Principles and Techniques of Modern Radar Systems
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Lecture - 34
Detection in Radar Receiver (Contd.)

Key Concepts: Derivation of impulse response of the matched filter for AWGN, Determination of maximum instantaneous SNR obtained from Matched filter

Welcome to this NPTEL lecture, on Principles and Techniques of Modern Radar System. So, we were discussing in the last class match filter, we have found an expression for match filter, we will continue more with that and we will find out exactly what is match filter why it is called match filter because. So, what I do, but before going there, I do some two more things so that later we can come back there. What I do this equation, equation 9 what I derived last time, now, that we can multiply both sides by something.

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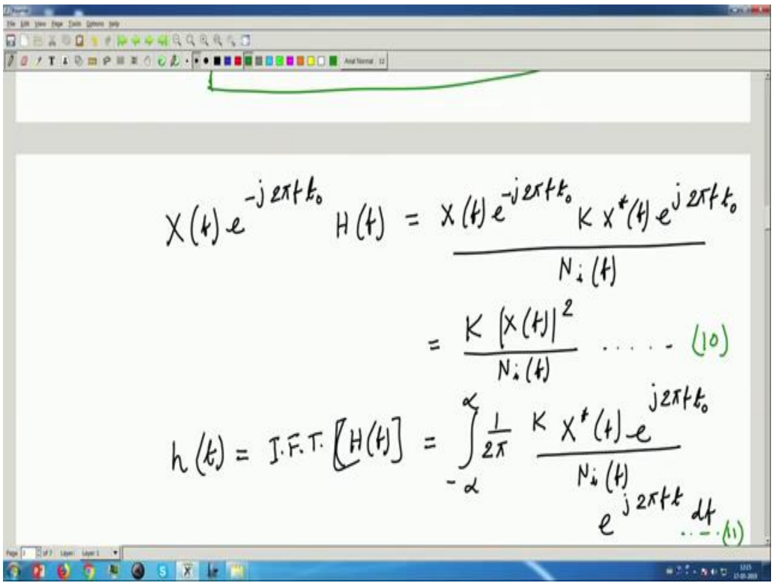

So, I will multiply by sorry, multiply by X f e to the power minus j 2 pi f t 0 both sides I will multiply. So, this side will be H f and that side will be. First let me put the multiplied thing that X f e to the power minus j 2 pi f t naught then, K X star f e to the power j 2 pi f t 0 by N i f. So, you know this side becomes nothing but, K X f square by N i f.

$$X(f)\,e^{-j2\pi f t_0}\,H(f) = \frac{X(f)\,e^{-j2\pi f t_0}\,K\,X^*(f)\,e^{j2\pi f t_0}}{N_i(f)}$$
$$= \frac{K\,|X(f)|^2}{N_i(f)}$$

So, this equation I am calling, what was the number I think it will be number 10 and you see, that from here I can easily again write an alternate expression or of H f because, H f again can be expressed in this form actually equation 9 and equation 10 they are two alternative presentation. So, we will be using that later that is why I am derive that otherwise there is nothing new in that and we can find from equation 9 what is the or from this equation that what from equation 9th we know the transfer function. So, we can also find the impulse response.

So, I can say that what is the impulse response of the match filter; again another very a thing that is. So, inverse Fourier transform of H f. So, I can say that inverse Fourier transform of H f and so, this is nothing but, minus infinity to plus infinity 1 by 2 pi H f into e to the power plus j 2 pi f t d f. So, now, if I put the value that K sorry 2 pi is already I have written. So, I will that K into X star f e to the power j 2 pi f t 0 divided by N i f. Then, e to the power j 2 pi f t d f.

$$h(t) = \text{I.F.T.}\left[H(f)\right] = \int_{-\infty}^{\infty} \frac{1}{2\pi}\,\frac{K\,X^*(f)\,e^{j2\pi f t_0}}{N_i(f)}\,e^{j2\pi f t}\,df$$

So, this I am calling equation number 11. So, this is the expression for the impulse response of the thing. Now this is for completeness sake we have done that, but actually generally we work in the Fourier domain. So, equation 9 is more important than this ok. Now our time is so, we have seen again let us look at 9 actually various instances we will have to refer back to this equation 9. So, what it says that the transfer function of the match filter is dependent both on the signal X t and the noise. So, we can now specialize this noise to a white noise case.

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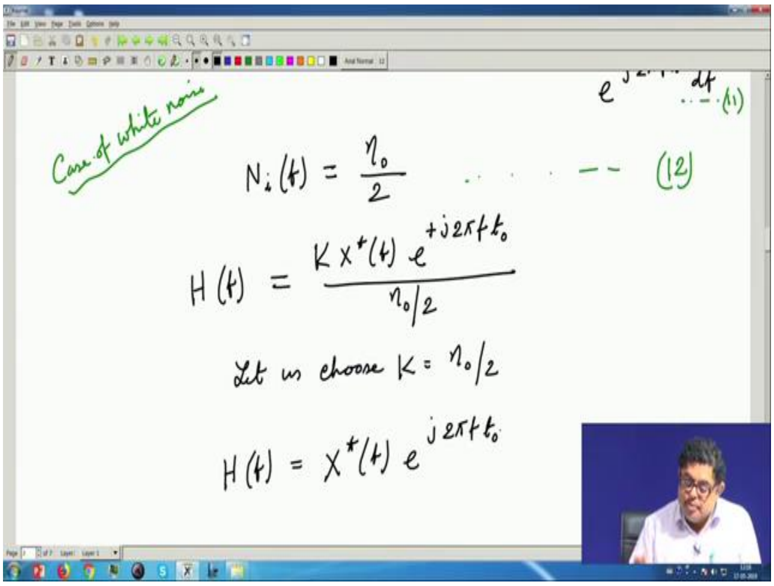

So, case of white noise so, we are specializing that N i f. So, if I specialize then we know that for white noise we can write what is the value of N i f? That is the noise spectral density we know white noise is flat for all frequencies. So, it is given by if we have the double sided spectral then it is eta naught by 2.

$$N_i(f) = \frac{\eta_0}{2}$$

So, with this we can so, this also let me a because this is an important thing. So, and then what will be for white noise case what is the match filter; it is K X star f e to the power plus j 2 pi f t 0 by eta naught by 2 ok.

$$H(f) = \frac{K X^*(f) e^{+j2\pi f t_0}}{\eta_0/2}$$

So, now you see K is an arbitrary constant. So, if we set K is equal to eta naught by 2 because, eta naught by 2 is a known thing. So, if I measure noise we can easily find what is eta naught. So, our next thing is let us choose K is equal to eta naught by 2 the moment we do that we get the value of H f is X star f e to the power j 2 pi f t naught. Okay.

$$H(f) = X^*(f) e^{j2\pi f t_0}$$

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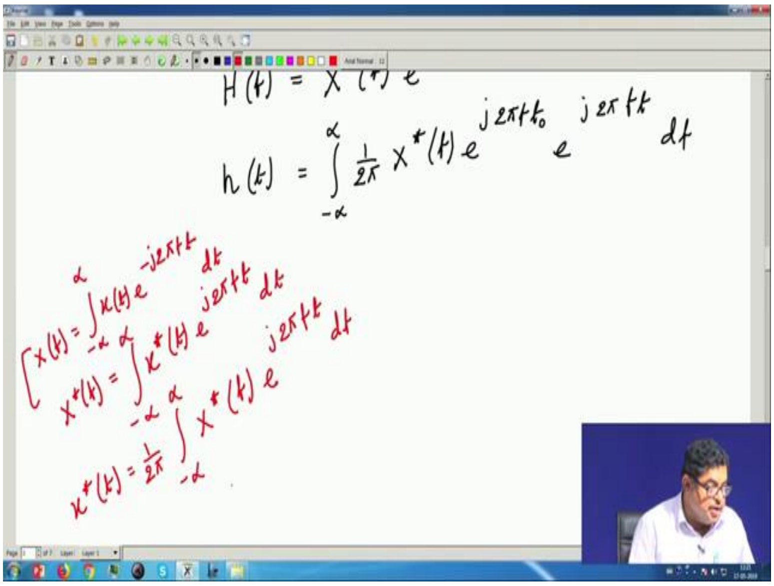

So, what is the impulse response for this; so, h t will be minus infinity to infinity 1 by 2 pi then H f; that means, X star f e to the power j 2 pi f t naught then e to the power j 2 pi f t d f; this we know. Now what is this?

$$h(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X^*(f)\, e^{j2\pi f t_0}\, e^{j2\pi f t}\, df$$

So, let us to understand that, actually I am making you come here that because there is X star. So, let us understand what is this we know that X f or let me use some other color so that, you will get it clear. So, we know that what is X f; X f is minus infinity to infinity x t e to the power minus j 2 pi f t d t this we know. So, what is X star f; X star f is the conjugate of that spectrum, so that will be x star t e to the power j 2 pi f t d t. So, this is from X f, X f is a complex number its conjugate is these two should be conjugated.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$X^*(f) = \int_{-\infty}^{\infty} x^*(t)\, e^{j2\pi f t}\, dt$$

Now what is x star t; it is inverse transform of these so, and what is that; 1 by 2 pi minus infinity to infinity, I can write X star f e to the power j 2 pi f t d f.

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(f)\, e^{j2\pi f t}\, df$$

This we know from the basic thing and then we can put the value of X star f 1 by 2 pi minus infinity to infinity.

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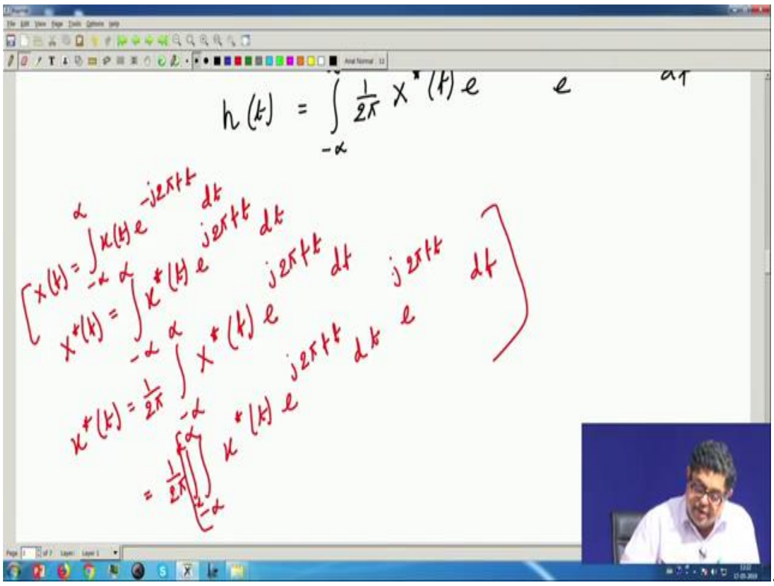

So, X star f is x star t e to the power j 2 pi f t d t and then you see e to the power. So, there will be another one e to the power, j 2 pi f t then I will have a d f or without going there let me take another shortcut because it already came to our that thing. So, x star t I have written like this.

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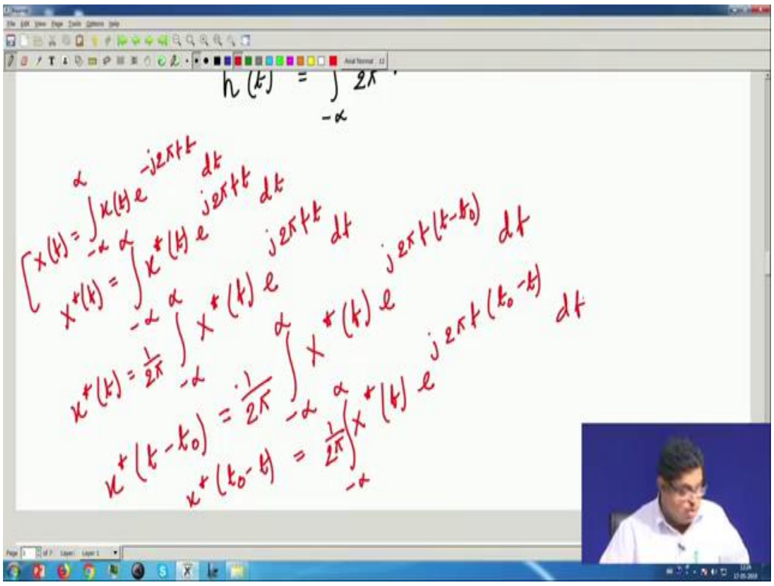

And what is x star t minus t naught that will be 1 by 2 pi minus infinity to infinity X star f e to the power j 2 pi f t minus t naught d f.

$$x^*(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(f)\, e^{j2\pi f(t-t_0)}\, df$$

Now if I show you this can you tell me this that, this and this they are same you see that this t minus t naught. So, can I now say that this is x t minus t naught but if I write what is x star t naught minus t then, that will be 1 by 2 pi minus infinity to infinity, X star f e to the power j 2 pi f t naught minus t d f which is nothing but our this thing if you look carefully.

$$x^*(t_0-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(f)\, e^{j2\pi f(t_0-t)}\, df$$

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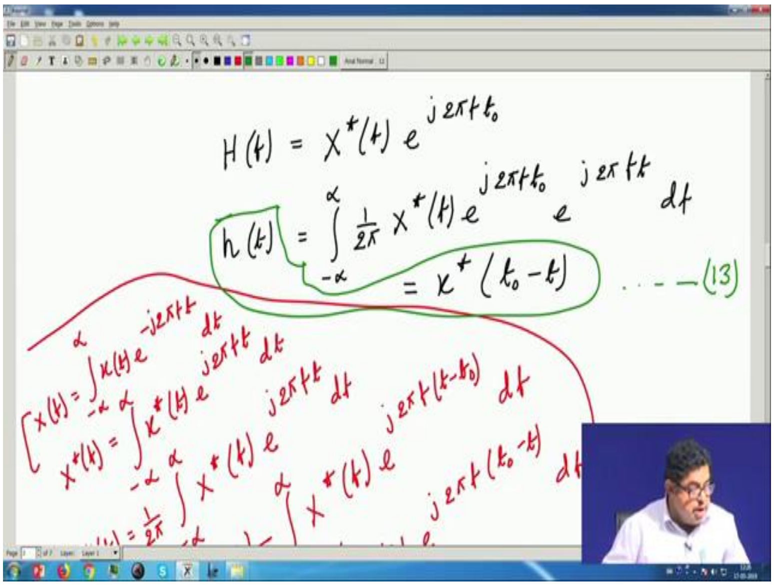

So, that thing I am now in that I can now write that this is nothing but, x star t naught minus t.

$$h(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X^{*}(f) e^{j 2\pi f t_0} e^{j 2\pi f t} df$$
$$= x^{*}(t_0 - t)$$

So, what is the impulse response of the match filter; impulse response of the match filter is related to the transmitted signal x t, but; obviously, the x that signal is inverted in time and then given a delay of t naught, but it is that is why. So, this equation is again important this h t; this I will call our equation number 13.

So, this equation tells us that h t is match to x t. So, that is why the name match filter so; that means, this filter is not a thing if you change the signal, the signal waveform transmitted signal waveform the filter impulse response will also change. That means, we will have to change the implementation of the filter ok, so that is why it is called a match filter equation 13 now clearly says that it is a match filter provided it has this expression provided, you assume white Gaussian noise; white noise Gaussian is not important, but white noise; that means, for a flat spectrum thing it is this match filter.

Now, we can find out because our main thing was what was the SNR? So, under this white noise we can now find out what is the maximum SNR. So, already we have found that thing.

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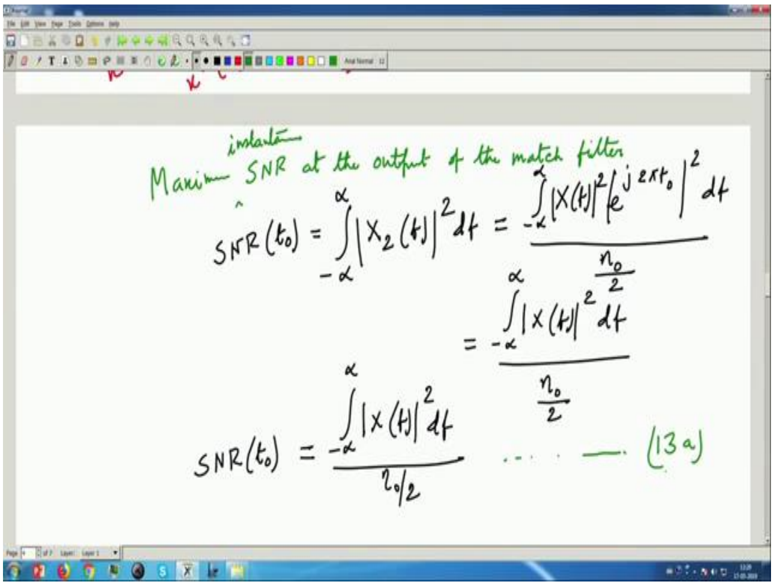

So, I can write that maximum SNR, maximum instantaneous SNR maximum instantaneous SNR at the output of the match filter. We know from the Schwarz inequality that time we have found what is the thing and that if I oh, sorry is that I can call that SNR at the time t is equal to t 0, that will be given by this because, we have seen from the Schwarz inequality when the maximum of that is X 2 f square. If you remember that I am again using the terminology; X 2 f square d f and we know what is our X 2 f.

If you see those we can easily write that this is nothing but X f square X f modulus square, then e to the power j 2 pi f t 0 that also square d f and this side will be that N i f which is eta naught by 2. So, this is you see this exponential term its modulus is 1 so, 1 square. So, now, it is nothing but minus infinity to infinity, X f square d f by eta naught by 2. Okay.

$$SNR(t_0) = \int_{-\infty}^{\infty} |X_2(t)|^2 dt = \frac{\int_{-\infty}^{\infty} |X(t)|^2 |e^{j2\pi t_0}|^2 dt}{\frac{\eta_0}{2}}$$

$$SNR(t_0) = \frac{\int_{-\infty}^{\infty} |X(t)|^2 dt}{\frac{\eta_0}{2}}$$

So, this is that I am again writing this equation this is also a very important equation that at time t is equal to t 0 what is the SNR instantaneous SNR that is given by. Same thing I am writing, because I will give it a equation number eta naught by 2.

So, let me give it the number is let us say 13 a because 13 gives us the impulse response this shows us the SNR at that time. Now we also know that what is the numerator of these 13 a. So, this is you see that X f is the Fourier spectral. So, can I say that, this is the signal energy numerator; so, X f square.

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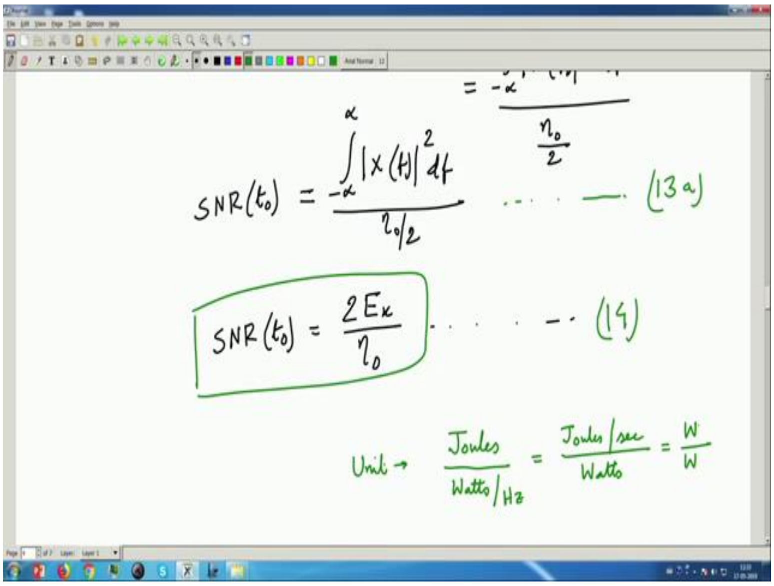

So, I can also write that SNR t 0 is this is if I call E x is the signal energy. So, then I can say 2 E x by eta naught.

$$SNR(t_0) = \frac{2E_x}{\eta_0}$$

So, this is again another very useful result; let me call it equation 14 and make it this. So, we know the power spectral density of noise two sided spectrum.

So, 2 E x by eta naught, now this is a important thing; you see compare the equation 13 what is equation 13; the match filter impulse response depends on the shape of the signal for white noise, it does not depend on anything else and what equation 14 says equation 14 says the SNR is at t is equal to 0 this instantaneous SNR at the output of the match filter is depends on the power spectral density of noise and also the signal's energy, it does not depend on signal shape. So, provided I have two signals with two different signal waveforms but with same energy the SNR will be same but match filter implementation h t will be different.

But match filter output that SNR of the SNR at the match filter output that does not depend on the shape of the signal; that depends only on the energy content of the signal. So, this is a very useful information also we need to check this equation dimensionally because, this equation actually will be using always that SNR is this. So, is it correct; let us see SNR we know it is a dimensionless quantity, but what is this E x. So, energy, so what is energy? Energy is unit. So, numerator I will say it is Joules and what is eta naught it is a power spectral density.

So, power spectral density is Watts per Hertz. So, can I say that Watts per Hertz means what; actually; that means, I can say it is the second, so this is Joules into second by Watts. So, Joules into second I sorry this was I have made a mistake here, that this was Watts per Hertz. So, if I go here it will be Joules per second by Watts Joules per second is again Watts so, Watts by Watts so, dimensionless. So, the left side is dimensionless right side also we show to be dimensionless.

$$\text{Unit} \rightarrow \quad \frac{\text{Joules}}{\text{Watts}/\text{Hz}} = \frac{\text{Joules}/\text{sec}}{\text{Watts}} = \frac{W}{W}$$

So, this now but one thing you see this SNR is actually varying and it is an instantaneous SNR. So, equation 14 also states you should remember this t naught that only if at the proper instant of time the time is given by t is equal to t naught what is t naught? That delay of the received signal compared to the transmitted signal; if you sample only that time that will be maximized and a; so, it also depends on the timing.

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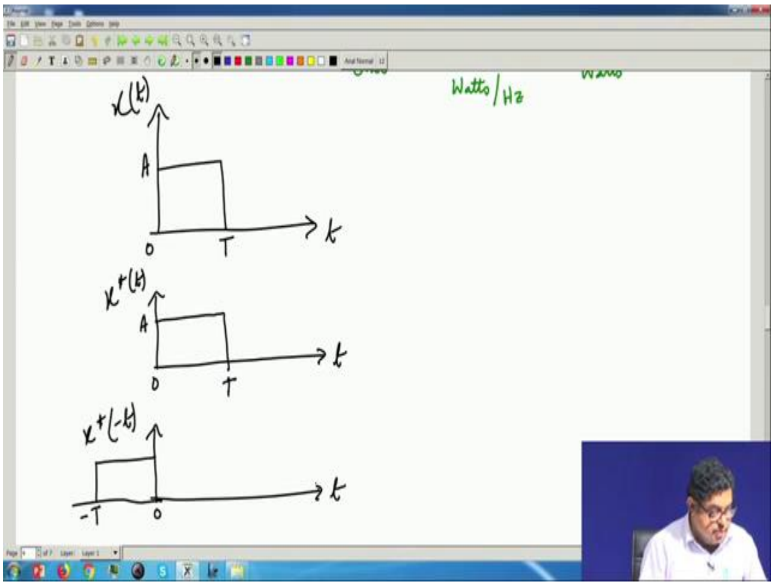

Now let us see the implication of all these equation 13 and 14 that; let us say I have a pulse, in pulse radar we send this. So, let us say that, I have a x t like this x t versus t. So, I have this 0 this is amplitude and let us say duration t it is a repetitive thing but let us see one pulse ok. So, what will be x star t? That will be same as this because x t is a real signal so, its complex conjugate is also this.

Now, what is x star minus t? That will be minus t means with here on the on this y about this y axis I will have to fold it. So, it will be so, if this is 0 then it is minus t ok.

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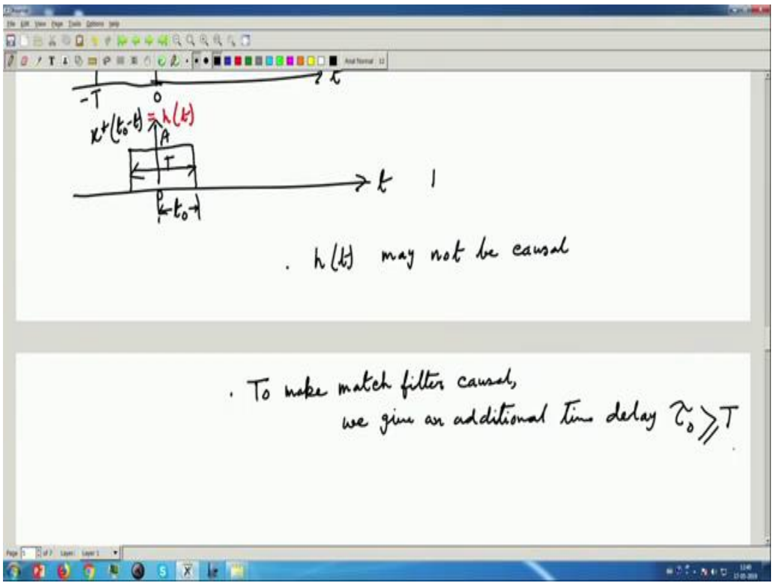

And what will be x star t naught minus t; that will be, that depending on the value of t naught I may have something this. So, what is t naught I can say that you see that time the end came here down this so, this is the value of t naught and what is this value still this value is capital T this amplitude is A in both the case but equation 13 equation 13 told me that this is also I can write that this is nothing but h t.

So, this is the impulse response. Now this is the if I have a pulse transmitted as x t then my impulse response I will have to make like this, but this has a problem, you see that if a impulse response is like this we call it non-causal because here the 0 is here. So, the transmitted signal was transmitted at x t but the impulse response has a value before the application of the signal. So, this diagram shows that there is a chance why because, that depends on value of t naught t naught is not in my hand; t naught is depending on the target. So, where the target is not in my hand, if the target is nearby then t naught will be small and I will have that.

Whereas if t naught was such that it was greater than capital T it was at least equal to capital T then this whole thing should have shifted and I would have got a causal transfer function. So, I can write that one thing is that h t may not be causal. Now this is a serious problem in electronics because if I do not have causal then I do not know what will be the output things etcetera. So, we generally do not work with casual sorry non causal systems, because we want that after the application of the cause the system should respond. So, that is why there is a extra delay additional time delay is added in the impulse response of the match filter to have thing.

So, that additional time delay, so I will say that to make match filter causal filter causal, we give a an additional time delay of let us call it tau naught. To make we give an additional time delay tau naught and what we require from that tau naught, that it should be at least equal to the duration of the signal capital T.

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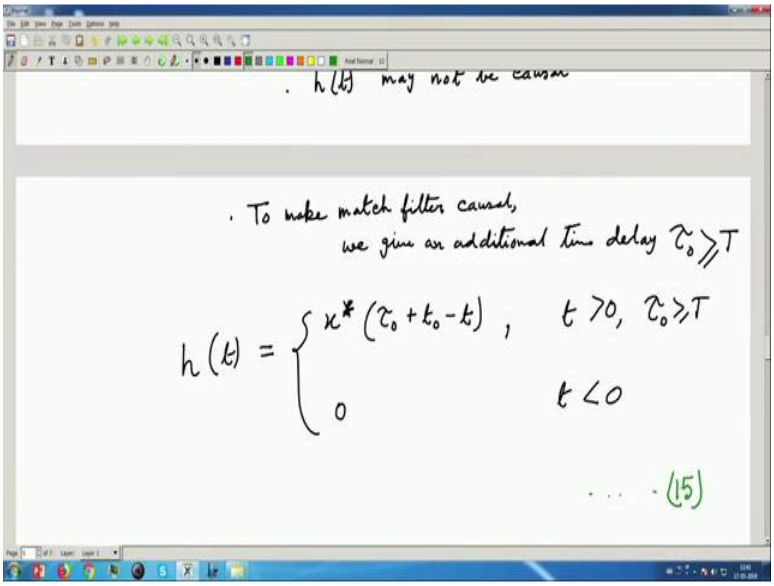

So, we are now changing that h t definition of equation 13. So, we are modifying and writing a causal impulse response of a match filter or a realizable match filter; because non causal systems are not always realizable. So, I am writing that it is x star x conjugate then I am giving that delay tau naught plus t naught minus t.

So, if I do that it will be always causal. So, this will be my a thing. So, this is t is greater than 0 and again writing explicitly tau naught is greater than t; and this is that at t less than 0 I am deliberately making h t 0 to make it non-causal and this is actually the expression we will be using.

$$h(t) = \begin{cases} x^*(\tau_0 + t_0 - t), & t > 0,\ \tau_0 \geq T \\ 0 & t < 0 \end{cases}$$

So, this I am calling my equation number 15. So, I can find out because now I have changed my impulse response, the you see the SNR value maximum SNR value does not get changed by this. Because, SNR value does not depend on this shape but one thing is dependent SNR value is dependent on the delay. So, now, SNR will be maximum at a time t is equal to tau naught plus t naught not simply t naught ok.

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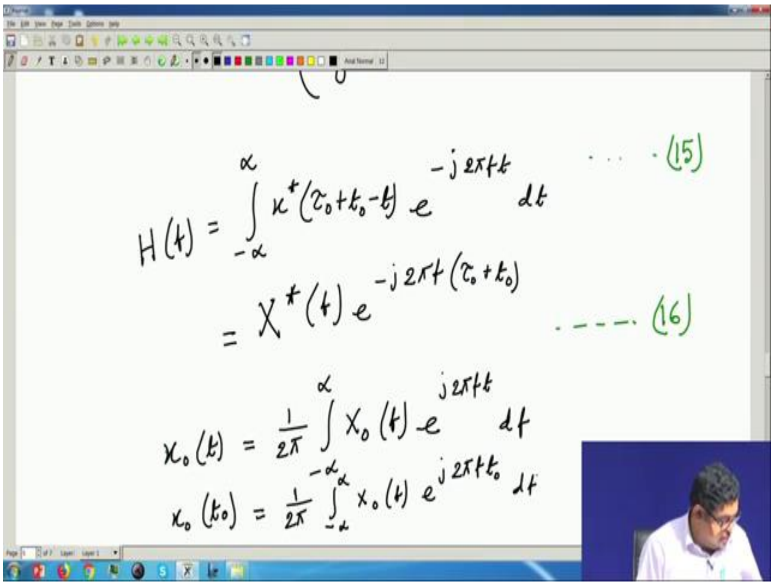

And I can now find the transfer function for the match filter sorry, that transfer function will be H f minus infinity to infinity x star tau naught plus t naught minus t; then e to the power minus Fourier transform j 2 pi f t d t and again you see that this x star is there. So, I have already shown you by that red a thing that how you can take those red things where are those this same thing you can use and if you put it here, for this extra thing you will see that finally, this becomes exactly same those things just do it and it will be X star f e to the power minus j 2 pi f t naught plus or that time I have written tau naught first, so tau naught plus t naught ok.

$$H(f) = \int_{-\infty}^{\infty} x^*(\tau_0+t_0-t)\, e^{-j2\pi f t}\, dt$$
$$= X^*(f)\, e^{-j2\pi f(\tau_0+t_0)}$$

So, you see that basically from the previous one you have these that I already said that your thing will change. So, that always instead of the round trip delay, I am our deliberately added tau naught that is also coming into the picture and let me call this equation number 16. So, now, what I can say that let us find out that what is the output signal value.

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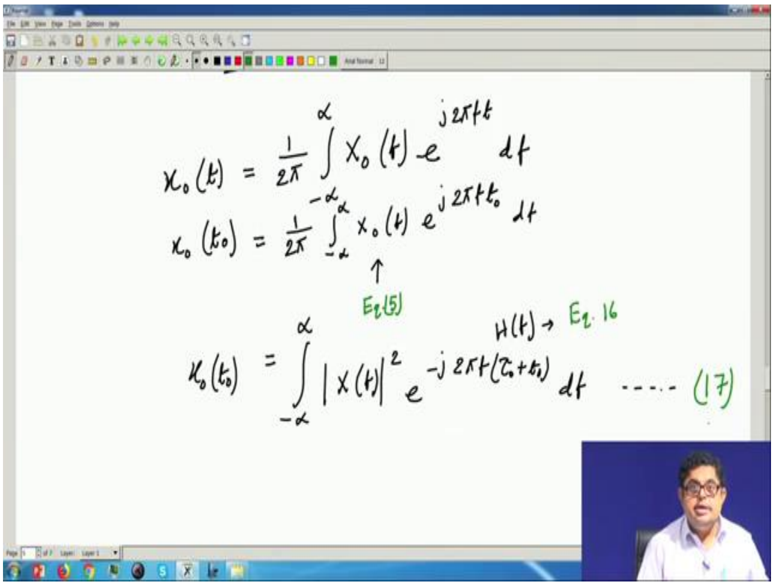

So, x naught t; that means, not nah x o t the output signal let us have that expression. So, we know it is nothing but 1 by 2 pi, these are all more manipulation to get some more extra insight. So, this will be X naught f e to the power j 2 pi f t d f and so, at a particular time I can say at t is equal to t naught what is the value; 1 by 2 pi 2 pi f t 0 d f.

$$x_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi f t}\, df$$

$$x_0(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi f t_0}\, df$$

Now here you see that this signal value at t naught that is here, now x naught f this one is already its value given in equation 5 put its value from equation number 5 and so what you will get is, this equation 5 if you see your notes you will see that it will involve a thing of H f the match filter’s thing; and that H f thing is just now given in equation 16. So, I will say that put this value and H f’s value by equation 16.

So, if you do that, then you will be able to get that this thing becomes X f square e to the power minus j 2 pi f again that tau naught plus t naught into d f this we will be able to do. So, I will write that x naught t naught is this, this I am calling equation number 17.

$$x_0(t_0) = \int_{-\infty}^{\infty} |X(f)|^2 e^{-j2\pi f(\tau_0+t_0)} df \quad ----- (17)$$

So, this says that output of the match filter should be sampled at a delay t naught then you will get the maximum SNR. Now so, here I end this today's lecture. So, we have seen what is match filter, the tutorials etcetera will give a problem that will tell you a particular type of thing signal and you may be asked to find out what is the match filter implementation.

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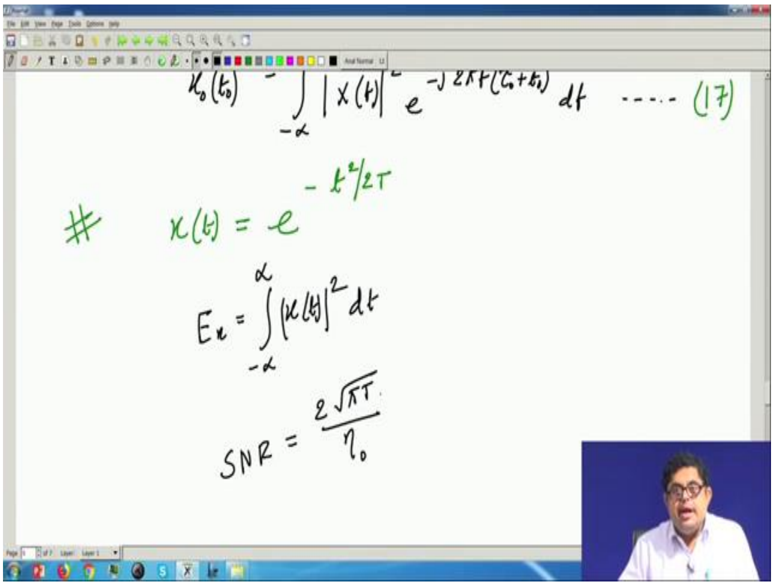

So, that and what is the maximum SNR that we will be able to see like, I can give you an example that in a problem example. Tutorial problem you can be given suppose a signal x t is Gaussian type of signal minus t square by 2 T.

$$x(t) = e^{-t^2/2T}$$

So, and let us say that the noise property will be also said that noise is white Gaussian noise zero mean Gaussian. So, if it this characteristic is said you may be asked to find out what is the maximum instantaneous SNR. I think I am giving you the answer because, these generally in classes we give. So, you can easily put it into the E x expression and find out because what will be E x; always E x is minus in.

Since time domain expression is given you can write it in time domain also that x t square d t.

$$E_x = \int_{-\alpha}^{\alpha} |x(t)|^2 dt$$

And that you can evaluate and once you know E x so, since this is known you can evaluate this and you will see that SNR will turn out to be 2 root over pi T by eta naught.

$$SNR = \frac{2\sqrt{\pi T}}{\eta_0}$$

So, this so, we are ending today but since actually communication people only does these but we will see that later that actually this output of the match filter that has some additional properties, that has some additional things which for radar is very important. We will try to see those properties, I am naming actually we will see that the relation between the transmitted and received signal that also is available from the match filter output. That we will prove next and that will have enormous implication for our resolution and other things also for detection.

Actually we will see that; that will help us to resolve that whether there is a target or not, because in detection theory actually what we are discussing is there is a target but when I receive a signal a question comes to the observer’s mind that is it due to signal or is it due to some noise that there is a good amount of thing because, the eco signal received is very weak and there is enormous noise because it has travelled enormous distance.

So, they are very comparable, so you will be always tempted to ask is it signal or is it noise. Now that question cannot be answered from any other thing but this match filter output will help us to say without ambiguity, whether there is signal or noise that we will discuss next. Actually that is the main purpose of going through match filter.

So, it not only maximizes SNR that is a good thing that is why in all modern day receiver circuits the match filter is used but for radar people this has got much more informations in that output and that if you can extract intelligently you can have lot lot more things to do; that we will discuss in the next class.

Thank you.