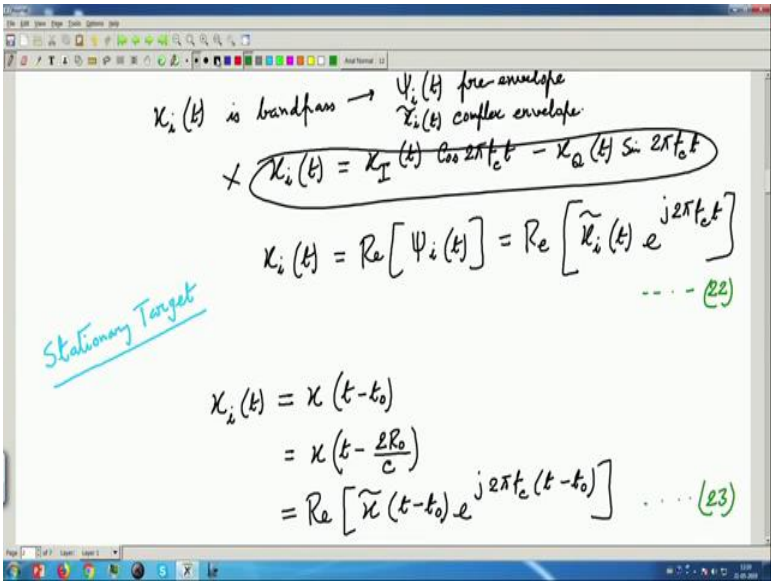

And so, its let us say that x i t is band pass so, if I say that psi i t is its pre envelope. All of you are familiar with this that we express pre-envelope concept and the real part of that actually is the complex envelope which is a low pass filter. So, and let us say that complex envelope of this, so complex envelope we generally put a tilde over the thing to a say that it is a complex envelop like phasor type of thing.

So, x i t let us say is the complex envelope. So, what is the relation, instead of writing this I can also write x i t as a in terms of complex envelop. That is nothing but, real part of psi i t and what is psi i t? This is just for recall that is why I am not giving any equation number here it is all of you can see these in your communication things. Haykin; Simon Haykin is a good book where these concepts are quite well written.

So, e to the power j 2 pi f c t.

$$x_i(t) = Re\left[\psi_i(t)\right] = Re\left[\tilde{x}_i(t)\, e^{j2\pi f_c t}\right]$$

Actually the beauty of complex envelop is that this is the complex envelope and x i t any time you can give but, you can see that the complex envelope is free of the carrier because, the carrier is not carrying the information. I mean it is carrying the information, but information is not here information is either in the phase or in the amplitude of the carrier. So, otherwise carrier is not useful for us. So, complex envelop is carrier less that is why it becomes a low pass signal. So, if we analyze complex envelope, we get all the information of the signal.

So, this is the means of throwing away the carrier that is why it is also this you see that carrier and the complex envelop, they are multiplication and they are separately written. Where as if I write like this you see carrier and the other parts signal part that is not separate that is why this is a better representation than this one. So, that is why we will not do this but this is a thing.

Now, let us specialize first I will see a stationary target we will develop the mathematics then will go to the moving target. That means, what I wrote as R t now my range is fixed let us say at stationary target there is a target at a fixed range. So, R naught there it is. So, let me write this that now I will discuss a stationary target.

So, to start with I am having this simplified assumption, later I will remove that assumption. So, stationary target, okay before that just give this thing a name so, let us say this is our equation 22; because you will be heavily using these the actual signal, real signal, and the complex signal of what is their relation in terms of the frequency carrier frequency, okay. So, for stationary target I comeback, so what is the received echo x i t for stationary target, that is nothing, but x t minus t 0 and that we can also write that what is t 0 that we know, t minus 2 R 0 by c ok.

So, now what is the complex in terms of complex envelope I can say that it is real part of x i or instead of x i if I write it directly in terms of a. So, I can say real x t minus t naught, e to the power j 2 pi f c t minus t naught.

$$
\begin{aligned}
x_i(t) &= x(t - t_0) \\
&= x\left(t - \frac{2R_0}{c}\right) \\
&= \mathrm{Re}\left[\tilde{x}(t - t_0)\, e^{j 2\pi f_c (t - t_0)}\right]
\end{aligned}
$$

This I can write by using… So x i t is what real part of, I should have written x i t but, I know that for stationary target x i t is a x t minus t 0 that I am using here. So, this equation we should give a name just this is our equation 23.

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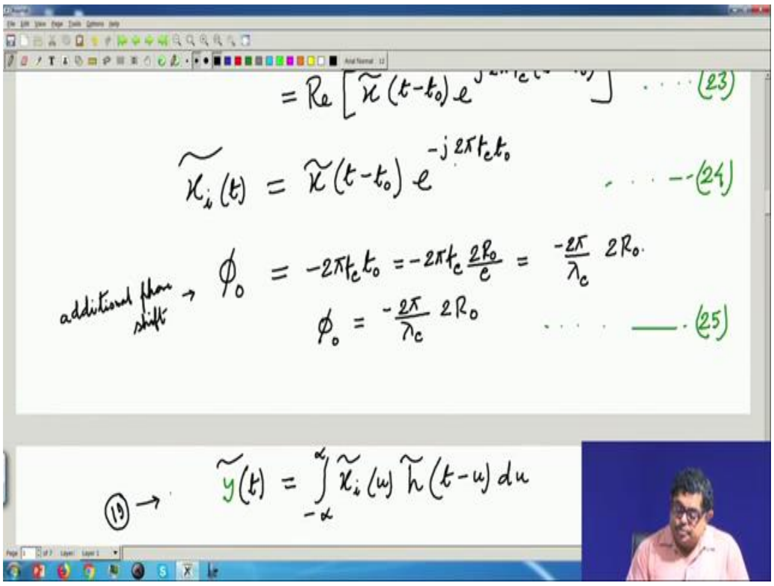

So, we can now say that what is the sorry, complex envelope for the received echo so, I can say that what is the complex; just look at here so, complex envelop of the received echo is what, just looking at here I can say x t minus t 0 e to the power minus j 2 pi f c t naught, just suppress the carrier part.

$$\tilde{x}_i(t) = \tilde{x}(t-t_0)\, e^{-j 2\pi f_c t_0}$$

So, this is important that what is the complex envelop of the received echo? In terms of the transmitted signal complex envelope, this is the relation.

So, this should be given a number 24. So, you can see that received complex echo is more than just the delayed version, of the transmitted complex envelop it has an additional phase shift, how much is the phase shift? Minus 2 pi f c t naught and the beauty is that contains actually the range information that I will show that. So, you see that, received echo signal just not the transmitted signal's delayed version, it also has a phase shift additional phase shift.

So, what is that phase shift? Let us call this phase shift as, what is called the phi naught. So, if I call it additional phase shift then I should write additional phase shift, because the circuits for radar the receiver circuitry they only see the complex envelopes they analyze on the basis of complex envelope. So, what they will do that, phi naught is what from here you can see that because phase is time invariant. So, here it is minus 2 pi f c t naught so, we know the value of t naught. So, what is t naught? I can say 2 R naught for a stationary target remember and that is what that is nothing but, now c by f c that will be lambda naught so or lambda c. So, the carrier lambda and then multiplied by 2 R naught.

So, I can say that, this is another equation that phi naught. So, additional phase shift is how much? It is minus 2 pi by lambda c into 2 R naught.

$$\phi_0 = -2\pi f_c t_0 = -2\pi f_c \frac{2R_0}{c} = -\frac{2\pi}{\lambda_c} 2R_0$$

$$\phi_0 = -\frac{2\pi}{\lambda_c} 2R_0$$

So, this is an again an important relation because, for a stationary target also you see we have an additional phase shift. So, now the question is a very small change in range, suppose the range R naught is changed but, you see that change in range R naught that can give rise to a large change in the phase.

So, that is why this phase term that will be treated as a random variable with uniform pdf in the interval 0 to 2 pi. This we have seen many times in the a that if the range is small then also this phase can be significant and you know that due to modulo 2 pi nature, so its becomes a random variable. So, this is the thing that phi naught is random and generally it is modeled as a uniform thing. Also, this phase needs to be unwrapped; that means, modulo two things should be understood and that is called unwrapping ok

But, let us continue our discussion that we are actually trying to find out that match filter output how what are the phases of that output, what is the magnitude part of that output that we are trying to find out for a stationary target. So, we can refer to equation 19 and we can put these values because, now I know the received echo. I also know that what was our equation 19; you can see that in equation 19 we have seen that what is the match filter output? If you give an input then for a match filter what will be the output so, that we have done.

So, I can say that let us go to the next page that, match filter output is always y t is equal to minus infinity to infinity; x i u now I will so, this is also the complex envelope of the match filter output. So, that is h t minus u d u. So, this is from 19 equation 19 gives us this.

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{x}_i(u)\,\tilde{h}(t-u)\,du$$

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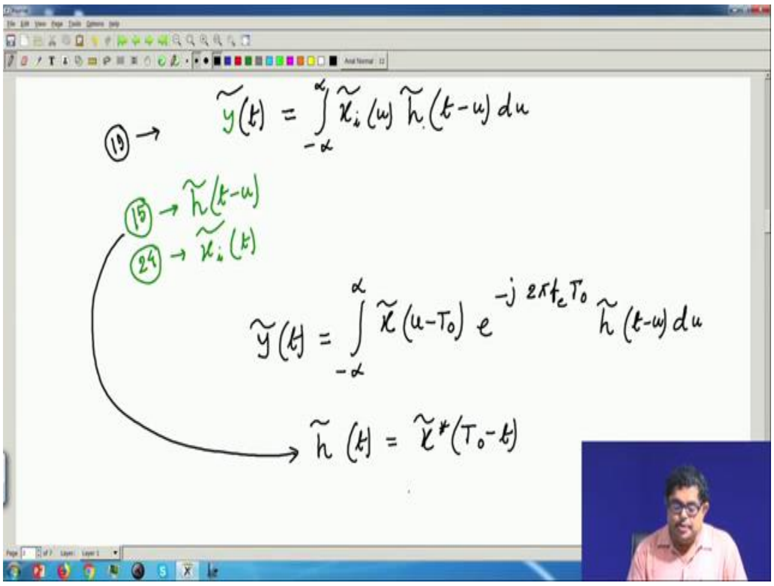

So, now here we can put that, we have the definition of match filter; that means, what is this in terms of the signal, that is given by equation 15. So, you can put equation 15, from equation 15 we will get the value of this h t minus u and from just equation 24, you can see equation 24 gives us x i. So, I will write that from 15 you can get this value from 24 you can get the value of x i its complex envelope value.

So, if we put that here, we get what that so, now, everything is complex envelope. So, that is why I am redoing everything. So, it will be minus infinity to infinity, then x u minus T 0 e to the power minus j 2 pi f c T 0 h; still I have not put the value. So, h t minus u d u.

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{x}(u-T_0)\, e^{-j 2\pi f_c T_0}\, \tilde{h}(t-u)\, du$$

But in 15 we have that value that we will have to do and then we get something but, h we know what is h? We know that h of t that we know that is this is the definition of the match filter.

$$\tilde{h}(t) = \tilde{x}^*(T_0 - t)$$

That means, equation 15 tells us this; but, we will have to find h t minus u. So, that we will do, but time is up. So, that will take up in the next class.

Thank you.